SPATIAL DECAY ESTIMATES OF SAINT-VENANT TYPE IN GENERALIZED THERMOELASTICITY

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Abstract—Decay estimates of Saint-Venant type are established within the context of dynamic generalized thermoelasticity. A thermoelastic body is deformed under null body loads and initial conditions and subject to non-zero boundary data only on a plane end. The behaviour is studied for the total energy stored in that part of the body beyond a distance z from the basis. An estimate is established that describes the spatial decay of end effects.

When the harmonically vibrations of a right cylinder are considered, it is shown that an exponential decay of activity holds away from the excited end provided that the exciting frequency is less than a certain critical frequency.

1. INTRODUCTION

This paper studies a class of practically important spatial decay and relatively new problems for the dynamic theory of generalized thermoelasticity furnished by Green and Lindsay [1]. Originally the feasibility of Saint-Venant principle for time-dependent problems was first suggested by Boley [2] in connection with transient heat conduction (governed by a parabolic differential equation). By means of various techniques, in [3–8] have been established spatial decay estimates for transient heat conduction. It was shown by Horgan *et al.* [7] and Chiriță [8] that the spatial decay rate in the transient problem is actually faster than that for the steady-state problem. Recently Quintanilla [9] has given a first result of decay estimate type for the hyperbolic heat equation.

An energy estimate was established by Chiriță [10] which shows that end effects for linear dynamic theory of classical thermoelasticity decay spatially with the distance z from the loaded end, the decay rate being controlled by the factor $\exp(-z/\nu(t)\sqrt{t})$, where the positive function $\nu(t)$ depends on the thermoelastic coefficients too.

In this paper the hyperbolic equations of the dynamic theory of generalized thermoelasticity are studied in order to obtain spatial energy bounds and decay estimates in connection with the following two problems:

- (a) a thermoelastic body R is deformed under null initial conditions and subject to non-zero boundary data only on a plane end S_0 ;
- (b) for a right cylindrical region of finite extent, one plane end is subject to an excitation which is harmonic in time; while the remainder of the boundary is clamped and thermally insulated.

In both problems it is assumed that any body loads are absent.

Confining attention to the problem (a), results are proved which state spatial bounds of end effects in a similar way to the results stated at [11–13]. It is shown that, at least for appropriate short times, the decay of end effects is faster than that predicted in [10] for classical theory of thermoelasticity.

When the problem (b) is considered, the results established in the present paper prove an

exponential decay of activity away from the excited end provided that the exciting frequency is less than a certain critical frequency.

2. BASIC EQUATIONS. PRELIMINARY CONSIDERATIONS

Let R be a closed, bounded, regular region in three-dimensional space whose boundary ∂R includes a plane portion S_0 . Choose Cartesian coordinates x_1, x_2, x_3 so that S_0 lies in the plane $x_3 = 0$, and suppose that R lies in the half-space $x_3 > 0$. Let S_z be the intersection with R of a plane $x_3 = z$, and let

$$R_z = \{\mathbf{x} \in R: x_3 > z\}, \quad (2.1)$$

and let L be the maximum value of x_3 on R . Also, the spatial argument and the time argument of a function will be omitted when there is no likelihood of confusion.

We suppose that the region R is occupied by an anisotropic and homogeneous thermoelastic medium. In the absence of the body force and heat supply fields, the field equations for the linear thermoelasticity are

$$\rho \ddot{u}_i = t_{ki,k}, \quad (2.2)$$

$$t_{ki} = t_{ik}, \quad (2.3)$$

$$\rho \theta_0 \dot{\eta} = -q_{i,i}, \quad (2.4)$$

in $R \times (0, \infty)$, in which u_i are the components of displacement, t_{ij} are the components of stress, q_i are the components of the heat conduction vector, η is the specific entropy, ρ is the constant reference density and θ_0 is the constant reference temperature. A superposed dot denotes $\partial/\partial t$ and the suffix “ k ” denotes $\partial/\partial x_k$.

