Rayleigh waves in Cosserat elastic materials

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Abstract

The present paper gives explicit solutions for surface waves propagation in a homogeneous half space filled with an isotropic Cosserat elastic material. Such solutions are important in the study of seismic waves in an earthquake, supposing that the bottom land is modeled as having a microstructure. To construct explicit expressions for the possible surface waves under consideration, we use the Stroh formalism. These solutions are further used to study the Rayleigh waves and to give the explicit equation for the Rayleigh surface wave speed (secular equation). Numerical calculations and graphics corresponding to the analytical solution are given for aluminium-epoxy composite.

Keywords: Cosserat elastic half space, Rayleigh waves, Stroh formalism, Secular equation

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1. Introduction

This paper is concerned with the seismic waves propagation in Cosserat elastic materials. The theory of elastic materials with microstructure goes back to the book of Cosserat & Cosserat (1909). After that, the theory of materials with microstructure became a subject of intensive study in literature (see, for example, Toupin (1962); Mindlin (1963, 1964); Eringen & Suhubi (1964a,b)). Eringen (1966) introduced the concept of micropolar continua, which is similar with Cosserat continua; additionally he introduced a conservation law for the microinertia tensor, as a special case of micromorphic continua (Eringen & Suhubi, 1964a,b).

It is well-known that the response of the material to external stimuli depends heavily on the motions of its inner structure. Classical elasticity ignores this effect by ascribing

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only translation degrees of freedom to material points of the body. In the micropolar continuum theory, the rotational degrees of freedom play a central role. Thus, we have six degrees of freedom, instead of the three ones considered in classical elasticity. Moreover, in micropolar theories, in order to characterize the force applied on the surface element, two tensors are used: an asymmetric stress tensor and a couple stress tensor. Crystals, composites, polymers, suspensions, blood, grid and multibar systems can be considered as examples of media with microstructure. In fact, nature abounds with many substances which point out the necessity for the considering of micromotions into the mechanical studies. A review of the historical developments as well as references to various contributions on the subject may be found in the monographs by Truesdell & Noll (1965), Nowacki (1986), Eringen (1999) and Ieşan (2004).

The classical theory of elasticity does not explain certain discrepancies that occur in the case of problems involving elastic vibrations of high frequency and short wavelength, that is, vibrations generated by ultrasonic waves. According to the book of Eringen (1999), if the ratio of the characteristic length associated with the external stimuli and the internal characteristic length is in the neighborhood of 1, then the response of constituent subcontinua becomes important. This is the reason why the short wavelength behavior departs drastically from experimental observations in classical elasticity. The micropolar effects become important in high-frequency and short wave-length regions of waves.

We have to outline that Kulesh et al. (2005, 2006) have studied the propagation of elastic surface waves in Cosserat medium and have sought the solutions in the form of wave packets determined by an arbitrary-shape Fourier spectrum. So, the solution is given in the form of Fourier integrals. In some previous papers (see, for example, (Eringen, 1999) and the papers cited therein) the authors have considered some lower bounds for the frequency and also for the wave-number; also, the attenuating coefficient and some conditions upon wave speed and upon wave-number are not explicitly given in the previous papers (see (Kulesh et al., 2006; Erofejev, 2003)). These are consequences of the methods used in their approaches. For the gradient type approach of microstructured solids the propagation of surface waves was studied by (Georgiadis & Velgaki, 2003; Georgiadis et al., 2004). Moreover, the class for which the generalized form of the secular equation has an admissible solution does not established yet.

Our main purpose is to construct new solutions for the waves propagation problem in a micropolar half space. In fact, we use the Stroh formalism (Stroh, 1962; Destrade, 2007) in order to obtain explicit expressions for the possible surface waves in concern and, moreover, we obtain a sextic equation with real coefficients for the propagation condition. Further, we give explicit expressions of the attenuating coefficients and explicit conditions upon wave speed. After that we give the exact expressions of three linear independent amplitude vectors. These amplitudes are characteristic for the coupled case of the elastic and microstructure effects. In the last part of the paper, these inhomogeneous plane wave solutions are used to study the Rayleigh surface waves (Rayleigh, 1885) in an isotropic Cosserat elastic half space. An explicit equation is also established for the Rayleigh wave speed (secular equation). This equation has a simple form, it is not a generalization of the secular equation from the classical elasticity and it is a special one valid for the genuine micropolar model. Moreover, by comparing it with other generalized forms of the secular

