SAINT-VENANT'S PROBLEM FOR COMPOSITE MICROPOLAR ELASTIC CYLINDERS

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Abstract—We consider Saint-Venant's problem in the case when the cross-section of the cylinder is occupied by different inhomogeneous and isotropic micropolar elastic solids. The results are used to study the deformation of a circular composite cylinder.

1. INTRODUCTION

THE DEFORMATION of homogeneous and isotropic micropolar elastic cylinders was studied in various papers [1–6]. In this paper we consider the case of a composite cylinder when the generic cross-section is occupied by different inhomogeneous and isotropic micropolar elastic solids. We assume that the elastic coefficients are independent of the axial coordinate. In the classical theory of elasticity this problem was considered in [7, 8].

In the first part of the paper we present a method of solving Saint-Venant's problem. Then, we use this method to study the deformation of a composite circular cylinder.

2. STATEMENT OF THE PROBLEM

Throughout this paper R denotes the interior of a right cylinder of the length l with the generic cross-section Σ and the lateral boundary B . We call ∂R the boundary of R , and designate by n_i the components of the outward unit normal of ∂R . Throughout this paper a rectangular Cartesian coordinate system Ox_k ($k = 1, 2, 3$) is used. The rectangular Cartesian coordinate frame is chosen such that the x_3 -axis is parallel to the generators of R and the x_1Ox_2 -plane contains one of the terminal cross-sections. We call $\Sigma^{(o)}$ the cross-section located at $x_3 = o$ and $\Sigma^{(l)}$ the cross-section which lies in the plane $x_3 = l$. We denote by L the boundary of the generic cross-section Σ .

We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1, 2) whereas Latin subscripts—over (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote differentiation with respect to the corresponding Cartesian coordinate.

In this paper we consider the linear theory of micropolar elasticity. The basic equation for isotropic and centrosymmetric solids are [9]:

Equilibrium equations, in absence of body forces and body couples

$$t_{ji,j} = 0, \quad m_{ji,j} + \epsilon_{irs} t_{rs} = 0, \quad (2.1)$$

constitutive equations

$$t_{ij} = \lambda e_{rr} \delta_{ij} + (\mu + \kappa) e_{ij} + \mu e_{ji}, \quad m_{ij} = \alpha \varphi_{r,r} \delta_{ij} + \beta \varphi_{i,j} + \gamma \varphi_{j,i}, \quad (2.2)$$

geometrical equations

$$e_{ij} = u_{j,i} + \epsilon_{ji} \varphi_r, \quad (2.3)$$

In these equations we have used the following notations: t_{ij} , components of the stress tensor; m_{ij} , components of the couple-stress tensor; u_i , components of the displacement vector; φ_i , components of the microrotation vector; e_{ij} , components of the strain tensor; δ_{ij} , Kronecker's delta; ϵ_{ijk} , alternating symbol; $\lambda, \mu, \kappa, \alpha, \beta, \gamma$, characteristic coefficients of the material.

We assume that Σ is a C^1 -smooth domain [10]. Let L_1 and L_2 be two disjoint subsets of L such that $L = L_1 \cup L_2$. Let Γ be a curve contained in Σ satisfying the conditions that $L_\rho \cup \Gamma$ ($\rho = 1, 2$) is the boundary of a domain Σ_ρ contained in Σ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Suppose that Σ_1 and Σ_2 are occupied by two elastic materials. Let $\lambda^{(\rho)}, \mu^{(\rho)}, \kappa^{(\rho)}, \alpha^{(\rho)}, \beta^{(\rho)}, \gamma^{(\rho)}$ be the elastic coefficients relative to Σ_ρ . We denote by R_ρ the domain occupied by the material with the elastic coefficients $\lambda^{(\rho)}, \dots, \gamma^{(\rho)}$. Throughout this paper we assume that

$$\lambda^{(\rho)} = \lambda^{(\rho)}(x_1, x_2), \mu^{(\rho)} = \mu^{(\rho)}(x_1, x_2), \dots, \gamma^{(\rho)} = \gamma^{(\rho)}(x_1, x_2). \quad (2.4)$$

The functions $\lambda^{(\rho)}, \dots, \gamma^{(\rho)}$ are supposed to belong to C^∞ and the elastic potential corresponding to the body which occupies R_ρ is assumed to be a positive definite quadratic form.

We can consider Σ as being occupied by an elastic medium which, in general, has elastic coefficients discontinuous along Γ . We denote by Π the surface of separation of the two materials.

The functions u_i, φ_i must be continuous in passing from one medium to another so that we have the conditions

$$\begin{aligned} [u_i]_1 &= [u_i]_2, [\varphi_i]_1 = [\varphi_i]_2, \\ [t_{\beta i}]_1 \nu_\beta &= [t_{\beta i}]_2 \nu_\beta, [m_{\beta i}]_1 \nu_\beta = [m_{\beta i}]_2 \nu_\beta, \quad \text{on } \Pi, \end{aligned} \quad (2.5)$$

where it has been indicated that the expressions in the parentheses are calculated for the domains R_1 and R_2 , respectively, and ν_β are the direction cosines of the vector normal to Γ , outward to Σ_1 .

The cylinder is supposed to be free of lateral loading so that we have the conditions

$$t_{\alpha i} n_\alpha = 0, m_{\alpha i} n_\alpha = 0, \quad \text{on } B. \quad (2.6)$$

Saint-Venant's problem consists in determining the equilibrium of the cylinder (in absence of body forces and body couples) when the load applied to one of the bases of the cylinder is statically equivalent to a force $P(P_i)$ and a couple $M(M_i)$ and is free of lateral loading.

We will have occasion to use three special problems $P^{(s)}$ ($s = 1, 2, 3$) of plane strain. In what follows we denote by $u_\alpha^{(s)}, \varphi_3^{(s)}, e_{ij}^{(s)}, t_{ij}^{(s)}, m_{ij}^{(s)}$ the functions $u_\alpha, \varphi_3, e_{ij}, t_{ij}, m_{ij}$ for the problems $P^{(s)}$.

The problems $P^{(s)}$ are characterized by the equations

$$\begin{aligned} t_{\beta\alpha,\beta}^{(\eta)} + (\lambda^{(\rho)} x_\eta)_{,\alpha} &= 0, t_{\beta\alpha,\beta}^{(3)} + \lambda_{,\alpha}^{(\rho)} = 0, \\ m_{\beta 3,\beta}^{(\eta)} + \epsilon_{3\alpha\beta} t_{\alpha\beta}^{(\eta)} + \epsilon_{3\alpha\eta} \beta_{,\alpha}^{(\rho)} &= 0, m_{\beta 3,\beta}^{(3)} + \epsilon_{3\alpha\beta} t_{\alpha\beta}^{(3)} = 0, \end{aligned} \quad (2.7)$$

$$t_{\alpha\beta}^{(s)} = \lambda^{(\rho)} e_{\eta\eta}^{(s)} \delta_{\alpha\beta} + (\mu^{(\rho)} + \kappa^{(\rho)}) e_{\alpha\beta}^{(s)} + \mu^{(\rho)} e_{\beta\alpha}^{(s)}, m_{\alpha 3}^{(s)} = \gamma^{(\rho)} \varphi_{3,\alpha}^{(s)}, \quad (2.8)$$

$$e_{\alpha\beta}^{(s)} = u_{\beta,\alpha}^{(s)} + \epsilon_{3\beta\alpha} \varphi_3^{(s)}, \quad \text{in } \Sigma_\rho, \quad (2.9)$$

and the following conditions

$$\begin{aligned} [u_\alpha^{(s)}]_1 &= [u_\alpha^{(s)}]_2, [\varphi_3^{(s)}]_1 = [\varphi_3^{(s)}]_2, [t_{\beta\alpha}^{(s)}]_1 \nu_\beta = [t_{\beta\alpha}^{(s)}]_2 \nu_\beta + p_\alpha^{(s)}, \\ [m_{\alpha 3}^{(s)}]_1 \nu_\alpha &= [m_{\alpha 3}^{(s)}]_2 \nu_\alpha + q^{(s)}, \quad \text{on } \Gamma, \end{aligned} \quad (2.10)$$

$$\begin{aligned} [t_{\beta\alpha}^{(\eta)} n_\beta]_\rho &= -\lambda^{(\rho)} x_\eta n_\alpha, [t_{\beta\alpha}^{(3)} n_\beta]_\rho = -\lambda^{(\rho)} n_\alpha, [m_{\alpha 3}^{(\eta)} n_\alpha]_\rho = \epsilon_{3\eta\nu} \beta_\nu^{(\rho)} n_\rho, \\ [m_{\alpha 3}^{(3)} n_\alpha]_\rho &= 0, \quad \text{on } L_\rho \end{aligned} \quad (2.11)$$

where

$$\begin{aligned} p_\alpha^{(\eta)} &= (\lambda^{(2)} - \lambda^{(1)}) x_\eta \nu_\alpha, p_\alpha^{(3)} = (\lambda^{(2)} - \lambda^{(1)}) \nu_\alpha, \\ q^{(\eta)} &= \epsilon_{3\eta\alpha} (\beta_\alpha^{(1)} - \beta_\alpha^{(2)}) \nu_\alpha, q^{(3)} = 0. \end{aligned} \quad (2.12)$$

It is easy to prove that the necessary and sufficient conditions for the existence of the solution are satisfied for each boundary value problem $P^{(s)}$. In what follows we assume that the functions $u_\alpha^{(s)}$, $\varphi_3^{(s)}$ are known.

