ON SAINT-VENANT'S PRINCIPLE IN MICROPOLAR ELASTICITY

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Abstract—A surface measure of the displacement is employed to obtain decay results of Saint-Venant's type in cylinders composed of a physically nonlinear micropolar elastic solid. Finite length cylinders as well as cylinders which are infinite in one direction are considered.

1. INTRODUCTION

Saint-Venant type inequalities show that certain measures of the displacement which depend upon the distance from the portion of the boundary at which the external tractions are applied, decrease exponentially with that distance. The commonest approach (initiated by Toupin [1]) is to measure the displacement by the elastic energy stored in part of the body which is relatively distant from the loaded end. Recently, however, certain surface measures of the displacement have been utilized for the same purpose (see [2–7], for instance). Reviews of the progress made in this area of research have been given in [8–11].

In [12] and [13] Toupin's method was employed to obtain decay results which illustrate the Saint-Venant principle for a cylinder of finite length composed of a micropolar, linearly elastic solid whose strain energy was positive definite. The case of a micropolar helical body was considered in [14] whereas in [15] the approach in [1] was adapted to the more general situation of linear elasticity with microstructure. To our knowledge the papers [12–15] are the only ones which deal with the Saint-Venant principle in the context of elasticity with microstructure.

In the present paper we extend the results recently obtained in [5] for the linearly elastic, semi-infinite cylinder to a class of physically nonlinear, micropolar elastic solids. Specifically, when a certain functional (which is evaluated at the solution of the problem and reduces to the total strain energy in the linear case) is bounded we show that a surface measure of an appropriately defined work function decreases exponentially with the distance from the loaded end. (As in [5] the lateral sides of the cylinder are taken to be free of tractions.) Additionally, we show that an analogous result also holds for cylinders of finite length provided that one of its ends (as well as its lateral surface) is unloaded. Our results for the finite cylinder are, in some respects, similar to those obtained in [12] and [13].

For linear, homogeneous, micropolar elastic solids our constitutive hypotheses are implied by the assumption that the strain energy is positive definite. Known examples from classical, as well as linear micropolar, elasticity indicate that this assumption is both realistic and consistent with the fact that the total energy associated with the deformation of an infinite body should be bounded. (Whilst solutions with infinite total energy should be discounted on physical grounds we are not aware of any constitutive assumption which automatically does so.) For sufficiently long, micropolar, linearly elastic cylinders therefore, we expect the Saint-Venant principle to be applicable.

2. PRELIMINARIES

Consider a prismatic cylinder of uniform cross-section D whose boundary is sufficiently smooth to admit application of the divergence theorem. The origin O of a rectangular Cartesian system $Ox_1x_2x_3$ is located at an interior point of the base cross-section with the x_3 -axis directed parallel to the generator. The region $\Omega \subset R^3$ occupied by the cylinder is therefore given by $\Omega = D \times [0, L]$ and its lateral surface is $\partial\Omega = \partial D \times [0, L]$, $L > 0$ being the length of the cylinder. We assume that D is star-shaped with respect to the point at which the x_3 -axis intersects D (see [16, p. 14] for the relevant definitions). The boundary ∂D of D may be represented by $r = r(\theta)$, $\theta \in [0, 2\pi]$, where (r, θ) are polar coordinates, and we define

$$r_1 \equiv \min_{\theta} r(\theta), \quad r_2 \equiv \max_{\theta} r(\theta). \quad (2.1)$$

Following [17] we introduce the linearized strain measures

$$e_{ij}(\mathbf{U}) \equiv u_{j,i} + \varepsilon_{ijk}\varphi_k, \quad \kappa_{ij}(\mathbf{U}) \equiv \varphi_{j,i}, \quad i, j, k = 1, 2, 3, \quad (2.2)$$

where u_j are the components of the displacement vector, φ_j are the components of the microrotation vector, $\mathbf{U}$ is short-hand for the six-dimensional vector field with components (u_i, φ_i) , ε_{ijk} is the alternating symbol and a comma followed by i denotes the derivative with respect to x_i . In what follows the italic indices will always assume the values 1, 2, 3 whereas the Greek indices will range over the values 1, 2. The summation convention over the repeated indices is adopted throughout this paper. Clearly $e_{ij}(\mathbf{U}) = 0$ and $\kappa_{ij}(\mathbf{U}) = 0$, simultaneously, if and only if

$$u_i = u_i^0 \equiv a_i + \varepsilon_{ijk}b_jx_k, \quad \varphi_i = \varphi_i^0 \equiv b_i, \quad (2.3)$$

where a_i, b_j are arbitrary constants. We refer to the vector $\mathbf{U}^0 = (u_i^0, \varphi_i^0)$ as a rigid micropolar displacement.

