Rank-one convexity and polyconvexity of Hencky-type energies

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We investigate a family of isotropic volumetric-isochorically decoupled strain energies based on the Hencky-logarithmic (true, natural) strain tensor $\log U$. The main result of this note is that for $n = 2$ the considered energies are rank-one convex for suitable values of two material parameters. We also conjecture that there are values of the material parameters such that the corresponding energies are polyconvex.

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1 Introduction

In a recent contribution [7], we have introduced a family of nonlinear elastic energies based on certain invariants of the Hencky tensor $\log U$, namely $\|\operatorname{dev}_n \log U\|^2$ and $(\operatorname{tr}(\log U))^2$, where $F = \nabla \varphi$ is the deformation gradient, $U = \sqrt{F^T F}$ is the right stretch tensor, $\log U$ is the referential (Lagrangian) logarithmic strain tensor, and $\operatorname{dev}_n X = X - \frac{1}{n} \operatorname{tr}(X) \cdot \mathbb{I}$ is the deviatoric part (the projection onto the traceless tensors) of the second order tensor $X \in \mathbb{R}^{n \times n}$ and $\|\cdot\|$ is the Frobenius tensor norm. The considered family of exponentiated Hencky-logarithmic strain type energies is given by

$$W_{eH}(F) := \begin{cases} \underbrace{\frac{\mu}{k} e^k \|\operatorname{dev}_n \log U\|^2 + \frac{\kappa}{2\hat{k}} e^{\hat{k}} (\operatorname{tr}(\log U))^2}_{\text{volumetric-isochoric split}} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0, \end{cases} \quad (1)$$

where $\mu > 0$ is the shear (distortional) modulus, $\kappa = \frac{2\mu+3\lambda}{3} > 0$ is the bulk modulus with the first Lamé constant λ , and $k, \hat{k}$ are dimensionless parameters. The immediate importance of the family (1) of free-energy functions is seen by looking at small (but not infinitesimally small) strains. For small elastic strains, W_{eH} approximates the classical quadratic Hencky strain energy $W_H(F) := \mu \|\operatorname{dev}_n \log U\|^2 + \frac{\kappa}{2} [\operatorname{tr}(\log U)]^2$, which is not rank-one convex. The Hencky energy W_H has been introduced by Heinrich Hencky starting from 1928 [4] and has since then acquired a unique status in finite elastostatics and especially in finite strain elasto-plasticity. The first axiomatic study on the nonlinear stress-strain function involving a logarithmic strain tensor is, however, due to the famous geologist George Ferdinand Becker in 1893 [9].

In plane elastostatics, i.e. $n = 2$, we prove the rank-one convexity of the proposed family W_{eH} for $k \geq \frac{1}{4}$ and $\hat{k} \geq \frac{1}{8}$. Moreover, in [7] we show that the corresponding Cauchy (true) stress - true strain relation is invertible for $n = 2, 3$ and we discuss the monotonicity of the Cauchy stress tensor as a function of the true strain tensor. We also prove that the rank-one convexity of the energies belonging to this family is not preserved in dimension $n = 3$ and that the energy $W(F) := e^k \|\log U\|^2$, $F \in \operatorname{GL}^+(n)$, $n \in \mathbb{N}$, $n \geq 2$, is not rank-one convex. An immediate application of W_{eH} in multiplicative elasto-plasticity [3, 11] is also proposed in [7].

2 Rank-one convexity in plane elastostatics

In order to study the rank-one convexity of the family of energies W_{eH} , the first step is to check if the volumetric response $\det F \mapsto e^{\hat{k}(\log \det F)^m}$ is convex. In arbitrary dimensions, the following result holds:

Proposition 2.1 *The function $\det F \mapsto e^{\hat{k}(\log \det F)^m}$, $F \in \operatorname{GL}^+(n)$, is convex in $\det F$ for $\hat{k} \geq \frac{1}{m(m+1)}$.*

The next step is to prove the rank-one convexity of the isochoric exponentiated Hencky energy $F \mapsto e^k \|\operatorname{dev}_2 \log U\|^2$ in plane elastostatics. We show that although $F \mapsto \|\operatorname{dev}_2 \log U\|^2$ is not rank-one convex, the function $F \mapsto e^{\|\operatorname{dev}_2 \log U\|^2}$ is indeed rank-one convex. In terms of the singular values λ_1, λ_2 of the right stretch tensor $U = \sqrt{F^T F}$, we have the following representation of the isochoric exponentiated Hencky energy:

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Lemma 2.2 Let $F \in \text{GL}^+(2)$ with singular values λ_1, λ_2 . Then $e^{k \|\text{dev}_2 \log U\|^2} = e^{k \|\log \frac{U}{\det U^{1/2}}\|^2} = g(\lambda_1, \lambda_2)$, where $g : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, $g(\lambda_1, \lambda_2) := e^{\frac{k}{2} \left(\log \frac{\lambda_1}{\lambda_2} \right)^2}$.