3. EXTENSION AND BENDING BY TERMINAL COUPLES

Let the loading applied on the end $\Sigma^{(o)}$ be statically equivalent to the force $P(o, o, P_3)$ and the moment $M(M_1, M_2, o)$. Thus, for $x_3 = o$ we have the following conditions

$$\int_{\Sigma} t_{3\alpha} d\sigma = 0, \quad (3.1)$$

$$\int_{\Sigma} t_{33} d\sigma = -P_3, \quad (3.2)$$

$$\int_{\Sigma} (x_\alpha t_{33} - \epsilon_{3\alpha\beta} m_{3\beta}) d\sigma = \epsilon_{3\alpha\beta} M_\beta, \quad (3.3)$$

$$\int_{\Sigma} (\epsilon_{3\alpha\beta} x_\alpha t_{3\beta} + m_{33}) d\sigma = 0. \quad (3.4)$$

The problem consists in the solving of the eqns (2.1)–(2.3) with the conditions (2.5), (2.6), (3.1)–(3.4). We seek the solution in the form

$$\begin{aligned} u_\alpha &= -\frac{1}{2} a_\alpha x_3^2 + \sum_{s=1}^3 a_s u_\alpha^{(s)}, \quad u_3 = (a_1 x_1 + a_2 x_2 + a_3) x_3, \quad \varphi_\alpha = \epsilon_{3\alpha\beta} a_\beta x_3, \\ \varphi_3 &= \sum_{s=1}^3 a_s \varphi_3^{(s)}, \end{aligned} \quad (3.5)$$

where $u_\alpha^{(s)}$, $\varphi_3^{(s)}$ are the solutions of the problems $P^{(s)}$ and a_s are unknown constants. From (3.5) and (2.2) we get

$$\begin{aligned} t_{\alpha\beta} &= \lambda^{(\rho)} (a_1 x_1 + a_2 x_2 + a_3) \delta_{\alpha\beta} + \sum_{s=1}^3 t_{\alpha\beta}^{(s)}, \\ t_{33} &= (\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}) (a_1 x_1 + a_2 x_2 + a_3) + \lambda^{(\rho)} \sum_{s=1}^3 a_s e_{\alpha\alpha}^{(s)}, \\ t_{\alpha 3} &= t_{3\alpha} = 0, \quad m_{\alpha\beta} = m_{33} = 0, \\ m_{\alpha 3} &= \epsilon_{3\alpha\nu} \beta^{(\rho)} a_\nu + \sum_{s=1}^3 a_s m_{\alpha 3}^{(s)}, \\ m_{3\alpha} &= \epsilon_{3\alpha\nu} \gamma^{(\rho)} a_\nu + \sum_{s=1}^3 a_s m_{3\alpha}^{(s)}. \end{aligned} \quad (3.6)$$

Using (2.7)–(2.11), (3.5) and (3.6) it follows that the conditions (2.5) and (2.6) are satisfied.

The conditions (3.1) and (3.4) are satisfied on the basis of the relations (3.6). From (3.2), (3.3) and (3.6) we obtain the following system for the unknown constants

$$A_{rs} a_s = B_r, \quad (3.7)$$

where

$$\begin{aligned} A_{\alpha\beta} &= \sum_{\rho=1}^2 \int_{\Sigma_\rho} \{x_\alpha [(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}) x_\beta + \lambda^{(\rho)} e_{\eta\eta}^{(\beta)}] - \epsilon_{3\alpha\lambda} (\epsilon_{3\lambda\beta} \gamma^{(\rho)} + m_{3\lambda}^{(\beta)})\} d\sigma, \\ A_{\alpha 3} &= \sum_{\rho=1}^2 \int_{\Sigma_\rho} \{x_\alpha [\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)} + \lambda^{(\rho)} e_{\eta\eta}^{(3)}] - \epsilon_{3\alpha\lambda} m_{3\lambda}^{(3)}\} d\sigma, \end{aligned}$$

$$\begin{aligned}
A_{3\alpha} &= \sum_{\rho=1}^2 \int_{\Sigma_\rho} [(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})x_\alpha + \lambda^{(\rho)}e_{\eta\eta}^{(\alpha)}] d\sigma, \\
A_{33} &= \sum_{\rho=1}^2 \int_{\Sigma_\rho} [\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)} + \lambda^{(\rho)}e_{\eta\eta}^{(3)}] d\sigma, \\
B_\alpha &= \epsilon_{3\alpha\beta} M_\beta, \quad B_3 = -P_3.
\end{aligned} \tag{3.8}$$

As in [7, 8] we can prove that $\det(A_{rs}) \neq 0$ so that the system (3.7) determines the constants a_i .

4. TORSION

Let the loading applied on the end $\Sigma^{(o)}$ be statically equivalent to the force $P(o, o, o)$ and the moment $M(o, o, M_3)$. In this case for $x_3 = o$, we have the following conditions

$$\int_{\Sigma} t_{3\alpha} d\sigma = 0, \tag{4.1}$$

$$\int_{\Sigma} t_{33} d\sigma = 0, \tag{4.2}$$

$$\int_{\Sigma} (x_\alpha t_{33} - \epsilon_{3\alpha\beta} m_{3\beta}) d\sigma = 0, \tag{4.3}$$

$$\int_{\Sigma} (\epsilon_{3\alpha\beta} x_\alpha t_{3\beta} + m_{33}) d\sigma = -M_3. \tag{4.4}$$

The problem consists in the solving of the eqns (2.1)–(2.3) with the conditions (2.5), (2.6) and (4.1)–(4.4). Using the results of [1] we try to solve the problem assuming that

$$u_\alpha = \epsilon_{3\beta\alpha} \tau x_\beta x_3, \quad u_3 = \tau \Phi(x_1, x_2), \quad \varphi_\alpha = \tau \Phi_\alpha(x_1, x_2), \quad \varphi_3 = \tau x_3, \tag{4.5}$$

where Φ, Φ_1, Φ_2 are unknown functions and τ is a unknown constant.

Let $V = (G, G_1, G_2)$ be an ordered triplet of functions G, G_1, G_2 . We introduce the notations

$$\begin{aligned}
T_\alpha V &= (\mu^{(\rho)} + \kappa^{(\rho)})G_{,\alpha} + \kappa^{(\rho)}\epsilon_{3\alpha\beta}G_\beta, \\
S_\alpha V &= \mu^{(\rho)}G_{,\alpha} + \kappa^{(\rho)}\epsilon_{3\beta\alpha}G_\beta, \\
M_{\alpha\beta} V &= \alpha^{(\rho)}G_{\nu,\nu}\delta_{\alpha\beta} + \beta^{(\rho)}G_{\alpha,\beta} + \gamma^{(\rho)}G_{\beta,\alpha},
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\mathcal{L}^{(\rho)} V &\equiv (T_\alpha V)_{,\alpha} = [(\mu^{(\rho)} + \kappa^{(\rho)})G_{,\alpha}]_{,\alpha} + \epsilon_{3\alpha\beta}(\kappa^{(\rho)}G_\beta)_{,\alpha}, \\
\mathcal{L}_\nu^{(\rho)} V &\equiv (M_{\beta\nu} V)_{,\beta} + \epsilon_{3\nu\beta}(T_\beta V - S_\beta V) = (\alpha^{(\rho)}G_{\lambda,\lambda})_{,\nu} + (\beta^{(\rho)}G_{\lambda,\nu})_{,\lambda} \\
&\quad + (\gamma^{(\rho)}G_{\nu,\lambda})_{,\lambda} + \epsilon_{\nu\beta 3}\kappa^{(\rho)}G_{,\beta} - 2\kappa^{(\rho)}G_{,\nu},
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\mathcal{B}^{(\rho)} V &\equiv (T_\alpha V)n_\alpha = (\mu^{(\rho)} + \kappa^{(\rho)})G_{,\alpha}n_\alpha + \kappa^{(\rho)}\epsilon_{3\alpha\beta}G_\beta n_\alpha, \\
\mathcal{B}_\nu^{(\rho)} V &\equiv (M_{\alpha\nu} V)n_\alpha = \alpha^{(\rho)}G_{\lambda,\lambda}n_\nu + \beta^{(\rho)}G_{\lambda,\nu}n_\lambda + \gamma^{(\rho)}G_{\nu,\lambda}n_\lambda.
\end{aligned} \tag{4.8}$$

Taking into account (2.2), (2.3) and (4.5) we obtain

$$\begin{aligned}
t_{\alpha\beta} &= 0, \quad t_{33} = 0, \\
t_{\alpha 3} &= \tau(T_\alpha \Lambda + \mu^{(\rho)}\epsilon_{3\beta\alpha}x_\beta), \\
t_{3\alpha} &= \tau[S_\alpha \Lambda + (\mu^{(\rho)} + \kappa^{(\rho)})\epsilon_{3\beta\alpha}x_\beta], \\
m_{\eta\nu} &= \tau(M_{\eta\nu} \Lambda + \alpha^{(\rho)}\delta_{\eta\nu}), \quad m_{\alpha 3} = m_{3\alpha} = 0, \\
m_{33} &= \tau(\alpha^{(\rho)}\Phi_{\nu,\nu} + \alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)}),
\end{aligned} \tag{4.9}$$

where $\Lambda = (\Phi, \Phi_1, \Phi_2)$.