The cylinder that occupies the region Ω is composed of a micropolar elastic material for which the stress-strain relations can be written in the form [17, 18]

$$t_{ij} = \frac{\partial W}{\partial e_{ij}}, \quad m_{ij} = \frac{\partial W}{\partial \kappa_{ij}}, \quad (2.4)$$

where $W = W(e_{ij}, \kappa_{mn}; x_k)$ is the strain-energy function per unit volume. We assume that there exist positive constants c_0 and c_1 such that

$$\int_{D(x_3)} \left[\frac{\partial W}{\partial e_{ij}} e_{ij} + \frac{\partial W}{\partial \kappa_{ij}} \kappa_{ij} \right] dA \geq c_0 \int_{D(x_3)} [e_{ij}e_{ij} + \kappa_{ij}\kappa_{ij}] dA, \quad (2.5)$$

$$\int_{D(x_3)} \left[\frac{\partial W}{\partial e_{ij}} e_{ij} + \frac{\partial W}{\partial \kappa_{ij}} \kappa_{ij} \right] dA \geq c_1 \int_{D(x_3)} \left[\frac{\partial W}{\partial e_{3i}} \frac{\partial W}{\partial e_{3i}} + \frac{\partial W}{\partial \kappa_{3i}} \frac{\partial W}{\partial \kappa_{3i}} \right] dA. \quad (2.6)$$

The notation

$$\int_{D(x_3)} h dA \equiv \int_D h(x_1, x_2, x_3) dx_1 dx_2, \quad (2.7)$$

which was employed in the formulation of (2.5) and (2.6), will be used in a similar fashion throughout this paper. The pointwise version of (2.5) (apparently first used in [19]) reduces in the linear case (i.e. when $W = (\frac{1}{2})A_{ijks}e_{ij}e_{ks} + B_{ijks}e_{ij}\kappa_{ks} + (\frac{1}{2})C_{ijks}\kappa_{ij}\kappa_{ks}$, where $A_{ijks} = A_{ksij}$, $C_{ijks} = C_{ksij}$, B_{ijks} are functions of x_1, x_2, x_3) to the well-known positive-definiteness condition

$$2W \geq c_0(e_{ij}e_{ij} + \kappa_{ij}\kappa_{ij}). \quad (2.8)$$

The counterpart of (2.6) in the linear theory is implied by

$$t_{ij}t_{ij} + m_{ij}m_{ij} \leq 2cW, \quad c = \text{const.}, \quad c > 0, \quad (2.9)$$

(where c is known as the maximum elastic moduli) which holds [13, p. 98] when the material functions A_{ijks} , B_{ijks} and C_{ijks} are bounded on Ω .

We assume that the lateral surface of the cylinder is free of tractions so that we have

$$s_i \equiv t_{ji}n_j = 0, \quad q_i \equiv m_{ji}n_j = 0, \quad \text{on } \partial D \times [0, L], \quad (2.10)$$

where n_j denote the components of the outwardly directed unit normal vector to $\partial\Omega$. Over the end $x_3 = 0$ the loads s_i and q_i (which, in general, are specified to be different from zero) are supposed to be self-equilibrated, i.e.,

$$\int_{D(x_3)} s_i \, dA = \int_{D(x_3)} [q_i + \varepsilon_{ijk} x_j s_k] \, dA = 0, \quad x_3 = 0. \quad (2.11)$$

If $L < \infty$ (i.e. the cylinder has finite length) the loads over the end $x_3 = L$ are assumed to be zero. The body force and body couple vectors are supposed to be zero throughout this paper.