Using this representation and the necessary and sufficient criteria for rank-one convexity given by Knowles and Sternberg [1] we establish the following result:

Proposition 2.3 If $k \geq \frac{1}{4}$, then the function $F \mapsto e^{k \|\text{dev}_2 \log U\|^2}$ is rank-one convex in $\text{GL}^+(2)$.

In view of the above auxiliary results we conclude:

Theorem 2.4 (planar rank-one convexity) The functions $W_{eH} : \mathbb{R}^{n \times n} \rightarrow \bar{\mathbb{R}}_+$ from the family of exponentiated Hencky type energies $W_{eH}(F)$ are rank-one convex for the two-dimensional situation $n = 2$, for $\mu > 0, \kappa > 0, k \geq \frac{1}{4}$ and $\hat{k} \geq \frac{1}{8}$.

We give the conjecture:

Conjecture 2.5 (planar polyconvexity) The functions $W_{eH} : \mathbb{R}^{n \times n} \rightarrow \bar{\mathbb{R}}_+$ from the family of exponentiated Hencky type energies $W_{eH}(F)$ are polyconvex for the two-dimensional situation $n = 2$, for $\mu > 0, \kappa > 0, k \geq \frac{1}{4}$ and $\hat{k} \geq \frac{1}{8}$.

In plane elasto-statics, the rank-one convex energy $W_{eH}(F)$ is applicable to the bending or shear of long strips and to all cases in which symmetry arguments can be applied to reduce the formulation to a planar deformation.

3 Outlook for three dimensions and conclusions

In [7] it is also shown that for all parameters $k > 0$ the energy function $F \mapsto \frac{\mu}{k} e^{k \|\text{dev}_3 \log U\|^2}$, $F \in \text{GL}^+(3)$, is not rank-one convex. Numerical tests suggest that the rank-one convexity domain of the distortional energy function $F \mapsto \frac{\mu}{k} e^{k \|\text{dev}_3 \log U\|^2}$, $F \in \text{GL}^+(3)$, is a large cone $\{U \in \text{PSym}(3) \mid \|\text{dev}_3 \log U\|^2 < 27\}$, while $k \geq \frac{3}{16}$ is the necessary condition for separate convexity of $e^{k \|\text{dev}_3 \log U\|^2}$ in the three-dimensional situation $n = 3$.

By purely differential geometric reasoning, in forthcoming papers [5, 6, 10] it will be shown that

$$\text{dist}_{\text{geod}}^2((\det F)^{1/n} \cdot \mathbb{1}, \text{SO}(n)) = \text{dist}_{\text{geod}, \mathbb{R}_+ \cdot \mathbb{1}}^2((\det F)^{1/n} \cdot \mathbb{1}, \mathbb{1}) = |\log \det F|^2,$$

$$\text{dist}_{\text{geod}}^2\left(\frac{F}{(\det F)^{1/n}}, \text{SO}(n)\right) = \text{dist}_{\text{geod}, \text{SL}(n)}^2\left(\frac{F}{(\det F)^{1/n}}, \text{SO}(n)\right) = \|\text{dev}_n \log U\|^2,$$

where $\text{dist}_{\text{geod}}^2$ is the canonical left invariant geodesic distance on the Lie-groups $\text{GL}(n)$, $\text{SL}(n)$ and $\mathbb{R}_+ \cdot \mathbb{1}$, see also [2, 10].

Hence, using this terminology, in the present note we have shown, for $\mu > 0, \kappa > 0, k \geq \frac{1}{4}$ and $\hat{k} \geq \frac{1}{8}$, the rank-one convexity of $W_{eH}(F) := \frac{\mu}{k} e^{k \text{dist}_{\text{geod}, \text{SL}(2)}^2\left(\frac{F}{(\det F)^{1/2}}, \text{SO}(2)\right)} + \frac{\kappa}{2k} e^{\hat{k} \text{dist}_{\text{geod}, \mathbb{R}_+ \cdot \mathbb{1}}^2(\det F^{1/2} \cdot \mathbb{1}, \text{SO}(2))}$. In a forthcoming paper [8] we will provide a detailed study of the polyconvexity of the energies W_{eH} .

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