When the reference solid has a centre of symmetry at each point, but is otherwise non-isotropic, then the constitutive relations are [1]

$$t_{ik} = k_{ikmn} e_{mn} + a_{ik}(\theta + \alpha \dot{\theta}), \quad (2.5)$$

$$q_i = -\theta_0 k_{ij} \theta_{,j}, \quad (2.6)$$

$$\rho \eta = a + d\theta + h\dot{\theta} - a_{ik} e_{ik}, \quad (2.7)$$

in $R \times (0, \infty)$, where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \quad (2.8)$$

The tensor coefficients in (2.5)–(2.7) are constants subject to the symmetry conditions

$$k_{ijmn} = k_{jimn} = k_{ijnm} = k_{mnij}, \quad a_{ij} = a_{ji}, \quad k_{ij} = k_{ji}. \quad (2.9)$$

The density ρ and temperature θ_0 are given constants subject to the conditions

$$\rho > 0, \quad \theta_0 > 0. \quad (2.10)$$

From the residual entropy inequality [1] we have the following conditions

$$d\alpha - h \geq 0, \quad k_{ij} \chi_i \chi_j \geq 0, \quad \text{for all arbitrary } \chi_i. \quad (2.11)$$

According to the conditions (2.11), throughout this paper we shall assume that

$$\alpha > 0, \quad h > 0, \quad d\alpha - h > 0; \quad (2.12)$$

moreover, we shall assume that the tensors (k_{ijmn}) and (k_{ij}) are positive definite.

Thus, let μ_M and μ_m denote the maximum and minimum elastic moduli for (k_{ijmn}) . Then

$$\mu_m e_{ij} e_{ij} \leq k_{ijmn} e_{ij} e_{mn} \leq \mu_M e_{ij} e_{ij}. \quad (2.13)$$

Further, it follows that

$$\mu_m^2 e_{ij} e_{ij} \leq k_{ijmn} k_{ijrs} e_{mn} e_{rs} \leq \mu_M^2 e_{ij} e_{ij}. \quad (2.14)$$

Now, let k_M and k_m denote the maximum and minimum conductivity moduli for (k_{ij}) . Then

$$k_m \chi_i \chi_i \leq k_{ij} \chi_i \chi_j \leq k_M \chi_i \chi_i, \quad (2.15)$$

and, moreover, we deduce that

$$k_m^2 \chi_i \chi_i \leq k_{ij} k_{ik} \chi_k \chi_j \leq k_M^2 \chi_i \chi_i. \quad (2.16)$$

Let ϵ be a positive number. For every symmetric second order tensor (A_{ij}) and (B_{ij}) , we have the inequality

$$(A_{ij} + B_{ij})(A_{ij} + B_{ij}) \leq (1 + \epsilon) A_{ij} A_{ij} + \left(1 + \frac{1}{\epsilon}\right) B_{ij} B_{ij}. \quad (2.17)$$

Therefore, the relations (2.5), (2.13), (2.14) and (2.17) yield

$$\begin{aligned} t_{ik} t_{ik} &= [k_{ijmn} e_{mn} + a_{ij}(\theta + \alpha \dot{\theta})][k_{ijrs} e_{rs} + a_{ij}(\theta + \alpha \dot{\theta})] \\ &\leq (1 + \epsilon) k_{ijmn} k_{ijrs} e_{mn} e_{rs} + \left(1 + \frac{1}{\epsilon}\right) a_{ij} a_{ij} (\theta + \alpha \dot{\theta})^2 \\ &\leq (1 + \epsilon) \mu^* k_{ijmn} e_{ij} e_{mn} + \left(1 + \frac{1}{\epsilon}\right) A (\theta + \alpha \dot{\theta})^2, \end{aligned} \quad (2.18)$$

where ϵ is a positive number that will be chosen later and

$$\mu^* = \frac{\mu_M^2}{\mu_m}, \quad A = a_{ij} a_{ij}. \quad (2.19)$$

Let

$$s_i = t_{ij} n_j, \quad (2.20)$$

denote the components of the surface traction on the boundary of the body with the outward unit normal $\mathbf{n}(n_i)$. In view of the relation (2.18) and since $n_i n_i = 1$, we get