Finally, we assume that under the above hypotheses the cylinder is in equilibrium so that there exists a vector function $\mathbf{U} \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ ($\bar{\Omega} \equiv \Omega \cup \partial\Omega$) which, in addition to the boundary conditions, satisfies the equilibrium equations [18]

$$\left(\frac{\partial W}{\partial e_{ji,j}} \right) = 0, \quad \left(\frac{\partial W}{\partial \kappa_{ji,j}} \right) + \varepsilon_{ijk} \frac{\partial W}{\partial e_{jk}} = 0, \quad \text{in } \Omega. \quad (2.12)$$

When this is the case the equations (2.11)_{1,2} are satisfied for all x_3 , $0 \leq x_3 \leq L$.

We consider the eigenvalue problem

$$\left. \begin{aligned} w_{\alpha,\rho\rho} + \varepsilon_{3\alpha\beta} \varphi_{,\beta} + \lambda w_\alpha &= 0, & \text{on } D, \\ \varphi_{,\rho\rho} - 2\varphi + \varepsilon_{3\alpha\beta} w_{\beta,\alpha} + \lambda \varphi &= 0, & \text{on } D, \end{aligned} \right\} \quad (2.13)$$

$$\left. \begin{aligned} (w_{\beta,\alpha} + \varepsilon_{3\beta\alpha} \varphi) n_\alpha &= 0, & \text{on } \partial D, \\ \varphi_{,\alpha} n_\alpha &= 0, & \text{on } \partial D, \end{aligned} \right\} \quad (2.14)$$

and note that it satisfies the hypotheses required by the Theorem 6.V in [20]. Thus, there exists an increasing sequence of positive eigenvalues (the first of which is zero) which tends to infinity. Associated with the problem (2.13), (2.14) there is the Rayleigh quotient

$$q_D(\mathbf{w}) \equiv \frac{\int_{D(x_3)} [(w_{\beta,\alpha} + \varepsilon_{3\beta\alpha} \varphi)(w_{\beta,\alpha} + \varepsilon_{3\beta\alpha} \varphi) + \varphi_{,\alpha} \varphi_{,\alpha}] \, dA}{\int_{D(x_3)} (w_\rho w_\rho + \varphi^2) \, dA} \geq 0, \quad (2.15)$$

where $\mathbf{w}$ stands for (w_1, w_2, φ) . Clearly, $q_D(\mathbf{w}) = 0$ if and only if

$$w_\rho = w_\rho^0 \equiv a_\rho + \varepsilon_{3\beta\rho} x_\beta b, \quad \varphi = \varphi^0 \equiv b, \quad a_\rho, b = \text{const.}, \quad (2.16)$$

in which case the corresponding eigenvalue is zero. As solutions of the type (2.16) are excluded by the normalization conditions

$$\int_{D(x_3)} w_\alpha \, dA = 0, \quad \int_{D(x_3)} (\varepsilon_{3\alpha\beta} x_\alpha w_\beta + \varphi) \, dA = 0, \quad (2.17)$$

it follows that

$$q_D(\mathbf{w}) \geq \lambda_0 > 0 \quad (2.18)$$

(where λ_0 denotes the smallest non-zero eigenvalue) for all $\mathbf{w}$ that satisfy (2.17).

Let $B(y, d)$ be the sphere of radius d and centre $y = (y_1, y_2, y_3)$ whose surface is denoted by $\partial B(y, d)$. Following [21] we can show that

$$\int_{B(y, d)} [(u_{j,i} + \varepsilon_{jik} \varphi_k)(u_{j,i} + \varepsilon_{jik} \varphi_k) + \varphi_{i,j} \varphi_{i,j}] dV \geq \lambda_1 \int_{\partial B(y, d)} (u_i u_i + \varphi_i \varphi_i) dS, \quad (2.19)$$

$$\lambda_1 = \text{const.}, \quad \lambda_1 > 0,$$

provided

$$\int_{\partial B(y, d)} u_i dS = \int_{\partial B(y, d)} (\varepsilon_{ijk} x_j u_k + \varphi_i) dS = 0. \quad (2.20)$$

The constant $\lambda_1 = \lambda_1(d)$ is the smallest non-zero eigenvalue associated with the following boundary-value problem of Stekloff type:

$$\left. \begin{aligned} u_{i,jj} + \varepsilon_{ijk} \varphi_{k,j} &= 0, & \text{on } B(y, d), \\ \varphi_{i,jj} + \varepsilon_{ijk} u_{k,j} - 2\varphi_i &= 0, & \text{on } B(y, d), \end{aligned} \right\} \quad (2.21)$$

$$\left. \begin{aligned} (u_{i,j} + \varepsilon_{ijk} \varphi_k) n_j - \lambda u_i &= 0, & \text{on } \partial B(y, d), \\ \varphi_{i,j} n_j - \lambda \varphi_i &= 0, & \text{on } \partial B(y, d). \end{aligned} \right\} \quad (2.22)$$

As above n_j denote the components of the outwardly oriented unit normal vector to $\partial B(y, d)$.