equation (Eringen, 1999; Kulesh et al., 2006; Erofeyev, 2003; Koebke & Weitsman, 1971) (see, also, (Chiriță & Ghiba, 2010)) this equation does not involve the complex form of the attenuating coefficients, and for this reason we consider that it has to be most appropriate for considerations in further studies. For a specified class of materials we prove that this equation always has at least one admissible solution. The conditions imposed upon the constitutive quantities are in concordance with those expressed by the positive definiteness of the internal energy and include the materials considered by Gauthier & Jahsman (1975) and Gauthier (1982) (see also (Eringen, 1999), Sections 5.11-5.13) in their experiments. In fact, from the illustrative graphics which we give within this paper one can see that it is possible to have only one admissible wave speed for each material. To obtain such solutions we do not impose any lower bounds to the frequency or to the wave-number. Moreover, the Rayleigh wave solution is also valid for a complementary class of materials with respect to those previously considered in the literature.

We have to notice that the Rayleigh wave problem in the classical theory of linear elasticity has been a subject of great interest in literature on the field (see, for example, the article by Hayes & Rivlin (1962) and the monographs by Jeffreys (1952) and Achenbach (1973)). Important contributions on this argument have been reported recently by Rahman & Barber (1995), Nkemzi (1997, 2008), Malischewsky (2000), Vinh & Ogden (2004), Li (2006), Destrade (2007), Vinh & Malischewsky (2007), Vinh & Malischewsky (2008) and Ting (2011a,b).

2. Basic equations of the Cosserat elastic model

Throughout this section B is a bounded regular region of three-dimensional Euclidean space. We let ∂B denote the boundary on B , and designate by $\mathbf{n}$ the outward unit normal on ∂B . We assume that the body occupying B is a linearly Cosserat elastic material. The body is referred to a fixed system of rectangular Cartesian axes Ox_i ($i = 1, 2, 3$). Throughout this paper Latin indices have the range 1, 2, 3, Greek indices have the range 1, 2 and the usual summation convention is employed. We use a subscript preceded by a comma for partial differentiation with respect to the corresponding coordinate and a superposed dot represents the derivative with respect to time variable.

The deformation of such a medium is described by two fields, namely, the displacement field $\mathbf{u}$ and the microrotation field $\boldsymbol{\varphi}$ over $B \times [0, \infty)$.

Within the framework of the linear theory, the constitutive equations for a homogeneous and isotropic Cosserat elastic solid are (Eringen, 1999; Ieșan, 2004)

$$\begin{aligned} t_{ij} &= \lambda e_{rr} \delta_{ij} + (\mu + \kappa) e_{ij} + \mu e_{ji}, \\ m_{ij} &= \alpha \varkappa_{rr} \delta_{ij} + \beta \varkappa_{ji} + \gamma \varkappa_{ij}, \end{aligned} \quad (2.1)$$

where t_{ij} are the components of the stress tensor, m_{ij} are the components of the couple stress tensor, $\lambda, \mu, \kappa, \alpha, \beta$ and γ are elastic moduli, δ_{ij} is the Kronecker delta and the strain measures e_{ij} and $\varkappa_{ij}$ are given by the geometrical equations

$$e_{ij} = u_{j,i} + \varepsilon_{jik} \varphi_k, \quad \varkappa_{ij} = \varphi_{j,i}, \quad (2.2)$$

u_i are the components of the displacement vector and φ_i are the components of the microrotation vector.

In the absence of the body force and the body couple, the equations of motion of the linear theory of Cosserat elastic materials are given by (Eringen, 1999; Ieşan, 2004)

$$\begin{aligned} t_{ji,j} &= \varrho \ddot{u}_i, \\ m_{ji,j} + \varepsilon_{irs} t_{rs} &= \varrho j \ddot{\varphi}_i, \end{aligned} \quad (2.3)$$

where ϱ is the density mass and j is the inertia coefficient.

The surface force and the surface couple at a regular point $\mathbf{x}$ of the boundary ∂B are given by

$$t_i = t_{ji} n_j, \quad m_i = m_{ji} n_j, \quad (2.4)$$

where $n_j = \cos(\mathbf{n}_x, Ox_j)$ and $\mathbf{n}_x$ is the unit vector of the outward normal to ∂B at $\mathbf{x}$.