The equilibrium eqns (2.1) reduce to

$$\mathcal{L}^{(\rho)}\Lambda = \epsilon_{\alpha\beta 3}(\mu^{(\rho)}x_\beta)_{,\alpha}, \mathcal{L}_\nu^{(\rho)}\Lambda = x_\nu\kappa^{(\rho)} - \alpha^{(\rho)}_{,\nu}, \quad \text{in } \Sigma_\rho, \quad (4.10)$$

and the boundary conditions (2.6) become

$$\mathcal{B}^{(\rho)}\Lambda = \mu^{(\rho)}\epsilon_{3\alpha\beta}x_\beta n_\alpha, \mathcal{B}_\nu^{(\rho)}\Lambda = -\alpha^{(\rho)}n_\nu, \quad \text{on } L_\rho. \quad (4.11)$$

The conditions (2.5) reduce to

$$\begin{aligned} [\Phi]_1 &= [\Phi]_2, [\Phi_\alpha]_1 = [\Phi_\alpha]_2, \\ [\mathcal{B}^{(1)}\Lambda](\nu) - [\mathcal{B}^{(2)}\Lambda](\nu) &= \epsilon_{\alpha\beta 3}(\mu^{(1)} - \mu^{(2)})x_\beta \nu_\alpha, \\ [\mathcal{B}_\eta^{(1)}\Lambda](\nu) - [\mathcal{B}_\eta^{(2)}\Lambda](\nu) &= (\alpha^{(2)} - \alpha^{(1)})\nu_\eta, \quad \text{on } \Gamma, \end{aligned} \quad (4.12)$$

where $[\mathcal{B}^{(\rho)}\Lambda](\nu)$ denotes the operator $\mathcal{B}^{(\rho)}\Lambda$ for $n_\alpha = \nu_\alpha$.

Following [10, 11] we can prove that the necessary and sufficient conditions for the existence of the solution of the boundary value problem (4.10)–(4.12) is

$$\sum_{\rho=1}^2 \left[\int_{\Sigma_\rho} \mathcal{L}^{(\rho)}\Lambda \, d\sigma - \int_{L_\rho} \mathcal{B}^{(\rho)}\Lambda \, ds \right] - \int_\Gamma (\mathcal{B}^{(1)}\Lambda - \mathcal{B}^{(2)}\Lambda) \, ds = 0. \quad (4.13)$$

It is easy to see that this condition is satisfied. We assume that the functions Φ, Φ_α are known.

Taking into account (4.9) it follows that the conditions (4.2) and (4.3) are satisfied. As in [1] we can prove that the conditions (4.1) are identically satisfied. From (4.4) we determine the constant τ . Using (4.9) the condition (4.4) reduce to

$$\tau D = -M_3, \quad (4.14)$$

where D is the torsional rigidity

$$\begin{aligned} D &= \sum_{\rho=1}^2 \int_{\Sigma_\rho} (\epsilon_{3\alpha\beta}x_\alpha \{ \mu^{(\rho)}\Phi_{,\beta} + \epsilon_{\nu\beta 3}[\kappa^{(\rho)}\Phi_\nu + (\mu^{(\rho)} + \kappa^{(\rho)})x_\nu] \} \\ &\quad + \alpha^{(\rho)}\Phi_{\nu,\nu} + \alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)}) \, d\sigma. \end{aligned} \quad (4.15)$$

As in [1] we can prove that $D \neq 0$. The relation (4.15) determines the constant τ . Thus the torsion problem is solved.

5. FLEXURE

Let the loading applied on the end $\Sigma^{(o)}$ be statically equivalent to the force $P(P_1, P_2, o)$ and the moment $M(o, o, o)$. For $x_3 = o$ we have the conditions

$$\int_\Sigma t_{3\alpha} \, d\sigma = -P_\alpha, \quad (5.1)$$

$$\int_\Sigma t_{33} \, d\sigma = 0, \quad (5.2)$$

$$\int_\Sigma (x_\alpha t_{33} - \epsilon_{3\alpha\beta} m_{3\beta}) \, d\sigma = 0, \quad (5.3)$$

$$\int_\Sigma (\epsilon_{3\alpha\beta} x_\alpha t_{3\beta} + m_{33}) \, d\sigma = 0. \quad (5.4)$$

We seek the solution of the problem (2.1)–(2.3), (2.5), (2.6) and (5.1)–(5.4) in the form

$$\begin{aligned}
u_\alpha &= -\frac{1}{6} b_\alpha x_3^3 + x_3 \sum_{s=1}^3 b_s u_\alpha^{(s)} + \epsilon_{3\beta\alpha} \tau x_\beta x_3, \\
u_3 &= \frac{1}{2} (b_1 x_1 + b_2 x_2 + b_3) x_3^2 + \tau \Phi + \Psi(x_1, x_2), \\
\varphi_\alpha &= \frac{1}{2} \epsilon_{3\alpha\beta} b_\beta x_3^2 + \tau \Phi_\alpha + \Psi_\alpha(x_1, x_2), \\
\varphi_3 &= x_3 \sum_{s=1}^3 b_s \varphi_3^{(s)} + \tau x_3,
\end{aligned} \tag{5.5}$$

where Φ, Φ_1, Φ_2 are the torsion functions considered in Section 4, $u_\alpha^{(s)}, \varphi_3^{(s)}$ are the solutions of the problems $P^{(s)}$ already considered in Section 3, Ψ, Ψ_1, Ψ_2 are unknown functions and b_i, τ are unknown constants. From (5.5), (2.3) and (2.2) we obtain

$$\begin{aligned}
t_{\alpha\beta} &= x_3 \sum_{s=1}^3 b_s t_{\alpha\beta}^{(s)} + \lambda^{(\rho)} x_3 (b_1 x_1 + b_2 x_2 + b_3) \delta_{\alpha\beta}, \\
t_{33} &= (\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}) (b_1 x_1 + b_2 x_2 + b_3) x_3 + \lambda^{(\rho)} x_3 \sum_{s=1}^3 b_s e_{\alpha\alpha}^{(s)}, \\
t_{\alpha 3} &= \tau (T_\alpha \Lambda + \mu^{(\rho)} \epsilon_{3\beta\alpha} x_\beta) + T_\alpha \Omega + \mu^{(\rho)} \sum_{s=1}^3 b_s u_\alpha^{(s)}, \\
t_{3\alpha} &= \tau [S_\alpha \Lambda + \epsilon_{3\beta\alpha} (\mu^{(\rho)} + \kappa^{(\rho)}) x_\beta] + S_\alpha \Omega + (\mu^{(\rho)} + \kappa^{(\rho)}) \sum_{s=1}^3 b_s u_\alpha^{(s)}, \\
m_{\eta\nu} &= \tau (M_{\eta\nu} \Lambda + \alpha^{(\rho)} \delta_{\eta\nu}) + M_{\eta\nu} \Omega + \alpha^{(\rho)} \delta_{\eta\nu} \sum_{s=1}^3 b_s \varphi_3^{(s)}, \\
m_{\alpha 3} &= \beta^{(\rho)} \epsilon_{3\alpha\beta} b_\beta x_3 + x_3 \sum_{s=1}^3 b_s m_{\alpha 3}^{(s)}, \\
m_{3\alpha} &= x_3 \sum_{s=1}^3 b_s m_{3\alpha}^{(s)} + \gamma^{(\rho)} \epsilon_{3\alpha\beta} b_\beta x_3, \\
m_{33} &= (\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)}) \left(\tau + \sum_{s=1}^3 b_s \varphi_3^{(s)} \right) + \alpha^{(\rho)} (\tau \Phi_{\alpha,\alpha} + \Psi_{\alpha,\alpha}),
\end{aligned} \tag{5.6}$$

where $\Omega = (\Psi, \Psi_1, \Psi_2)$.