We now choose d so that (i) $B(z, d) \subset \Omega$, $z \equiv (0, 0, x_3 + d)$ and (ii) every line segment joining $(0, 0, x_3 + 2d)$ to any point in D (where D is such that $(0, 0, x_3) \in D$) is contained in Ω . Then, for every sufficiently smooth vector function $\omega = (\omega_1, \omega_2, \omega_3)$ in $\bar{\Omega}$, we have [22]

$$\int_{D(x_3)} \omega_3^2 dA \leq \frac{k_1}{4} \int_{\Omega \setminus B(z, d)} (\omega_{i,j} + \omega_{j,i})(\omega_{i,j} + \omega_{j,i}) dV + k_2 \int_{\partial B(z, d)} \omega_i \omega_i dS, \quad (2.23)$$

where

$$k_1 \equiv d \left[4(1 + p^2)^3 + (4 + p^2)^2 \left(2 + \frac{1}{2} p^2 \right) \right], \quad k_2 \equiv 4(5 + 2p^2), \quad p \equiv \frac{r_2}{d}. \quad (2.24)$$

A possible choice for d is evidently $d = r_1$. As noted in [5] the inequality (2.23) remains valid for $z = (0, 0, x_3 - d)$ provided that the point $(0, 0, x_3 + 2d)$ is replaced by $(0, 0, x_3 - 2d)$, $x_3 \geq 2d$.

Using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ we see that

$$\frac{1}{4} (\omega_{i,j} + \omega_{j,i})(\omega_{i,j} + \omega_{j,i}) \leq \omega_{i,j} \omega_{i,j} \quad (2.25)$$

and thus (2.23) leads to

$$\int_{D(x_3)} \omega_k^2 dA \leq k_1 \int_{\Omega \setminus B(z, d)} \omega_{i,j} \omega_{i,j} dV + k_2 \int_{\partial B(z, d)} \omega_i \omega_i dS, \quad k = 1, 2, 3. \quad (2.26)$$

3. THE CROSS-SECTIONAL WORK FUNCTION

In this section we establish some properties of the cross-sectional work function

$$I_U(x_3) \equiv \int_{D(x_3)} [t_{3i}(U) u_i + m_{3i}(U) \varphi_i] dA, \quad (3.1)$$

where $\mathbf{U}$ is a solution of the considered boundary-value problem and where $t_{3i}(\mathbf{U})$ and $m_{3i}(\mathbf{U})$ have their obvious significance. To this end we re-write (3.1) in the form

$$I_{\mathbf{U}}(x_3) = I_{\mathbf{U}}^{(1)}(x_3) + I_{\mathbf{U}}^{(2)}(x_3), \quad (3.2)$$

where

$$I_{\mathbf{U}}^{(1)}(x_3) \equiv \int_{D(x_3)} [t_{3\alpha}(\mathbf{U})u_{\alpha} + m_{33}(\mathbf{U})\varphi_3] \, dA \quad (3.3)$$

and

$$I_{\mathbf{U}}^{(2)}(x_3) \equiv \int_{D(x_3)} [t_{33}(\mathbf{U})u_3 + m_{3\alpha}(\mathbf{U})\varphi_{\alpha}] \, dA \quad (3.4)$$

and note that both $I_{\mathbf{U}}^{(\alpha)}(x_3)$, $\alpha = 1, 2$, are invariant over superposed rigid micropolar displacements. To verify the last statement we write

$$\mathbf{V} = \mathbf{U}^0 + \mathbf{U}, \quad \mathbf{V} \equiv (v_i, \psi_i) \quad (3.5)$$

where $\mathbf{U}^0$ is an arbitrary rigid micropolar displacement and, in view of the obvious identities

$$e_{ij}(\mathbf{V}) = e_{ij}(\mathbf{U}), \quad \kappa_{ij}(\mathbf{V}) = \kappa_{ij}(\mathbf{U}), \quad (3.6)$$

and

$$t_{ij}(\mathbf{V}) = t_{ij}(\mathbf{U}), \quad m_{ij}(\mathbf{V}) = m_{ij}(\mathbf{U}), \quad (3.7)$$

we obtain

$$I_{\mathbf{U}}^{(1)}(x_3) = \int_{D(x_3)} [t_{3\alpha}(\mathbf{V})v_{\alpha} + m_{33}(\mathbf{V})\psi_3] \, dA - \int_{D(x_3)} [t_{3\alpha}(\mathbf{U})u_{\alpha}^0 + m_{33}(\mathbf{U})\varphi_3^0] \, dA. \quad (3.8)$$