In the case of isotropic linear micropolar elastic materials, the internal energy W per unit volume is given by

$$2W(\mathbf{e}, \boldsymbol{\varkappa}) = \lambda e_{rr} e_{ss} + (\mu + \kappa) e_{ij} e_{ij} + \mu e_{ij} e_{ji} + \alpha \varkappa_{rr} \varkappa_{ss} + \beta \varkappa_{ji} \varkappa_{ij} + \gamma \varkappa_{ij} \varkappa_{ij}. \quad (2.5)$$

The necessary and sufficient conditions for the internal energy density W to be a positive definite quadratic form in terms of e_{ij} and $\varkappa_{ij}$ are

$$\begin{aligned} 3\lambda + 2\mu + \kappa &> 0, \quad 2\mu + \kappa > 0, \quad \kappa > 0, \\ 3\alpha + \beta + \gamma &> 0, \quad \gamma + \beta > 0, \quad \gamma - \beta > 0. \end{aligned} \quad (2.6)$$

Further, we also suppose that ϱ and $J = \varrho j$ are strictly positive numbers. Our analysis is within this framework.

In the present paper, we shall consider the region B to be the half space Σ : $x_2 \geq 0$ subject to a process of plane strain parallel to the $x_1 Ox_2$ plane, that is

$$\begin{aligned} u_\beta &= u_\beta(x_1, x_2, t), \quad u_3 = 0, \\ \varphi_\beta &= 0, \quad \varphi_3 = \varphi(x_1, x_2, t), \quad (x_1, x_2, x_3) \in \Sigma, \quad t \geq 0. \end{aligned} \quad (2.7)$$

So, all the quantities defined above are independent of x_3 , and we have the relevant non-zero strain measures

$$e_{\alpha\beta} = u_{\beta,\alpha} + \varepsilon_{\beta\alpha 3} \varphi, \quad \varkappa_{\alpha 3} = \varphi_{,\alpha}, \quad (2.8)$$

while the equations of motion (2.3) are reduced to

$$\begin{aligned} t_{\beta\alpha,\beta} &= \varrho \ddot{u}_\alpha, \\ m_{\beta 3,\beta} + t_{12} - t_{21} &= J \ddot{\varphi}, \end{aligned} \quad (2.9)$$

where

$$\begin{aligned} t_{\alpha\beta} &= \lambda e_{\varrho\varrho} \delta_{\alpha\beta} + (\mu + \kappa) e_{\alpha\beta} + \mu e_{\beta\alpha}, \\ m_{\varrho 3} &= \gamma \varkappa_{\varrho 3}. \end{aligned} \quad (2.10)$$

3. Stroh formalism for plane waves

We will treat the system of partial differential equations $\mathcal{S}$ defined by (2.8), (2.9) and (2.10) in the considered homogeneous micropolar elastic half space by means of the plane waves. Without loss in generality we will study the waves propagating along the x_1 -axis with speed v and wave-number k in the x_1 -direction. Thus, we seek solutions of the above half space problem of the form

$$\{u_1, u_2, \varphi\}(x_1, x_2, t) = \{U_1, U_2, \phi\}(x_2)e^{ik(x_1-vt)}, \quad (3.1)$$

and

$$\{t_{11}, t_{22}, t_{12}, t_{21}, m_{13}, m_{23}\}(x_1, x_2, t) = i\{T_{11}, T_{22}, T_{12}, T_{21}, M_{13}, M_{23}\}(x_2)e^{ik(x_1-vt)}, \quad (3.2)$$

where $i = \sqrt{-1}$ is the unit complex.

The system $\mathcal{S}$ lead to a first-order differential system

$$\mathcal{U}'(x_2) = ik\mathcal{N}\mathcal{U}(x_2), \quad (3.3)$$

for

$$\mathcal{U}(x_2) = \{U_1, U_2, \phi, T_{21}, T_{22}, M_{23}\}^T(x_2), \quad (3.4)$$

where

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{N}_3 & \bar{\mathcal{N}}_1^T \end{pmatrix}, \quad (3.5)$$

$$\mathcal{N}_1 = \begin{pmatrix} 0 & -1 + \frac{\kappa}{c_{66}} & \frac{i\kappa}{kc_{66}} \\ -\frac{c_{12}}{c_{11}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.6)$$

$$\mathcal{N}_2 = \begin{pmatrix} \frac{1}{kc_{66}} & 0 & 0 \\ 0 & \frac{1}{kc_{11}} & 0 \\ 0 & 0 & \frac{1}{k\gamma} \end{pmatrix}, \quad (3.7)$$