Taking into account the relations (2.7)–(2.11) and (4.9)–(4.12) it follows that the equilibrium eqns (2.1) and the conditions (2.5) and (2.6) reduce to

$$\begin{aligned}
\mathcal{L}^{(\rho)} \Omega &= - \sum_{s=1}^3 b_s [\lambda^{(\rho)} e_{\alpha\alpha}^{(s)} + (\mu^{(\rho)} u_\alpha^{(s)})_{,\alpha}] - (\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}) (b_1 x_1 + b_2 x_2 + b_3), \\
\mathcal{L}_\nu^{(\rho)} \Omega &= -\gamma^{(\rho)} \epsilon_{3\nu\beta} b_\beta - \sum_{s=1}^3 b_s [(\alpha^{(\rho)} \varphi_3^{(s)})_{,\nu} + m_{3\nu}^{(s)} - \epsilon_{\nu\beta 3} \kappa^{(\rho)} u_\beta^{(s)}], \quad \text{in } \Sigma_\rho, \\
[\Psi]_1 &= [\Psi]_2, [\Psi_\alpha]_1 = [\Psi_\alpha]_2,
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
[\mathcal{B}^{(1)} \Omega](\nu) - [\mathcal{B}^{(2)} \Omega](\nu) &= (\mu^{(2)} - \mu^{(1)}) \nu_\alpha \sum_{s=1}^3 b_s u_\alpha^{(s)}, \\
[\mathcal{B}_\eta^{(1)} \Omega](\nu) - [\mathcal{B}_\eta^{(2)} \Omega](\nu) &= (\alpha^{(2)} - \alpha^{(1)}) \nu_\eta \sum_{s=1}^3 b_s \varphi_3^{(s)}, \quad \text{on } \Gamma,
\end{aligned} \tag{5.8}$$

$$\mathcal{B}^{(\rho)} \Omega = -\mu^{(\rho)} n_\alpha \sum_{s=1}^3 b_s u_\alpha^{(s)}, \mathcal{B}_\nu^{(\rho)} \Omega = -\alpha^{(\rho)} n_\nu \sum_{s=1}^3 b_s \varphi_3^{(s)}, \quad \text{on } L_\rho. \tag{5.9}$$

The necessary and sufficient conditions (4.13) for the existence of the solution of the

boundary value problem (5.7)–(5.9) become

$$A_{3s}b_s = 0, \quad (5.10)$$

where A_{3s} are given by (3.8).

Let us consider the conditions (5.1). As in the case of homogeneous cylinders we can prove the relation

$$\int_{\Sigma} t_{3\alpha} d\sigma = \int_{\Sigma} (x_{\alpha} t_{33} - \epsilon_{3\alpha\beta} m_{3\beta})_{,3} d\sigma.$$

Thus, taking into account (5.6), the conditions (5.1) reduce to

$$A_{\alpha s}b_s = -P_{\alpha}. \quad (5.11)$$

The system (5.10) and (5.11) determines the constants b_i . The condition (5.10) is satisfied so that in what follows we assume the function Ψ , Ψ_{α} to be known.

From (5.6) and (5.4) we obtain

$$\begin{aligned} \tau D = & - \sum_{\rho=1}^2 \int_{\Sigma_{\rho}} \left\{ \epsilon_{3\alpha\beta} x_{\alpha} \left[\mu^{(\rho)} \Psi_{,\beta} + \epsilon_{\nu\beta 3} \kappa^{(\rho)} \Psi_{,\nu} + (\mu^{(\rho)} + \kappa^{(\rho)}) \sum_{s=1}^3 b_s u_{\beta}^{(s)} \right] \right. \\ & \left. + (\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)}) \sum_{s=1}^3 b_s \varphi_3^{(s)} + \alpha^{(\rho)} \Psi_{\alpha,\alpha} \right\} d\sigma, \end{aligned} \quad (5.12)$$

where D is given by (4.15). The relation (5.12) determines the constant τ .

The conditions (5.2) and (5.3) are identically satisfied on the basis of the relation (5.6). In this way the problem is reduced to solving plane problems.

6. TORSION OF A COMPOSITE CIRCULAR CYLINDER

In what follows we use the method given in Sections 2–4 in order to study the deformation of a composite circular cylinder. We assume that the domain Σ_1 is bounded by two concentric circles L and Γ on radius R_1 and r_2 , respectively, where $r_2 < r_1$. The domain Σ_2 is bounded by the circle Γ . Axis Ox_3 of the coordinate system is taken along the axis of the cylinder.

We consider the case when the domains Σ_1 and Σ_2 are occupied by two different homogeneous and isotropic micropolar elastic materials.

In what follows we will use the cylindrical coordinate system (r, θ, z) and denote by $\bar{e}_r$, $\bar{e}_{\theta}$, $\bar{e}_z$ the unit base vectors of this system.

Let us denote

$$\bar{u} = u_r \bar{e}_r + u_{\theta} \bar{e}_{\theta} + u_z \bar{e}_z, \quad \bar{\varphi} = \varphi_r \bar{e}_r + \varphi_{\theta} \bar{e}_{\theta} + \varphi_z \bar{e}_z.$$

Let $\hat{\Phi} = (\Phi, \Phi_r, \Phi_{\theta})$ be an ordered triplet of functions Φ , Φ_r , Φ_{θ} . The eqns (4.10) can be written in the form

$$L_z^{(\rho)} \hat{\Phi} = 0, \quad M_r^{(\rho)} \hat{\Phi} = \frac{1}{2} (s^{(\rho)})^2 r^3, \quad M_{\theta}^{(\rho)} \hat{\Phi} = 0, \quad \text{in } \Sigma_{\rho}, \quad (6.1)$$

where

$$L_z^{(\rho)} \hat{\Phi} = (D_r^2 + D_{\theta}^2) \Phi + e^{(\rho)} r (D_r + 1) \Phi_{\theta} - e^{(\rho)} r D_{\theta} \Phi_r,$$

$$M_r^{(\rho)} \hat{\Phi} = (D_r^2 + b^{(\rho)} D_{\theta}^2 - (s^{(\rho)})^2 r^2 - 1) \Phi_r + [(1 - b^{(\rho)}) D_r - (1 + b^{(\rho)})] D_{\theta} \Phi_{\theta} + \frac{1}{2} (s^{(\rho)})^2 r D_{\theta} \Phi,$$

$$M^{(\rho)}\hat{\Phi} = [b^{(\rho)}(D_r^2 - 1) - r^2(s^{(\rho)})^2 + D_\theta^2]\hat{\Phi}_\theta + [(1 - b^{(\rho)})D_r + (1 + b^{(\rho)})]D_\theta\hat{\Phi}_r - \frac{1}{2}r(s^{(\rho)})^2D_r\hat{\Phi},$$

$$D_r = r\frac{d}{dr}, \quad D_\theta = \frac{d}{d\theta},$$

$$e^{(\rho)} = \frac{\kappa^{(\rho)}}{\mu^{(\rho)} + \kappa^{(\rho)}}, (s^{(\rho)})^2 = \frac{2\kappa^{(\rho)}}{\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)}}, b^{(\rho)} = \frac{\gamma^{(\rho)}}{\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)}}.$$

From the conditions (4.11) and (4.12) it follows

$$\begin{aligned} [\hat{\Phi}]_1 &= [\hat{\Phi}]_2, \quad T_z^{(1)}\hat{\Phi} - T_z^{(2)}\hat{\Phi} = 0, \\ S_r^{(1)}\hat{\Phi} - S_r^{(2)}\hat{\Phi} &= \alpha^{(2)} - \alpha^{(1)}, \quad S_\theta^{(1)}\hat{\Phi} - S_\theta^{(2)}\hat{\Phi} = 0, \quad \text{on } \Gamma, \\ T_z^{(1)}\hat{\Phi} &= 0, \quad S_r^{(1)}\hat{\Phi} = -\alpha^{(1)}, \quad S_\theta^{(1)}\hat{\Phi} = 0, \quad \text{on } L, \end{aligned} \quad (6.2)$$

where

$$\begin{aligned} T_z^{(\rho)}\hat{\Phi} &= \frac{1}{r}(\mu^{(\rho)} + \kappa^{(\rho)})D_r\hat{\Phi} + \kappa^{(\rho)}\hat{\Phi}_\theta, \quad rS_\theta^{(\rho)}\hat{\Phi} = \gamma^{(\rho)}D_r\hat{\Phi}_\theta \\ &\quad + \beta^{(\rho)}(D_\theta\hat{\Phi}_r - \hat{\Phi}_\theta), \\ rS_r^{(\rho)}\hat{\Phi} &= (\alpha^{(\rho)} + \beta^{(\rho)} + \gamma^{(\rho)})D_r\hat{\Phi}_r + \alpha^{(\rho)}(D_\theta\hat{\Phi}_\theta + \hat{\Phi}_r). \end{aligned}$$