Taking into account (2.3) and using the fact that (2.11)_{1,2} are valid for any x_3 , we see that the second integral in (3.8) is zero and thus deduce that

$$I_{\mathbf{U}}^{(1)}(x_3) = I_{\mathbf{V}}^{(1)}(x_3). \quad (3.9)$$

A similar argument shows that $I_{\mathbf{U}}^{(2)}(x_3)$ is also invariant over superposed rigid micropolar displacements.

Combining (3.9) and (3.7) we find

$$I_{\mathbf{U}}^{(1)}(x_3) = \int_{D(x_3)} [t_{3\alpha}(\mathbf{U})v_{\alpha} + m_{33}(\mathbf{U})\psi_3] \, dA, \quad (3.10)$$

and similarly

$$I_{\mathbf{U}}^{(2)}(x_3) = \int_{D(x_3)} [t_{33}(\mathbf{U})v_3 + m_{3\rho}(\mathbf{U})\psi_{\rho}] \, dA. \quad (3.11)$$

Using the Schwarz inequality and the arithmetic–geometric mean inequality in (3.10) and (3.11) we obtain (for arbitrary $\alpha > 0$)

$$|I_{\mathbf{U}}^{(1)}(x_3)| \leq \frac{\alpha}{2} \int_{D(x_3)} [t_{3\rho}(\mathbf{U})t_{3\rho}(\mathbf{U}) + m_{33}(\mathbf{U})^2] \, dA + \frac{1}{2\alpha} \int_{D(x_3)} (v_{\rho}v_{\rho} + \psi_3^2) \, dA \quad (3.12)$$

and

$$|I_{\mathbf{U}}^{(2)}(x_3)| \leq \frac{\alpha}{2} \int_{D(x_3)} [t_{33}(\mathbf{U})^2 + m_{3\rho}(\mathbf{U})m_{3\rho}(\mathbf{U})] \, dA + \frac{1}{2\alpha} \int_{D(x_3)} (v_3^2 + \psi_{\rho}\psi_{\rho}) \, dA, \quad (3.13)$$

respectively.

We now choose $\mathbf{V}$ in (3.12) so that

$$\int_{D(x_3)} v_\alpha \, dA = 0, \quad \int_{D(x_3)} (x_1 v_2 - x_2 v_1 + \psi_3) \, dA = 0. \quad (3.14)$$

In view of (3.5) and (2.3) this clearly amounts to choosing the constants a_i, b_j such that (3.14) is satisfied. On combining (3.12), (2.18), (2.15), (2.2), (3.6), (2.5) and (2.4) we find

$$\begin{aligned} |I_U^{(1)}(x_3)| &\leq \frac{\alpha}{2} \int_{D(x_3)} [t_{3\rho}(\mathbf{U})t_{3\rho}(\mathbf{U}) + m_{33}(\mathbf{U})^2] \, dA \\ &\quad + (2\lambda_0 c_0 \alpha)^{-1} \int_{D(x_3)} [t_{ij}(\mathbf{U})e_{ij}(\mathbf{U}) + m_{ij}(\mathbf{U})\kappa_{ij}(\mathbf{U})] \, dA. \end{aligned} \quad (3.15)$$

Let $z = (0, 0, x_3 - d)$, $0 < d \leq r_1$, $x_3 + 2d \leq x_3 \leq L$, $x_3 \geq 0$, and let $B(z, d)$ be the sphere of radius d with the centre at z . Since D is star-shaped we can employ the inequality (2.23) (see [22] and also [5]) with $\omega_i = v_i$, $i = 1, 2, 3$, and $\Omega = \Omega(x_3, x_3)$ where we have denoted

$$\Omega(z_3^1, z_3^2) \equiv \{z = (z_1, z_2, z_3) \in \Omega: z_3^1 \leq z_3 \leq z_3^2\}. \quad (3.16)$$