$$\mathcal{N}_3 = \begin{pmatrix} -k(c^0 - \varrho v^2) & 0 & 0 \\ 0 & -k\left[\kappa\left(2 - \frac{\kappa}{c_{66}}\right) - \varrho v^2\right] & -i\kappa\left(2 - \frac{\kappa}{c_{66}}\right) \\ 0 & i\kappa\left(2 - \frac{\kappa}{c_{66}}\right) & -\gamma k - \frac{\kappa}{k}\left(2 - \frac{\kappa}{c_{66}}\right) + Jv^2k \end{pmatrix}, \quad (3.8)$$

and

$$c_{11} = \lambda + 2\mu + \kappa, \quad c_{12} = \lambda, \quad c_{66} = \mu + \kappa, \quad c^0 = c_{11} - \frac{c_{12}^2}{c_{11}}, \quad (3.9)$$

Here, $\bar{\mathcal{N}}_1^T$ denotes the transposed matrix of the complex conjugate of the matrix $\mathcal{N}_1$.

Moreover, we require that the solutions to be attenuated in the direction x_2 , so that they are decaying with distance from the plane surface $x_2 = 0$, that is we require that

$$\lim_{x_2 \rightarrow \infty} \{u_1, u_2, \varphi, t_{11}, t_{22}, t_{12}, t_{21}, m_{13}, m_{23}\}(x_1, x_2, t) = 0, \quad (3.10)$$

for all $x_1 \in \mathbb{R}$ and $t \in [0, \infty)$.

Thus, we seek solutions of the above differential system in the form

$$\mathcal{U}(x_2) = \mathcal{V}e^{ikrx_2}, \quad (3.11)$$

with

$$\text{Im}(r) > 0, \quad (3.12)$$

where $\mathcal{V} = \{v_1, v_2, \chi, S_{21}, S_{22}, N_{23}\}^T$ is a constant vector and r is a complex parameter. Therefore, $\mathcal{V}$ and r satisfy the following eigenvalue problem

$$\mathcal{N}\mathcal{V} = r\mathcal{V}. \quad (3.13)$$

The inequality (3.12) ensures that (3.10) holds true.

The characteristic equation corresponding to the above eigenvalue problem is

$$r^6 + P_1r^4 + P_2r^2 + P_3 = 0, \quad (3.14)$$

where

$$\begin{aligned} P_1 &= C_1 + C_2 + C_3, \\ P_2 &= C_1C_2 + C_2C_3 + C_3C_1 - \frac{\varrho\kappa^2v^2}{\gamma k^2c_{66}^2}, \\ P_3 &= C_1C_2C_3 - \frac{\varrho\kappa^2v^2}{\gamma k^2c_{66}^2}C_1, \\ C_1 &= 1 - \frac{\varrho v^2}{c_{11}}, \quad C_2 = 1 - \frac{\varrho v^2}{c_{66}}, \\ C_3 &= 1 + \frac{\kappa(2c_{66} - \kappa)}{\gamma k^2c_{66}} - \frac{Jv^2}{\gamma}. \end{aligned} \quad (3.15)$$

It is easy to see that the characteristic equation can be written in the following form

$$(r^2 + C_1) \left[(r^2 + C_2)(r^2 + C_3) - \frac{\varrho\kappa^2v^2}{\gamma k^2c_{66}^2} \right] = 0, \quad (3.16)$$

whose solutions satisfy

$$\begin{aligned} r_1^2 &= -C_1, \\ r_{2,3}^2 &= \frac{1}{2} \left[-(C_2 + C_3) \pm \sqrt{(C_2 - C_3)^2 + \frac{4\varrho\kappa^2v^2}{\gamma k^2c_{66}^2}} \right]. \end{aligned} \quad (3.17)$$

The above quantities are admissible roots if and only if the inequality (3.12) is satisfied. So, we must have

$$C_1 > 0, \quad C_2 + C_3 > 0, \quad C_2 C_3 > \frac{\varrho \kappa^2 v^2}{\gamma k^2 c_{66}^2}. \quad (3.18)$$