Let us determine the solution of the problem (6.1) and (6.2). We seek this solution as function only of r . It is easy to verify that a particular solution of (6.1) is given by

$$\bar{\Phi} = 0, \quad \bar{\Phi}_r = -\frac{1}{2}r, \quad \bar{\Phi}_\theta = 0. \quad (6.3)$$

The homogeneous solution of (6.1) can be obtained as in [12]. We obtain

$$\begin{aligned} \Phi^o &= -e^{(2)}A_1 \int I_1(\delta^{(2)}r) dr + A_2, \\ \Phi_r^o &= A_3I_1(s^{(2)}r), \quad \Phi_\theta^o = A_1I_1(\delta^{(2)}r), \quad \text{for } 0 \leq r \leq r_2, \\ \Phi^o &= -e^{(1)} \int [B_1I_1(\delta^{(1)}r) + B_4K_1(\delta^{(1)}r)] dr + B_5 \left[1 + \frac{e^{(1)}\kappa^{(1)}}{\gamma^{(1)}\delta^{(1)2}} \right] \ln r + B_2, \\ \Phi_r^o &= B_3I_1(s^{(1)}r) + B_6K_1(s^{(1)}r), \\ \Phi_\theta^o &= B_1I_1(\delta^{(1)}r) + B_4K_1(\delta^{(1)}r) - \frac{\kappa^{(1)}}{\gamma^{(1)}\delta^{(1)2}} \cdot \frac{1}{r}B_5, \quad \text{for } r_2 \leq r \leq r_1, \end{aligned} \quad (6.4)$$

where A_i , B_i , B_{i+3} ($i = 1, 2, 3$) are unknown constants, $I_n(x)$, $K_n(x)$ are the modified Bessel functions of order n , and

$$(\delta^{(\rho)})^2 = (2 - e^{(\rho)})\sigma^{(\rho)}, \quad \sigma^{(\rho)} = \frac{\kappa^{(\rho)}}{\gamma^{(\rho)}}.$$

By substituting (6.3) and (6.4) in the conditions (6.2) we obtain for the unknown constants the following values

$$\begin{aligned} A_1 &= B_1 = B_4 = B_5 = 0, \quad A_2 = B_2, \\ B_3 &= (Q_5Q_7 - Q_4Q_8)/(Q_3Q_7 - Q_4Q_6), \\ B_6 &= (Q_3Q_8 - Q_5Q_6)/(Q_3Q_7 - Q_4Q_6), \\ A_3 &= Q_1B_3 + Q_2B_6, \end{aligned} \quad (6.5)$$

where

$$\begin{aligned}
 Q_1 &= I_1(s^{(1)}r_2)/I_1(s^{(2)}r_2), \quad Q_2 = K_1(s^{(1)}r_2)/I_1(s^{(2)}r_2), \\
 Q_3 &= Q_1[(\alpha^{(2)} + \beta^{(2)} + \gamma^{(2)})I_1'(s^{(2)}r_2) + \alpha^{(2)}I_1(s^{(2)}r_2)/(s^{(2)}r_2)] \\
 &\quad - [(\alpha^{(1)} + \beta^{(1)} + \gamma^{(1)})I_1'(s^{(1)}r_2) + \alpha^{(1)}I_1(s^{(1)}r_2)/(s^{(1)}r_2)]s^{(1)}/s^{(2)}, \\
 Q_4 &= Q_2[(\alpha^{(2)} + \beta^{(2)} + \gamma^{(2)})I_1'(s^{(2)}r_2) + \alpha^{(2)}I_1(s^{(2)}r_2)/(s^{(2)}r_2)] \\
 &\quad - [(\alpha^{(1)} + \beta^{(1)} + \gamma^{(1)})K_1'(s^{(1)}r_2) + \alpha^{(1)}K_1(s^{(1)}r_2)/(s^{(1)}r_2)]s^{(1)}/s^{(2)}, \\
 Q_5 &= [(\beta^{(2)} + \gamma^{(2)}) - (\beta^{(1)} + \gamma^{(1)})]/(2s^{(2)}), \quad Q_6 = (\alpha^{(1)} + \beta^{(1)} + \gamma^{(1)})I_1'(s^{(1)}r_1) + \alpha^{(1)}r_1 \\
 &\quad + \alpha^{(1)}I_1(s^{(1)}r_1)/(s^{(1)}r_1), \\
 Q_7 &= (\alpha^{(1)} + \beta^{(1)} + \gamma^{(1)})K_1'(s^{(1)}r_1) + \alpha^{(1)}K_1(s^{(1)}r_1)/(s^{(1)}r_1), \quad Q_8 = (\beta^{(1)} + \gamma^{(1)})/(2s^{(1)}), \\
 I_1'(r) &= \frac{dI_1}{dr}, \quad K_1'(r) = \frac{dK_1}{dr}.
 \end{aligned}$$

The solution of problem (6.1), (6.2) has the form

$$\begin{aligned}
 \Phi &= A_2, \quad \Phi_r = -\frac{1}{2}r + A_3I_1(s^{(2)}r), \quad \Phi_\theta = 0, \quad \text{for } 0 \leq r \leq r_2, \\
 \Phi &= A_2, \quad \Phi_r = -\frac{1}{2}r + B_3I_1(s^{(1)}r) + B_6K_1(s^{(1)}r), \quad \Phi_\theta = 0, \quad \text{for } r_2 \leq r \leq r_1.
 \end{aligned} \quad (6.6)$$

Therefore, the solution of the torsion problem for the composite circular cylinder is

$$\begin{aligned}
 u_r &= 0, \quad u_\theta = \tau rz, \quad u_z = \tau \Phi, \\
 \varphi_r &= \tau \Phi_r, \quad \varphi_\theta = \tau \Phi_\theta, \quad \varphi_z = \tau z,
 \end{aligned} \quad (6.7)$$

where

$$\begin{aligned}
 \tau &= -M_3/D, \\
 D &= \frac{1}{4}(2\mu^{(2)} + \kappa^{(2)})\pi r_2^4 + (\beta^{(2)} + \gamma^{(2)})\pi r_2^2 + \frac{2}{s^{(2)}}\pi \kappa^{(2)}A_3r_2^2I_2(s^{(2)}r_2) \\
 &\quad + 2\pi \alpha^{(2)}A_3r_2I_1(s^{(2)}r_2) + \frac{1}{4}(2\mu^{(1)} + \kappa^{(1)})\pi(r_1^4 - r_2^4) + (\beta^{(1)} + \gamma^{(1)})\pi(r_1^2 - r_2^2) \\
 &\quad + \{B_3[r_1^2I_2(s^{(1)}r_1) - r_2^2I_2(s^{(1)}r_2)] - B_6[r_1^2K_2(s^{(1)}r_1) - r_2^2K_2(s^{(1)}r_2)]\}2\pi \kappa^{(1)}/s^{(1)} \\
 &\quad + 2\pi \alpha^{(1)}\{B_3[r_1I_1(s^{(1)}r_1) - r_2I_1(s^{(1)}r_2)] + B_6[r_1K_1(s^{(1)}r_1) - r_2K_1(s^{(1)}r_2)]\}.
 \end{aligned}$$

It is easy to see that the constant A_2 characterizes a rigid translation.