We obtain

$$\int_{D(x_3)} v_3^2 \, dA \leq k_1 \int_{\Omega(x_3, x_3) \setminus B(z, d)} e_{ij}(\mathbf{V})e_{ij}(\mathbf{V}) \, dV + k_2 \int_{\partial B(z, d)} v_i v_i \, dS \quad (3.17)$$

where use has also been made of the inequality [19]

$$e_{ij}(\mathbf{V})e_{ij}(\mathbf{V}) \geq \frac{1}{4} (v_{i,j} + v_{j,i})(v_{i,j} + v_{j,i}). \quad (3.18)$$

Similarly, we find from (2.26)

$$\int_{D(x_3)} \psi_\rho \psi_\rho \, dA \leq 2k_1 \int_{\Omega(x_3, x_3) \setminus B(z, d)} \kappa_{ij}(\mathbf{V})\kappa_{ij}(\mathbf{V}) \, dV + 2k_2 \int_{\partial B(z, d)} \psi_i \psi_i \, dS. \quad (3.19)$$

Choosing $\mathbf{V}$ such that

$$\int_{\partial B(z, d)} v_i \, dS = 0, \quad \int_{\partial B(z, d)} (\varepsilon_{ijk} x_j v_k + \psi_i) \, dS = 0, \quad (3.20)$$

and making use of (3.17), (3.19), (2.19), (2.2) and (3.6) we deduce

$$\int_{D(x_3)} (v_3^2 + \psi_\rho \psi_\rho) \, dA \leq 2m_1 \int_{\Omega(x_3, x_3)} [e_{ij}(\mathbf{U})e_{ij}(\mathbf{U}) + \kappa_{ij}(\mathbf{U})\kappa_{ij}(\mathbf{U})] \, dV, \quad (3.21)$$

where $m_1 \equiv \max(k_1, k_2 \lambda_1^{-1})$. The inequality (3.21), together with (2.5), (2.4) and (3.13), now yields

$$\begin{aligned} |I_U^{(2)}(x_3)| &\leq \frac{\alpha}{2} \int_{D(x_3)} [t_{33}(\mathbf{U})^2 + m_{3\rho}(\mathbf{U})m_{3\rho}(\mathbf{U})] \, dA \\ &\quad + m_1 (c_0 \alpha)^{-1} \int_{\Omega(x_3, x_3)} [t_{ij}(\mathbf{U})e_{ij}(\mathbf{U}) + m_{ij}(\mathbf{U})\kappa_{ij}(\mathbf{U})] \, dV. \end{aligned} \quad (3.22)$$

Using (2.23) with $z = (0, 0, x_3 + d)$, $0 < d \leq r_1$, $0 \leq x_3 \leq L - 2d$ and $\Omega = \Omega(x_3, x_3 + h)$, $2d \leq h \leq L - x_3$, we obtain in a similar manner

$$\int_{D(x_3)} (v_3^2 + \psi_\rho \psi_\rho) \, dA \leq 2m_1 \int_{\Omega(x_3, x_3 + h)} [e_{ij}(\mathbf{U})e_{ij}(\mathbf{U}) + \kappa_{ij}(\mathbf{U})\kappa_{ij}(\mathbf{U})] \, dV \quad (3.23)$$

and, consequently

$$|I_U^{(2)}(x_3)| \leq \frac{\alpha}{2} \int_{D(x_3)} [t_{33}(\mathbf{U})^2 + m_{3\rho}(\mathbf{U})m_{3\rho}(\mathbf{U})] dA \\ + m_1(c_0\alpha)^{-1} \int_{\Omega(x_3, x_3+h)} [t_{ij}(\mathbf{U})e_{ij}(\mathbf{U}) + m_{ij}(\mathbf{U})\kappa_{ij}(\mathbf{U})] dV. \quad (3.24)$$