These conditions are equivalent with

$$C_1 > 0, \quad C_2 > 0, \quad C_3 > 0, \quad C_2 C_3 > \frac{\varrho \kappa^2 v^2}{\gamma k^2 c_{66}^2}, \quad (3.19)$$

and further they are satisfied if and only if the wave speed v is so that

$$v^2 < \min\{\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4\}, \quad (3.20)$$

or

$$\mathfrak{c}_5 < v^2 < \min\{\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3\}, \quad (3.21)$$

where

$$\begin{aligned} \mathfrak{c}_1 &= \frac{c_{11}}{\varrho}, \\ \mathfrak{c}_2 &= \frac{c_{66}}{\varrho}, \end{aligned} \quad (3.22)$$

$$\begin{aligned} \mathfrak{c}_3 &= \frac{\gamma}{J} \left[1 + \frac{\kappa(2c_{66} - \kappa)}{\gamma k^2 c_{66}} \right] \\ \mathfrak{c}_4 &= \frac{1}{2} \left(\mathfrak{c}_2 + \mathfrak{c}_3 + \frac{\varkappa^2}{k^2 J c_{66}} - \sqrt{\Delta} \right), \end{aligned} \quad (3.23)$$

$$\mathfrak{c}_5 = \frac{1}{2} \left(\mathfrak{c}_2 + \mathfrak{c}_3 + \frac{\varkappa^2}{k^2 J c_{66}} + \sqrt{\Delta} \right), \quad (3.24)$$

and

$$\Delta = (\mathfrak{c}_2 - \mathfrak{c}_3)^2 + 2(\mathfrak{c}_2 + \mathfrak{c}_3) \frac{\varkappa^2}{k^2 J c_{66}} + \left(\frac{\varkappa^2}{k^2 J c_{66}} \right)^2 > 0. \quad (3.25)$$

Furthermore, it follows from the relations (3.22)-(3.25) that

$$\mathfrak{c}_4 < \mathfrak{c}_2 < \mathfrak{c}_5 \quad (3.26)$$

and hence the possibility that (3.21) be true is avoided. Moreover, we have

$$\mathfrak{c}_3 - \mathfrak{c}_4 = \frac{1}{2} \left(\sqrt{(\mathfrak{c}_2 - \mathfrak{c}_3)^2 + \left(\frac{\varkappa^2}{k^2 J c_{66}} \right)^2} + 2(\mathfrak{c}_2 + \mathfrak{c}_3) \frac{\varkappa^2}{k^2 J c_{66}} - (\mathfrak{c}_2 - \mathfrak{c}_3) - \frac{\varkappa^2}{k^2 J c_{66}} \right) > 0, \quad (3.27)$$

and, therefore, we obtain

$$\mathfrak{c}_3 > \mathfrak{c}_4. \quad (3.28)$$

Concluding, from (3.20), (3.26) and (3.28), we deduce that the wave speed v is admissible if

$$v^2 < \mathfrak{c} \equiv \min\{\mathfrak{c}_1, \mathfrak{c}_4\}. \quad (3.29)$$

In the present paper we consider a homogeneous and isotropic body and so we have explicit expressions for the coefficients of the sextic equation. So we can obtain the eigenvalue r and we can find explicit conditions upon the wave speed v by direct calculation. However, we have to mention that for more complex situations (as, for example, an inhomogeneous or anisotropic body), the method described in (Ting, 2011a,b) can be used to prove that the propagation condition has real coefficients. Moreover, if the case of exponentially graded materials is considered, the impedance matrix method of Barnett and Lothe (Ting, 2011a,b) is highly recommended.

Corresponding to the eigenvalue r_1 of the matrix $\mathcal{A}$, we find the eigenvector

$$\begin{aligned} v_1^{(1)} &= \frac{1}{k}, \\ v_2^{(1)} &= \frac{1}{k}r_1, \\ \chi^{(1)} &= 0, \\ S_{21}^{(1)} &= (c_{11} - c_{12})r_1, \\ S_{22}^{(1)} &= c_{11}r_1^2 + c_{12}, \\ N_{23}^{(1)} &= 0, \end{aligned} \tag{3.30}$$

while, corresponding to the wave speeds $r_{2,3}$, we find the following sets of eigenvectors

$$\begin{aligned} v_1^{(2,3)} &= \frac{\kappa}{c_{66}k}r_{2,3}, \\ v_2^{(2,3)} &= -\frac{\kappa}{c_{66}k}, \\ \chi^{(2,3)} &= -i(r_{2,3}^2 + C_2), \\ S_{21}^{(2,3)} &= \frac{\kappa}{c_{66}}(c_{12} - C_1c_{11}), \\ S_{22}^{(2,3)} &= \frac{\kappa}{c_{66}}(c_{12} - c_{11})r_{2,3}, \\ N_{23}^{(2,3)} &= -ik\gamma r_{2,3}(r_{2,3}^2 + C_2). \end{aligned} \tag{3.31}$$