7. EXTENSION AND BENDING

Let $U = (u_r, u_\theta, \varphi_z)$ be an ordered triplet of functions u_r, u_θ, φ_z . We introduce the notations

$$\begin{aligned}
 L_r^{(\rho)}U &= (D_r^2 + c^{(\rho)}D_\theta^2 - 1)u_r + [(1 - c^{(\rho)})D_r - (1 + c^{(\rho)})]D_\theta u_\theta + d^{(\rho)}rD_\theta \varphi_z, \\
 L_\theta^{(\rho)}U &= [(1 - c^{(\rho)})D_r + (1 + c^{(\rho)})]D_\theta u_r + [c^{(\rho)}(D_r^2 - 1) + D_\theta^2]u_\theta - d^{(\rho)}rD_\theta \varphi_z, \\
 M_z^{(\rho)}U &= -\sigma^{(\rho)}rD_\theta u_r + \sigma^{(\rho)}r(D_r + 1)u_\theta + (D_r^2 + D_\theta^2 - 2\sigma^{(\rho)}r^2)\varphi_z,
 \end{aligned} \quad (7.1)$$

$$T_r^{(\rho)}U = [(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})D_r u_r + \lambda^{(\rho)}(D_\theta u_\theta + u_r)]/r,$$

$$T_\theta^{(\rho)}U = \frac{1}{r}[(\mu^{(\rho)} + \kappa^{(\rho)})D_r u_\theta + \mu^{(\rho)}(D_\theta u_r - u_\theta)] - \kappa^{(\rho)}\varphi_z,$$

$$S_z^{(\rho)}U = \frac{1}{r}\gamma^{(\rho)}D_r \varphi_z, \quad (7.2)$$

where

$$c^{(\rho)} = \frac{\mu^{(\rho)} + \kappa^{(\rho)}}{\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}}, d^{(\rho)} = \frac{\kappa^{(\rho)}}{\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}}.$$

Using these notations, the problems $P^{(s)}$ can be written in the form

$$L_r^{(\rho)} U^{(1)} = -(1 - 2c^{(\rho)} + d^{(\rho)})r^2 \cos \theta, L_\theta^{(\rho)} U^{(1)} = (1 - 2c^{(\rho)} + d^{(\rho)})r^2 \sin \theta, M_z^{(\rho)} U^{(1)} = 0, \quad \text{in } \Sigma_\rho, \quad (7.3)$$

$$\begin{aligned} [U^{(1)}]_1 &= [U^{(1)}]_2, T_r^{(2)} U^{(1)} - T_r^{(1)} U^{(1)} = (\lambda^{(1)} - \lambda^{(2)})r_2 \cos \theta, \\ T_\theta^{(2)} U^{(1)} - T_\theta^{(1)} U^{(1)} &= 0, S_z^{(2)} U^{(1)} - S_z^{(1)} U^{(1)} = (\beta^{(2)} - \beta^{(1)}) \sin \theta, \quad \text{on } \Gamma, \\ T_r^{(1)} U^{(1)} &= -\lambda^{(1)}r_1 \cos \theta, T_\theta^{(1)} U^{(1)} = 0, S_z^{(1)} U^{(1)} = \beta^{(1)} \sin \theta, \quad \text{on } L, \end{aligned} \quad (7.4)$$

$$L_r^{(\rho)} U^{(2)} = -(1 - 2c^{(\rho)} + d^{(\rho)})r^2 \sin \theta, L_\theta^{(\rho)} U^{(2)} = -(1 - 2c^{(\rho)} + d^{(\rho)})r^2 \cos \theta, M_z^{(\rho)} U^{(2)} = 0, \quad \text{in } \Sigma_\rho, \quad (7.5)$$

$$\begin{aligned} [U^{(2)}]_1 &= [U^{(2)}]_2, T_r^{(2)} U^{(2)} - T_r^{(1)} U^{(2)} = (\lambda^{(1)} - \lambda^{(2)})r_2 \sin \theta, \\ T_\theta^{(2)} U^{(2)} - T_\theta^{(1)} U^{(2)} &= 0, S_z^{(2)} U^{(2)} - S_z^{(1)} U^{(2)} = (\beta^{(1)} - \beta^{(2)}) \cos \theta, \quad \text{on } \Gamma, \\ T_r^{(1)} U^{(2)} &= -\lambda^{(1)}r_1 \sin \theta, T_\theta^{(1)} U^{(2)} = 0, S_z^{(1)} U^{(2)} = -\beta^{(1)} \cos \theta, \quad \text{on } L, \end{aligned} \quad (7.6)$$

$$L_r^{(\rho)} U^{(3)} = 0, L_\theta^{(\rho)} U^{(3)} = 0, M_z^{(\rho)} U^{(3)} = 0, \quad \text{in } \Sigma_\rho, \quad (7.7)$$

$$\begin{aligned} [U^{(3)}]_1 &= [U^{(3)}]_2, T_r^{(2)} U^{(3)} - T_r^{(1)} U^{(3)} = \lambda^{(1)} - \lambda^{(2)}, \\ T_\theta^{(2)} U^{(3)} - T_\theta^{(1)} U^{(3)} &= 0, S_z^{(2)} U^{(3)} - S_z^{(1)} U^{(3)} = 0, \quad \text{on } \Gamma, \\ T_r^{(1)} U^{(3)} &= -\lambda^{(1)}, T_\theta^{(1)} U^{(3)} = 0, S_z^{(1)} U^{(3)} = 0, \quad \text{on } L, \end{aligned} \quad (7.8)$$

where $U^{(s)} = (u_r^{(s)}, u_\theta^{(s)}, \varphi_z^{(s)})$.

Let us determine the solutions of the problems $P^{(s)}$. First we consider the problems $P^{(\beta)}$ ($\beta = 1, 2$). We seek the solutions of these problems in the form

$$U^{(\beta)} = (A_1^{(\beta)} \cos \theta + A_2^{(\beta)} \sin \theta, B_1^{(\beta)} \cos \theta + B_2^{(\beta)} \sin \theta, C^{(\beta)} \cos \theta + C_2^{(\beta)} \sin \theta), \quad (7.9)$$

where $A_\alpha^{(\beta)}, B_\alpha^{(\beta)}, C_\alpha^{(\beta)}$ are functions of r . From (7.3)–(7.6) we obtain

$$\begin{aligned} A_1^{(1)} &= B_1^{(1)} = C_1^{(1)} = A_1^{(2)} = B_2^{(2)} = C_2^{(2)} = 0, \\ A_1^{(1)} &= A_2^{(2)} = v_r, B_2^{(1)} = -B_1^{(2)} = v_\theta, C_2^{(1)} = -C_1^{(2)} = \psi_z, \end{aligned} \quad (7.10)$$

where v_r, v_θ, ψ_z satisfy the following equations

$$\begin{aligned} (D_r^2 - 1 - c^{(\rho)})v_r + [(1 - c^{(\rho)})D_r - (1 + c^{(\rho)})]v_\theta + d^{(\rho)}r\psi_z &= -(1 + d^{(\rho)} - 2c^{(\rho)})r^2, \\ [(1 - c^{(\rho)})D_r + (1 + c^{(\rho)})]v_r - [c^{(\rho)}(D_r^2 - 1) - 1]v_\theta + d^{(\rho)}rD_r\psi_z &= -(1 + d^{(\rho)} - 2c^{(\rho)})r^2, \\ (D_r^2 - 1 - 2\sigma^{(\rho)}r^2)\psi_z + \sigma^{(\rho)}r(D_r + 1)v_\theta + \sigma^{(\rho)}rv_r &= 0, \quad \text{in } \Sigma_\rho, \end{aligned} \quad (7.11)$$

and the conditions

$$\begin{aligned} [W]_1 &= [W]_2, \mathcal{T}_r^{(2)} W - \mathcal{T}_r^{(1)} W = (\lambda^{(1)} - \lambda^{(2)})r_2, \\ \mathcal{T}_\theta^{(2)} W - \mathcal{T}_\theta^{(1)} W &= 0, \mathcal{S}_z^{(2)} W - \mathcal{S}_z^{(1)} W = \beta^{(2)} - \beta^{(1)}, \quad \text{on } r = r_2, \\ \mathcal{T}_r^{(1)} W &= -\lambda^{(1)}r_1, \mathcal{T}_\theta^{(1)} W = 0, \mathcal{S}_z^{(1)} W = \beta^{(1)}, \quad \text{on } r = r_1. \end{aligned} \quad (7.12)$$

In the above relations we have used the notations

$$\begin{aligned}
W &= (v_r, v_\theta, \psi_z), \\
\mathcal{F}_r^{(\rho)} W &= [(\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)})D_r v_r + \lambda^{(\rho)}(v_r + v_\theta)]/r, \\
\mathcal{F}_\theta^{(\rho)} W &= [(\mu^{(\rho)} + \kappa^{(\rho)})D_r v_\theta - \mu^{(\rho)}(v_r + v_\theta) - r\kappa^{(\rho)}\psi_z]/r, \quad \mathcal{F}_z^{(\rho)} W = \frac{\gamma^{(\rho)}}{r} D_r \psi_z.
\end{aligned}$$

A particular solution of (7.11) is given by

$$\bar{v}_r = -\frac{1}{8}r^2, \quad \bar{v}_\theta = -\frac{5}{8}r^2, \quad \bar{\psi}_z = -r. \quad (7.13)$$