$$s_i s_i \leq (1 + \epsilon) \mu^* k_{ijmn} e_{ij} e_{mn} + \left(1 + \frac{1}{\epsilon}\right) A^* \frac{h}{\alpha} (\theta + \alpha \dot{\theta})^2, \quad (2.21)$$

where

$$A^* = A \frac{\alpha}{h}. \quad (2.22)$$

On the other hand, we have

$$\frac{1}{\theta_0^2} q_i q_i = k_{ij} \theta_j k_{ik} \theta_{,k} \leq k_M^2 \theta_{,i} \theta_{,i} \leq \frac{k_M^2}{k_m} k_{ij} \theta_j \theta_{,i} = k^* \alpha k_{ij} \theta_j \theta_{,i}, \quad (2.23)$$

where

$$k^* = \frac{k_M^2}{\alpha k_m}. \quad (2.24)$$

Let

$$q = q_i n_i. \quad (2.25)$$

Then, by means of the relation (2.23) and since $n_i n_i = 1$, it results

$$\frac{1}{\theta_0^2} q^2 \leq k^* \alpha k_{ij} \theta_{,j} \theta_{,i}. \quad (2.26)$$

For future convenience, for $\pi = \{u_i, \theta\}$ we introduce the following functional

$$\begin{aligned} E_\pi(z, t) = & \frac{1}{2} \int_{R_z} \{ \rho \dot{u}_i(t) \dot{u}_i(t) + k_{ijmn} u_{i,j}(t) u_{m,n}(t) + \alpha k_{ij} \theta_{,j}(t) \theta_{,i}(t) + \frac{h}{\alpha} (\theta(t) + \alpha \dot{\theta}(t))^2 \\ & + \left(d - \frac{h}{\alpha} \right) \theta^2(t) + 2 \int_0^t [k_{ij} \theta_{,j}(\tau) \theta_{,i}(\tau) + (d\alpha - h) \dot{\theta}^2(\tau)] d\tau \} dv. \end{aligned} \quad (2.27)$$

Obviously, $E_\pi(z, t)$ represents the total energy stored in that part of the body beyond a distance z from the basis S_0 .

3. A DYNAMIC SAINT-VENANT'S PRINCIPLE

Let the body be subject to null initial conditions so that

$$u_i(x, 0) = 0, \quad \dot{u}_i(x, 0) = 0, \quad \theta(x, 0) = 0, \quad \dot{\theta}(x, 0) = 0, \quad \text{on } R. \quad (3.1)$$

We further suppose that the body is deformed under the action of non-zero boundary conditions only on the plane end face S_0 . Therefore, we assume the following boundary conditions

$$s_i = 0, \quad q = 0 \quad \text{on } (\partial R \setminus S_0) \times (0, \infty). \quad (3.2)$$

To complete the statement of a boundary-initial value problem for the body R , we must prescribe the boundary data on S_0 . We shall specify this later.

In what follows we shall assume that $A > 0$. For $A = 0$ it results $a_{ij} = 0$ so that the thermal effects are independent of the mechanical deformation and they can be treated separately by means of the same procedure.

THEOREM 3.1. *Assume $\pi = \{u_i, \theta\}$ satisfies the equations (2.2)–(2.8), the initial conditions (3.1) and the boundary conditions (3.2). Let (z_0, t_0) be fixed so that $0 \leq z_0 \leq L$ and $t_0 \geq 0$. Then*