4. Rayleigh waves

In this section we consider that the Cosserat elastic material occupying the half space $x_2 \geq 0$ is homogeneous and isotropic and we consider Rayleigh waves on the Earth's surface. We assume that the body force and the body couple are absent and the boundary $x_2 = 0$ is traction free and couple traction free, that is, we assume

$$\begin{aligned} t_{21}(x_1, 0, x_3, t) &= 0, \quad t_{22}(x_1, 0, x_3, t) = 0, \\ m_{23}(x_1, 0, x_3, t) &= 0 \quad \text{for all } x_1, x_3 \in \mathbb{R}, t \in [0, \infty). \end{aligned} \tag{4.1}$$

Moreover, we require that the solutions be attenuated in the direction x_2 , so that they are decaying with the distance from the plane surface, that is we require that (3.10) holds true.

Without loss of generality we will study the waves propagating along the x_1 -axis. We will treat the above problem by means of the inhomogeneous plane waves as developed in the previous section. Thus, we seek solutions of the above free half space problem as a linear combination of the eigenvectors $\mathcal{V}^{(j)} = \{v_1^{(j)}, v_2^{(j)}, \chi^{(j)}, S_{21}^{(j)}, S_{22}^{(j)}, N_{23}^{(j)}\}^T$, $j = 1, 2, 3$, that is we seek a solution in the following form

$$\mathcal{V} = a_1 \mathcal{V}^{(1)} + a_2 \mathcal{V}^{(2)} + a_3 \mathcal{V}^{(3)}, \quad (4.2)$$

where a_1, a_2 and a_3 are parameters which will be determined in order that the boundary conditions (4.1) to be satisfied. More exactly, these unknown parameters are solutions of the following algebraic system

$$\begin{aligned} S_{21} &= a_1 S_{21}^{(1)} + a_2 S_{21}^{(2)} + a_3 S_{21}^{(3)} = 0, \\ S_{22} &= a_1 S_{22}^{(1)} + a_2 S_{22}^{(2)} + a_3 S_{22}^{(3)} = 0, \\ N_{23} &= a_1 N_{23}^{(1)} + a_2 N_{23}^{(2)} + a_3 N_{23}^{(3)} = 0. \end{aligned} \quad (4.3)$$

This homogeneous algebraic linear system has non-trivial solutions, if and only if

$$\Delta_0 = \begin{vmatrix} (c_{11} - c_{12})r_1 & \kappa(c_{12} - c_{11}C_1) & \kappa(c_{12} - c_{11}C_1) \\ c_{11}r_1^2 + c_{12} & \kappa(c_{12} - c_{11})r_2 & \kappa(c_{12} - c_{11})r_3 \\ 0 & -ik\gamma c_{66}r_2(r_2^2 + C_2) & -ik\gamma c_{66}r_3(r_3^2 + C_2) \end{vmatrix} = 0. \quad (4.4)$$

On the other hand, in view of the condition (3.29), the roots r_s , $s = 1, 2, 3$, can not be equal. So, the relation (4.4) implies

$$r_1 r_2 r_3 (r_2 + r_3) (c_{11} - c_{12})^2 + (r_2^2 + r_3^2 + r_2 r_3 + C_2) (c_{12} - c_{11} C_1)^2 = 0. \quad (4.5)$$

In view of (3.12) and (3.14)–(3.19), we have that

$$\begin{aligned} r_1 &= i\sqrt{C_1}, \\ r_2 r_3 &= -\sqrt{P_2 - C_1 C_2 - C_1 C_3}, \\ r_2^2 + r_3^2 &= -(C_2 + C_3), \\ r_2 + r_3 &= i\sqrt{C_2 + C_3 + 2\sqrt{P_2 - C_1 C_2 - C_1 C_3}}, \\ r_1 r_2 r_3 &= -i\sqrt{P_3}. \end{aligned} \quad (4.6)$$

It follows that the system (4.3) has a non-trivial solution if and only if the wave speed v satisfies the inequality (3.29) and the following equation

$$\begin{aligned} R(v) &\equiv \sqrt{P_3(v) \left(C_2(v) + C_3(v) + 2\sqrt{P(v)} \right)} (c_{11} - c_{12})^2 - \\ &\quad - \left(C_3(v) + \sqrt{P(v)} \right) (c_{12} - c_{11} C_1(v))^2 = 0, \end{aligned} \quad (4.7)$$

where

$$P(v) = P_2(v) - C_1(v)C_2(v) - C_1(v)C_3(v). \quad (4.8)$$

In the Section 5 we exemplify the above analysis with some numerical results relating the materials considered by Gauthier & Jahsman (1975) and Gauthier (1982) in their experiments. For the quantities which they have considered there and for $k = 1$, the above function has the graphic given by Figure 1. This graphic proves that for the respective micropolar elastic material the wave speed is $v \approx 0.873217$.