The homogeneous solutions of (7.11) can be derived from the general solutions given in [12]. We find

$$\begin{aligned}
v_r^o &= C_1 + (2 - 6c^{(2)} + 3d^{(2)})C_2 r^2 + \frac{e^{(2)}}{2\delta^{(2)}} C_3 [I_2(\delta^{(2)}r) - I_0(\delta^{(2)}r)], \\
v_\theta^o &= -C_1 - (6 - 2c^{(2)} + d^{(2)})C_2 r^2 + \frac{e^{(2)}}{\delta^{(2)}} C_3 I_1'(\delta^{(2)}r), \\
\psi_z^o &= -8C_2 r + C_3 I_1(\delta^{(2)}r), \quad \text{for } 0 \leq r \leq r_2, \\
v_r^o &= D_1 + (2 - 6c^{(1)} + 3d^{(1)})D_2 r^2 + \frac{e^{(1)}}{2\delta^{(1)}} \{D_3 [I_2(\delta^{(1)}r) - I_0(\delta^{(1)}r)] \\
&\quad - D_4 [K_2(\delta^{(1)}r) - K_0(\delta^{(1)}r)]\} + D_5 r^{-2} - \frac{1}{2} d^{(1)} D_6 + c^{(1)}(2 + 2c^{(1)} - d^{(1)})D_6 \ln r, \\
v_\theta^o &= -D_1 - (6 - 2c^{(1)} + d^{(1)})D_2 r^2 + \frac{e^{(1)}}{\delta^{(1)}} [D_3 I_1'(\delta^{(1)}r) + D_4 K_1'(\delta^{(1)}r)] \\
&\quad + D_5 r^{-2} - \frac{1}{2} [2(1 - c^{(1)})(2c^{(1)} - d^{(1)}) + d^{(1)}]D_6 - c^{(1)}(2 + 2c^{(1)} - d^{(1)})D_6 \ln r, \\
\psi_z^o &= -8D_2 r + D_3 I_1(\delta^{(1)}r) + D_4 K_1(\delta^{(1)}r) - 2c^{(1)}D_6 r^{-1}, \quad \text{for } r_2 \leq r \leq r_1,
\end{aligned} \quad (7.14)$$

where C_i, D_i, D_{i+3} ($i = 1, 2, 3$) are unknown constants.

Substituting these values in (7.12) we find

$$\begin{aligned}
D_1 &= C_1 + 2r_2^2[(2 - 2c^{(2)} + d^{(2)})C_2 - (2 - 2c^{(1)} + d^{(1)})D_2] - \frac{e^{(2)}}{2\delta^{(2)}} C_3 I_0(\delta^{(2)}r_2) \\
&\quad + \frac{e^{(1)}}{2\delta^{(1)}} [D_3 I_0(\delta^{(1)}r_2) - D_4 K_0(\delta^{(1)}r_2)], \\
D_2 &= H_1 + H_2 D_4 + H_3 D_5, \quad D_3 = H_4 + H_5 D_4 + H_6 D_5, \\
D_4 &= (H_{10}H_{11} - H_8H_{12})/(H_7H_{10} - H_8H_9), \quad D_5 = (H_7H_{12} - H_9H_{11})/(H_7H_{10} - H_8H_9), \\
D_6 &= 0, \\
C_2 &= H_{13} + H_{14}D_2 + H_{15}D_3 + H_{16}D_4 + H_{17}D_5, \\
C_3 &= H_{18} + H_{19}D_2 + H_{20}D_3 + H_{21}D_4 + H_{22}D_5,
\end{aligned} \quad (7.15)$$

where

$$\begin{aligned}
H_1 &= \left[-\frac{1}{4} \delta^{(1)} r_1^2 I_1'(\delta^{(1)}r_1) - \frac{e^{(1)}}{\delta^{(1)}} \cdot \frac{\beta^{(1)} + \gamma^{(1)}}{\gamma^{(1)}} I_2(\delta^{(1)}r_1) \right] / \Delta_1, \\
H_2 &= e^{(1)} [I_1'(\delta^{(1)}r_1) K_2(\delta^{(1)}r_1) + K_1'(\delta^{(1)}r_1) I_2(\delta^{(1)}r_1)] / \Delta_1, \\
H_3 &= -2\delta^{(1)} I_1'(\delta^{(1)}r_1) / (r_1^2 \Delta_1), \quad H_4 = 2r_1^2 \left[\frac{\beta^{(1)} + \gamma^{(1)}}{\gamma^{(1)}} (2 - 2c^{(1)} + d^{(1)}) - 1 \right] / \Delta_1,
\end{aligned}$$

$$H_5 = \left[-2(2 - 2c^{(1)} + d^{(1)})\delta^{(1)}r_1^2 K_1'(\delta^{(1)}r_1) + \frac{8e^{(1)}}{\delta^{(1)}} K_2(\delta^{(1)}r_1) \right] / \Delta_1,$$

$$H_6 = -16/(r_1^2 \Delta_1),$$

$$\Delta_1 = 8 \frac{e^{(1)}}{\delta^{(1)}} I_2(\delta^{(1)}r_1) + 2(2 - 2c^{(1)} + d^{(1)})\delta^{(1)}r_1^2 I_1'(\delta^{(1)}r_1);$$

$$H_7 = \frac{e^{(1)}}{\delta^{(1)}} [K_2(\delta^{(1)}r_2) - H_5 I_2(\delta^{(1)}r_2)] + 2r_2^2 [(2 + 2c^{(1)} - d^{(1)})H_2 - (2 + 2c^{(2)} - d^{(2)})$$

$$\times (H_{16} + H_5 H_{15} + H_2 H_{14})] + \frac{e^{(2)}}{\delta^{(2)}} I_2(\delta^{(2)}r_2)(H_{21} + H_5 H_{20} + H_2 H_{19}),$$

$$H_8 = -2r_2^2 - \frac{e^{(1)}}{\delta^{(1)}} H_6 I_2(\delta^{(1)}r_2) + 2r_2^2 [(2 + 2c^{(1)} - d^{(1)})H_3 - (2 + 2c^{(2)} - d^{(2)})$$

$$\times (H_{17} + H_6 H_{15} + H_3 H_{14})] + \frac{e^{(2)}}{\delta^{(2)}} I_2(\delta^{(2)}r_2)(H_{22} + H_6 H_{20} + H_3 H_{19}),$$

$$H_9 = -K_1(\delta^{(1)}r_2) + 8r_2 H_2 - I_1(\delta^{(1)}r_2)H_5 - 8r_2(H_{16} + H_5 H_{15} + H_2 H_{14})$$

$$+ I_1(\delta^{(2)}r_2)(H_{21} + H_5 H_{20} + H_2 H_{19}),$$

$$H_{10} = 8r_2 H_3 - I_1(\delta^{(1)}r_2)H_6 - 8r_2(H_{17} + H_6 H_{15} + H_3 H_{14}) + I_1(\delta^{(2)}r_2)(H_{22} + H_6 H_{20} + H_3 H_{19}),$$

$$H_{11} = -2r_2^2(2 + 2c^{(1)} - d^{(1)})H_1 + \frac{e^{(1)}}{\delta^{(1)}} I_2(\delta^{(1)}r_2)H_4 + 2(2 + 2c^{(2)} - d^{(2)})r_2^2(H_{13} + H_1 H_{14} + H_4 H_{15})$$

$$- \frac{e^{(2)}}{\delta^{(2)}} I_2(\delta^{(2)}r_2)(H_{18} + H_1 H_{19} + H_4 H_{20}),$$

$$H_{12} = -8r_2 H_1 + I_1(\delta^{(1)}r_2)H_4 + 8r_2(H_{13} + H_1 H_{14} + H_4 H_{15}) - I_1(\delta^{(2)}r_2)(H_{18} + H_1 H_{19} + H_4 H_{20}),$$

$$H_{13} = \left[(\chi - 1) \frac{e^{(2)}(\beta^{(2)} + \gamma^{(2)})}{\delta^{(2)}\gamma^{(2)}} I_2(\delta^{(2)}r_2) + \frac{1}{4}(\omega - 1)\delta^{(2)}r_2^2 I_1'(\delta^{(2)}r_2) \right] / \Delta_2,$$

$$H_{14} = \left[8\epsilon \frac{e^{(2)}}{\delta^{(2)}} I_2(\delta^{(2)}r_2) + 2\omega(2 - 2c^{(1)} + d^{(1)})\delta^{(2)}r_2^2 I_1'(\delta^{(2)}r_2) \right] / \Delta_2,$$

$$H_{15} = \left[-\epsilon e^{(2)} \frac{\delta^{(1)}}{\delta^{(2)}} I_2(\delta^{(2)}r_2) I_1'(\delta^{(1)}r_2) + \omega e^{(1)} \frac{\delta^{(2)}}{\delta^{(1)}} I_2(\delta^{(1)}r_2) I_1'(\delta^{(2)}r_2) \right] / \Delta_2,$$

$$H_{16} = \left[\epsilon e^{(2)} \frac{\delta^{(1)}}{\delta^{(2)}} I_2(\delta^{(2)}r_2) K_1'(\delta^{(1)}r_2) + \omega e^{(1)} \frac{\delta^{(2)}}{\delta^{(1)}} K_2(\delta^{(1)}r_2) I_1'(\delta^{(2)}r_2) \right] / \Delta_2,$$