In view of (2.10)_{1,2} we can write, for $0 \leq y_3 < x_3 \leq L$,

$$I_U(x_3) - I_U(y_3) = \int_{D(x_3)} [t_{3i}(\mathbf{U})n_3u_i + m_{3i}(\mathbf{U})n_3\varphi_i] dA \\ + \int_{D(y_3)} [t_{3i}(\mathbf{U})n_3u_i + m_{3i}(\mathbf{U})n_3\varphi_i] dA \\ + \int_{\partial D \times [y_3, x_3]} [t_{ji}(\mathbf{U})n_ju_i + m_{ji}(\mathbf{U})n_j\varphi_i] dS \quad (3.25)$$

which reduces to

$$I_U(x_3) - I_U(y_3) = \int_{\partial\Omega(y_3, x_3)} [t_{ji}(\mathbf{U})u_i + m_{ji}(\mathbf{U})\varphi_i]n_j dS. \quad (3.26)$$

An application of the divergence theorem, and use of (2.2), (2.4) and (2.12), now yields

$$I_U(x_3) - I_U(y_3) = \int_{\Omega(y_3, x_3)} [t_{ji}(\mathbf{U})e_{ji}(\mathbf{U}) + m_{ji}(\mathbf{U})\kappa_{ji}(\mathbf{U})] dV \quad (3.27)$$

which implies

$$\frac{d}{dx_3} [I_U(x_3)] = \int_{D(x_3)} [t_{ji}(\mathbf{U})e_{ji}(\mathbf{U}) + m_{ji}(\mathbf{U})\kappa_{ji}(\mathbf{U})] dA. \quad (3.28)$$

From (2.4) and (2.5) we find that

$$\frac{d}{dx_3} [I_U(x_3)] \geq 0. \quad (3.29)$$

Combining (3.2), (3.15), (3.22), (3.24), (2.6), (2.4), (3.27) and (3.28) we obtain (for arbitrary $\alpha > 0$)

$$|I_U(x_3)| \leq \frac{1}{2} [\alpha c_1^{-1} + (\lambda_0 c_0 \alpha)^{-1}] \frac{d}{dx_3} [I_U(x_3)] \\ + m_1(c_0\alpha)^{-1} [I_U(x_3) - I_U(x_3)], \quad x_3 + 2d \leq x_3 \leq L, \quad (3.30)$$

and

$$|I_U(x_3)| \leq \frac{1}{2} [\alpha c_1^{-1} + (\lambda_0 c_0 \alpha)^{-1}] \frac{d}{dx_3} [I_U(x_3)] + m_1(c_0\alpha)^{-1} [I_U(x_3 + h) - I_U(x_3)], \\ 0 \leq x_3 \leq L - 2d, \quad 2d \leq h \leq L - x_3, \quad (3.31)$$

respectively.

4. ANALYSIS

In this section we study the implications of the inequalities (3.30) and (3.31) for the case of the semi-infinite cylinder as well as for the case when the cylinder has a finite length L .

When the cylinder has finite length L the assumption that the end $x_3 = L$ is free of tractions implies that $I_U(L) = 0$ and we deduce from (3.29) that $I_U(x_3) \leq 0$ for all $x_3 \in [0, L]$. In the case

of the semi-infinite cylinder (3.29) implies that either (i) $I_U(x_3) \leq 0$, $x_3 \in [0, \infty)$ or (ii) there exists $x_3^* \in [0, \infty)$ so that $I_U(x_3) > 0$ for $x_3 \in [x_3^*, \infty)$.

We consider now the case of the semi-infinite cylinder and assume that there exists the number x_3^* with the properties listed above. The inequality (3.30), with $\alpha = 2m_1/c_0$ and $x_3 = x_3^*$, then implies

$$\frac{d}{dx_3} [I_U(x_3)] - v_1^2 I_U(x_3) \geq 0, \quad x_3 \geq 2d + x_3^*, \quad v_1^2 \equiv \frac{2c_0 c_1 m_1 \lambda_0}{4m_1^2 \lambda_0 + c_1 c_0}, \quad (4.1)$$

which by integration leads to

$$I_U(x_3) \geq I_U(2d + x_3^*) \exp[v_1^2(x_3 - 2d - x_3^*)], \quad x_3 \geq 2d + x_3^*. \quad (4.2)$$