1: The graphic of $R(v)$ on the interval $[0, 0.925689]$.

Further, inspired by this graphic, we prove that the secular equation (4.7) has at least a solution for each isotropic micropolar elastic material having a positive definite internal energy. To this end we first note that

$$\lim_{v \rightarrow \sqrt{c_1}} P_3(v) = 0, \quad \lim_{v \rightarrow \sqrt{c_4}} P_3(v) = 0, \quad (4.9)$$

and further, we conclude from (4.7) that

$$\lim_{v \rightarrow \sqrt{c_1}} R(v) < 0, \quad \lim_{v \rightarrow \sqrt{c_4}} R(v) < 0. \quad (4.10)$$

On the other hand, we have to remark that

$$R(0) = 0. \quad (4.11)$$

Let us now consider the function

$$\begin{aligned} Q(v) \equiv R(v) & \left[\sqrt{P_3(v) \left(C_3(v) + C_2(v) + 2\sqrt{P(v)} \right) (c_{11} - c_{12})^2 +} \right. \\ & \left. + \left(C_3(v) + \sqrt{P(v)} \right) (c_{12} - c_{11}C_1(v))^2 \right]. \end{aligned} \quad (4.12)$$

In view of the relation (4.11) we deduce that $Q(0) = 0$. Moreover, if we develop $Q(\varepsilon)$,

with ε a very small parameter, then we find

$$\begin{aligned}
Q(\varepsilon) = & \frac{\varrho(\kappa - 2c_{66})^3}{k^4\gamma^2c_{11}c_{66}^2} \{ [\kappa(2c_{66} - \kappa) + k^2\gamma c_{66}] [(2c_{66} - \kappa)(\kappa(2c_{66} - \kappa) + 2k^2\gamma c_{66}) - \\
& - 2c_{11}(\kappa(2c_{66} - \kappa) + k^2\gamma(\kappa + 2c_{66}))] + \\
& + 2k^2\gamma c_{66} [-c_{11}(2\kappa(2c_{66} - \kappa) + k^2\gamma(\kappa + 2c_{66})) + \\
& + (2c_{66} - \kappa)(\kappa(2c_{66} - \kappa) + k^2\gamma c_{66})] \sqrt{C_3(0)} \} \varepsilon^2 + O(\varepsilon^3).
\end{aligned} \tag{4.13}$$

It is easy to see that, if

$$\lambda + \mu > 0, \tag{4.14}$$

then, $Q(\varepsilon)$ is a positive quantity for any $\varepsilon > 0$ sufficiently small. Let $\varepsilon_0 > 0$ be so that $Q(\varepsilon_0) > 0$. Then, for materials characterized by the inequality (4.14), we have

$$R(\varepsilon_0) > 0. \tag{4.15}$$

On the other hand, in this case, we have

$$\mathfrak{c}_4 < \mathfrak{c}_1. \tag{4.16}$$

The function R is continuous on $(\varepsilon_0, \sqrt{\mathfrak{c}_4})$ and hence, in view of the relations (4.10) and (4.15), we can conclude that there is at least one admissible solution of the equation (4.7) in the interval $(\varepsilon_0, \sqrt{\mathfrak{c}_4})$.

We have to outline that, in the classical elasticity, the solution of the corresponding equation for Rayleigh surface waves speed (the secular equation) has been studied numerically by Rayleigh (1885). Jeffreys (1952) and Hayes & Rivlin (1962) also have studied the secular equation (see also the recent works by Rahman & Barber (1995), Nkemzi (1997, 2008), Malischewsky (2000), Vinh & Ogden (2004), Li (2006), Destrade (2007), Vinh & Malischewsky (2007), Vinh & Malischewsky (2008) and Ting (2011a,b)). The secular equation (4.7) also poses difficulties and the determination of the exact numerical solutions can be obtained as in the next section. Because we have the exact analytical solutions, the errors which can appear in the calculus become very small. But, by comparing with other generalized form of the secular equation ((Eringen, 1999), page 180) (see, also, (Chiriță & Ghiba, 2010)), this equation does not involve the complicated form of the attenuating coefficients, and we consider that it will be more appropriate for considerations in further studies.