$$H_{17} = 2\omega \delta^{(2)} I_1'(\delta^{(2)}r_2) / (r_2^2 \Delta_2),$$

$$H_{18} = 2r_2^2 \left[\omega - 1 + (2 - 2c^{(2)} + d^{(2)}) \frac{\beta^{(2)} + \gamma^{(2)}}{\gamma^{(2)}} (1 - \chi) \right] / \Delta_2,$$

$$H_{19} = 16r_2^2 [\omega(2 - 2c^{(1)} + d^{(1)}) - \epsilon(2 - 2c^{(2)} + d^{(2)})] / \Delta_2,$$

$$H_{20} = \left[8\omega \frac{e^{(1)}}{\delta^{(1)}} I_2(\delta^{(1)}r_2) + 2\epsilon(2 - 2c^{(2)} + d^{(2)})\delta^{(1)}r_2^2 I_1'(\delta^{(1)}r_2) \right] / \Delta_2,$$

$$H_{21} = \left[-8\omega \frac{e^{(1)}}{\delta^{(1)}} K_2(\delta^{(1)}r_2) + 2\epsilon(2 - 2c^{(2)} + d^{(2)})\delta^{(1)}r_2^2 K_1'(\delta^{(1)}r_2) \right] / \Delta_2,$$

$$H_{22} = 16\omega / (r_2^2 \Delta_2),$$

$$\Delta_2 = 8 \frac{e^{(2)}}{\delta^{(2)}} I_2(\delta^{(2)}r_2) + 2\delta^{(2)}r_2^2(2 - 2c^{(2)} + d^{(2)})I_1'(\delta^{(2)}r_2);$$

$$\omega = \frac{2\mu^{(1)} + \kappa^{(1)}}{2\mu^{(2)} + \kappa^{(2)}}, \epsilon = \frac{\gamma^{(1)}}{\gamma^{(2)}}, \chi = \frac{\beta^{(1)} + \gamma^{(1)}}{\beta^{(2)} + \gamma^{(2)}}.$$

Therefore, the solutions of the problems $P^{(\beta)}$ are

$$\begin{aligned}
u_r^{(1)} &= \left(-\frac{1}{8}r^2 + v_r^o\right) \cos \theta, u_\theta^{(1)} = \left(-\frac{5}{8}r^2 + v_\theta^o\right) \sin \theta, \varphi_z^{(1)} = (-r + \psi_z^o) \sin \theta, \\
u_r^{(2)} &= \left(-\frac{1}{8}r^2 + v_r^o\right) \sin \theta, u_\theta^{(2)} = \left(\frac{5}{8}r^2 - v_\theta^o\right) \cos \theta, \varphi_z^{(2)} = (r - \psi_z^o) \cos \theta,
\end{aligned} \quad (7.16)$$

where $v_r^o, v_\theta^o, \psi_z^o$ are given by (7.14).

The constant C_1 characterizes a rigid translation.

Let us consider now the problem $P^{(3)}$ defined by (7.7) and (7.8). We seek the solution of (7.7) as functions only of r . We obtain [12]

$$\begin{aligned}
u_r^{(3)} &= E_1 r, u_\theta^{(3)} = \frac{e^{(2)}}{\delta^{(2)}} E_2 I_1(\delta^{(2)} r), \varphi_z^{(3)} = E_2 I_o(\delta^{(2)} r), \quad 0 \leq r \leq r_2, \\
u_r^{(3)} &= F_1 r + F_3 r^{-1}, u_\theta^{(3)} = \frac{e^{(1)}}{\delta^{(1)}} [F_2 I^1(\delta^{(1)} r) - F_4 K_1(\delta^{(1)} r)] + F_5 r^{-1}, \\
\varphi_z^{(3)} &= F_2 I_o(\delta^{(1)} r) + F_4 K_o(\delta^{(1)} r), \quad r_2 \leq r \leq r_1,
\end{aligned} \quad (7.17)$$

where $E_1, E_2, F_1, \dots, F_4$ are unknown constants. If we introduce the displacements (7.17) in (7.8), we obtain the solution of the problem $P^{(3)}$ in the form

$$\begin{aligned}
u_r^{(3)} &= E_1 r, u_\theta^{(3)} = 0, \varphi_z^{(3)} = 0, \quad 0 \leq r \leq r_2, \\
u_r^{(3)} &= F_1 r + F_3 r^{-1}, u_\theta^{(3)} = 0, \varphi_z^{(3)} = 0, \quad r_2 \leq r \leq r_1,
\end{aligned} \quad (7.18)$$

where

$$\begin{aligned}
F_3 &= (\nu^{(1)} - \nu^{(2)}) \left\{ \frac{2\mu^{(1)} + \kappa^{(1)}}{2\lambda^{(2)} + 2\mu^{(2)} + \kappa^{(2)}} (r_2^{-2} - r_1^{-2}) + r_1^{-2} + r_2^{-2} - 2\nu^{(1)} r_1^{-2} \right\}^{-1}, \\
F_1 &= -\nu^{(1)} + (1 - 2\nu^{(1)}) F_3 r_1^{-2}, E_1 = -\nu^{(1)} + [r_2^{-2} + r_1^{-2} - 2\nu^{(1)} r_1^{-2}] F_3, \\
E_2 &= F_2 = F_4 = 0, \nu^{(\rho)} = \lambda^{(\rho)} / (2\lambda^{(\rho)} + 2\mu^{(\rho)} + \kappa^{(\rho)}).
\end{aligned} \quad (7.19)$$

The solution for the extension and bending of the composite circular cylinder is given by

$$\begin{aligned}
u_r &= (a_1 \cos \theta + a_2 \sin \theta) \left(-\frac{1}{2} z^2 - \frac{1}{8} r^2 + v_r^o \right) + a_3 u_r^{(3)}, \\
u_\theta &= (a_1 \sin \theta - a_2 \cos \theta) \left(\frac{1}{2} z^2 - \frac{5}{8} r^2 + v_\theta^o \right), \\
u_z &= (a_1 \cos \theta + a_2 \sin \theta) r z + a_3 z, \\
\varphi_r &= -(a_1 \sin \theta - a_2 \cos \theta) z, \\
\varphi_\theta &= -(a_1 \cos \theta + a_2 \sin \theta) z, \\
\varphi_z &= (a_1 \sin \theta - a_2 \cos \theta) (-r + \psi_z^o),
\end{aligned} \quad (7.20)$$

where $v_r^o, v_\theta^o, \psi_z^o$ are given by (7.14), $u_r^{(3)}$ is given by (7.18) and the constants $a_i (i = 1, 2, 3)$ can be determined from the system (3.7). From (7.16), (7.18) and (3.8) it follows

$$\begin{aligned}
A_{12} &= A_{21} = 0, A_{3\alpha} = A_{\alpha 3} = 0, \\
A_{33} &= \pi(\lambda^{(2)} + 2\mu^{(2)} + \kappa^{(2)} + 2\lambda^{(2)} E_1) r_2^2 + \pi(\lambda^{(1)} + 2\mu^{(1)} + \kappa^{(1)} + 2\lambda^{(1)} F_1) (r_1^2 - r_2^2), \\
A_{11} &= A_{22} = \pi(2\mu^{(2)} + \kappa^{(2)}) r_2^4 / 4 + \pi(\beta^{(2)} + \gamma^{(2)}) r_2^2 - \pi\beta^{(2)} r_2 [C_3 I_1(\delta^{(2)} r_2) \\
&\quad - 8C_2 r_2] - 2\pi\lambda^{(2)} (2c^{(2)} - d^{(2)}) C_2 r_2^4 + \pi\lambda^{(1)} D_5 (r_1 - r_2) \\
&\quad + \pi(2\mu^{(1)} + \kappa^{(1)}) (r_1^4 - r_2^4) / 4 + \pi(\beta^{(1)} + \gamma^{(1)}) (r_1^2 - r_2^2) - \pi\beta^{(1)} \{ D_3 [r_1 I_1(\delta^{(1)} r_1) \\
&\quad - r_2 I_1(\delta^{(1)} r_2)] - 8D_2 (r_1^2 - r_2^2) + D_4 [r_1 K_1(\delta^{(1)} r_1) - r_2 K_1(\delta^{(1)} r_2)] - 2\pi\lambda^{(1)} (2c^{(1)} - d^{(1)}) D_2 (r_1^4 - r_2^4),
\end{aligned}$$

so that we obtain

$$a_1 = \frac{1}{A_{11}} M_2, a_2 = -\frac{1}{A_{11}} M_1, a_3 = -\frac{1}{A_{33}} P_3.$$

From the above results it is easy to derive the solution in the case of homogeneous cylinders.

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