The latter shows that $I_U(x_3)$ tends to infinity as x_3 tends to infinity and from (3.27) we see that

$$E(x_3) \equiv \int_{\Omega(x_3, \infty)} [t_{ji}(\mathbf{U}) e_{ji}(\mathbf{U}) + m_{ji}(\mathbf{U}) \kappa_{ji}(\mathbf{U})] dV \quad (4.3)$$

is unbounded. We note that in the case of linear micropolar elasticity $E(x_3)$ is twice the total energy associated with $\Omega(x_3, \infty)$ and thus the situation in which it becomes unbounded is not physically realistic. We consider then the semi-infinite cylinder with $I_U(x_3) \leq 0$, for all x_3 , and observe that (3.31) now reduces to

$$\frac{d}{dx_3} [I_U(x_3)] + v_1^2 I_U(x_3) \geq 0, \quad x_3 \in [0, \infty), \quad (4.4)$$

which yields

$$-I_U(x_3) \leq -I_U(0) \exp(-v_1^2 x_3), \quad x_3 \in [0, \infty). \quad (4.5)$$

From (4.5) we see that $I_U(\infty) = 0$ so that, in view of (3.27), we conclude that $E(x_3)$ is bounded. Furthermore, (3.27) gives

$$E(x_3) = -I_U(x_3). \quad (4.6)$$

Combining the latter with (3.23), (2.5) and (4.5) we obtain

$$\int_{D(x_3)} (v_3^2 + \psi_\rho \psi_\rho) dA \leq -2m_1 c_0^{-1} I_U(0) \exp(-v_1^2 x_3), \quad x_3 \in [0, \infty), \quad (4.7)$$

where the vector $\mathbf{V}$ given by (3.5) has been chosen so that the equations (3.20) are satisfied. Choosing $\mathbf{V}$ to satisfy (3.14) and employing (2.18), (2.15), (2.5), (3.6) and (3.7) we find

$$\int_{D(x_3)} (v_\rho v_\rho + \psi_3^2) dA \leq (\lambda_0 c_0)^{-1} \int_{D(x_3)} [t_{ij}(\mathbf{U}) e_{ij}(\mathbf{U}) + m_{ij}(\mathbf{U}) \kappa_{ij}(\mathbf{U})] dA. \quad (4.8)$$

On integrating in (4.8) between x_3 and ∞ and using (4.3), (4.6) and (4.5) we deduce

$$\int_{\Omega(x_3, \infty)} (v_\rho v_\rho + \psi_3^2) dV \leq -(\lambda_0 c_0)^{-1} I_U(0) \exp(-v_1^2 x_3), \quad x_3 \in [0, \infty). \quad (4.9)$$

We emphasize that, according to the above discussion, two different vectors $\mathbf{V}$ [given by (3.5) where $\mathbf{U}$ is the solution to the considered boundary-value problem] satisfy the estimates (4.7) and (4.9).

As remarked before, our assumptions imply that for a cylinder of finite length L we have $I_U(x_3) \leq 0$. Then the inequality in (4.4) is still valid (for $x_3 \in [0, L - 2d]$) and thus we deduce that (4.5)₁ and (4.7)₁ are also valid for $x_3 \in [0, L - 2d]$. The inequality (4.9)₁ is replaced in this situation by

$$\int_{\Omega(x_3, L-2d)} (v_\rho v_\rho + \psi_3^2) dV \leq -(\lambda_0 c_0)^{-1} I_U(0) \exp(-v_1^2 x_3). \quad (4.10)$$

Since $I_U(x_3)$ is non-decreasing it follows that, for $x_3 \in [L - 2d, L]$, the effects of applied tractions as described by the left-hand sides of (4.5), (4.7) and (4.10) are $O\{\exp[-v_1^2(L - 2d)]\}$. We note that in this case (3.27) implies

$$I_U(x_3) = - \int_{\Omega(x_3, L)} [t_{ji}(\mathbf{U})e_{ji}(\mathbf{U}) + m_{ji}(\mathbf{U})\kappa_{ji}(\mathbf{U})] dV \quad (4.11)$$

so that $I_U(x_3)$ also represents a volumetric measure of the displacement [which, in the linear case, reduces to twice the total energy of the part of the cylinder contained in $\Omega(x_3, L)$]. For the finite cylinder therefore the inequality (4.5) is of the type obtained in [12] and [13]. We note, however, that the exponent v_1^2 which occurs in this inequality is different from that arrived at in [12] and [13]. Unlike that in [12] and [13] this exponent does not depend upon the size ℓ of an arbitrarily chosen cylindrical slice $\Omega(x_3, x_3 + \ell)$, $\ell > 0$.

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