5. Some numerical applications for experimental data

In this section we give numerical computations and some representations for the quantities obtained in the previous sections associated with the aluminium-epoxy composite considered by Gauthier (1982) (see also (Eringen, 1999)). For such a material we have

$$\begin{aligned}
\lambda = 7,59 \text{ G Pa}, \quad \mu = 1,89 \text{ G Pa}, \quad \frac{\kappa}{\mu} = 0,00788, \\
\frac{\kappa}{\rho} = 0,0067 \text{ mm}(\mu\text{s})^{-1}, \quad j = 0,0196 \text{ mm}^2, \quad \frac{\gamma}{\mu j} = 7,11.
\end{aligned} \tag{5.1}$$

In what follows, we give first of all the table of values for the wave speed v , solution of the secular equation (4.7), for different values of the wave number k and we identify in each case the value of the upper bound, $\sqrt{\mathfrak{c}_4}$.

Table 1. Surface wave speed

k	v	$\sqrt{c_4}$	k	v	$\sqrt{c_4}$
0.001	0.870495	0.923901	1	0.873217	0.925689
0.01	0.870519	0.923914	2	0.873242	0.925707
0.05	0.870979	0.92419	3	0.873246	0.92571
0.1	0.871755	0.924683	4	0.87324783	0.925711
0.2	0.872616	0.925264	5	0.87324857	0.92571154
0.3	0.872926	0.925481	6	0.87324898	0.92571183
0.4	0.873058	0.925575	7	0.87324922	0.92571201
0.5	0.873124	0.925622	8	0.87324968	0.92571234
0.6	0.873161	0.925649	9	0.87324949	0.92571220
0.7	0.873184	0.925665	10	0.87324957	0.92571226
0.8	0.873199	0.925676	50	0.87324988	0.92571249
0.9	0.87321	0.925684	$10^5 \geq k \geq 10^2$	0.87324990	0.92571249

We can observe that for k greater than 4 the ratio $\frac{v}{\sqrt{c_4}}$ is stabilized at a value around 0.943327. For large values of the wave number we will have the largest possible wave speed and thus the largest frequency.

In the Section 4, we have proved that in general we have at least one solution of the equation (4.7). But from the Figure 1, we can see that for the materials considered by Gauthier (1982), we have only one solution of this equation. So, the speed is uniquely determined.

The Figures 2, 3 and 4 provide the behavior of the displacement and the microrotation fields. Let us remark that we have maximum (minimum) values of the microrotation component in the neighborhood of minimum (maximum) values of the macrodisplacement component u_2 .

A significant attenuation of the amplitude of the surface wave with respect to the deepness in the direction of the x_2 can be observed. If we look to the behaviour of the components u_1 , u_2 and φ , from the above figures we can see that u_1 has a rapid decay with depth while the decays of u_2 of φ are milder.

In the Figures 5, 6 and 7 we represent the behavior of the traction and of the couple traction.

A minimum of the couple traction occurs around a maximum value of the component S_{21} of the stress tensor and vice versa. A similar decaying effect with respect to the deepness x_2 , like that for the displacement and microrotation components, can be observed for the stress and couple stress components.

2: The component u_1 of the displacement vector for $k = 1$ and the time $t = 10$, on $(x_1, x_2) \in [0, 7] \times [0, 8)$.

3: The component u_2 of the displacement vector for $k = 1$ and the time $t = 10$, on $(x_1, x_2) \in [0, 7] \times [0, 8)$.

4: The component φ of the microrotation vector for $k = 1$ and the time $t = 10$, on $(x_1, x_2) \in [0, 7] \times [0, 8)$.

5: S_{21} for $k = 1$ and the time $t = 10$, on $(x_1, x_2) \in [0, 7] \times [0, 8)$.

6: S_{22} for $k = 1$ and the time $t = 10$, on $(x_1, x_2) \in [0, 7] \times [0, 8)$.

7: N_{23} for $k = 1$ and the time $t = 10$, on $(x_1, x_2) \in [0, 7] \times [0, 8)$.

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