(i) *for $0 \leq z \leq z_0$ and $\max[0, t_0 + c^{-1}(z - z_0)] \leq t \leq t_0 - c^{-1}(z - z_0)$, we have*

$$E_\pi(z, t) \geq E_\pi(z_0, t_0), \quad (3.3)$$

(ii) *while for $z_0 \leq z \leq L$ and $\max[0, t_0 - c^{-1}(z - z_0)] \leq t \leq t_0 + c^{-1}(z - z_0)$, we have*

$$E_\pi(z, t) \leq E_\pi(z_0, t_0), \quad (3.4)$$

where

$$c = \left[\frac{(1 + \epsilon_0) \mu^*}{\rho} \right]^{1/2}, \quad (3.5)$$

and ϵ_0 is the positive root of the algebraic equation

$$\epsilon^2 + \epsilon \left(1 - \frac{A^*}{\mu^*} - \frac{k^* \rho \alpha}{h \mu^*} \right) - \frac{A^*}{\mu^*} = 0. \quad (3.6)$$

PROOF. We first define the function

$$\Phi(z, t) = - \int_0^t \int_{S_z} (s_i(s) \dot{u}_i(s) - \frac{1}{\theta_0} q(s)(\theta(s) + \alpha \dot{\theta}(s))) \, da \, ds, \quad (3.7)$$

and observe that for $0 \leq z^* \leq z \leq L$, the relations (2.2)–(2.8), (3.1) and (3.2) give

$$\begin{aligned} \Phi(z, t) = & - \int_0^t \int_{S_{z^*}} (s_i(s) \dot{u}_i(s) - \frac{1}{\theta_0} q(s)(\theta(s) + \alpha \dot{\theta}(s))) \, da \, ds \\ & - \frac{1}{2} \int_{z^*}^z \int_{S_\eta} \left\{ \rho \dot{u}_i(t) \dot{u}_i(t) + k_{ijmn} e_{ij}(t) e_{mn}(t) + \alpha k_{ij} \theta_{,i}(t) \theta_{,j}(t) + \frac{h}{\alpha} (\theta(t) + \alpha \dot{\theta}(t))^2 \right. \\ & \left. + \left(d - \frac{h}{\alpha} \right) \theta^2(t) + 2 \int_0^t [k_{ij} \theta_{,i}(\tau) \theta_{,j}(\tau) + (d\alpha - h) \dot{\theta}^2(\tau)] \, d\tau \right\} \, dv. \end{aligned} \quad (3.8)$$

Direct differentiation of the relations (3.7) and (3.8) gives

$$\begin{aligned} \frac{\partial \Phi}{\partial z}(z, t) = & - \frac{1}{2} \int_{S_z} \left\{ \rho \dot{u}_i(t) \dot{u}_i(t) + k_{ijmn} e_{ij}(t) e_{mn}(t) + \alpha k_{ij} \theta_{,i}(t) \theta_{,j}(t) \right. \\ & \left. + \frac{h}{\alpha} (\theta(t) + \alpha \dot{\theta}(t))^2 + \left(d - \frac{h}{\alpha} \right) \theta^2(t) + 2 \int_0^t [k_{ij} \theta_{,i}(\tau) \theta_{,j}(\tau) + (d\alpha - h) \dot{\theta}^2(\tau)] \, d\tau \right\} \, da, \end{aligned} \quad (3.9)$$

and

$$\frac{\partial \Phi}{\partial t}(z, t) = - \int_{S_z} [s_i(t) \dot{u}_i(t) - \frac{1}{\theta_0} q(t)(\theta(t) + \alpha \dot{\theta}(t))] \, da. \quad (3.10)$$

A use of Schwarz's inequality, the arithmetic–geometric mean inequality and the estimates (2.21) and (2.26) lead to

$$\begin{aligned} \left| \frac{\partial \Phi}{\partial t}(z, t) \right| \leq & \frac{1}{2} \int_{S_z} \left\{ \gamma \rho \dot{u}_i(t) \dot{u}_i(t) + \frac{1}{\gamma \rho} (1 + \epsilon) \mu^* k_{ijmn} e_{ij}(t) e_{mn}(t) \right. \\ & \left. + \frac{k^*}{\delta} \alpha k_{ij} \theta_{,i}(t) \theta_{,j}(t) + \left[\frac{1}{\gamma \rho} \left(1 + \frac{1}{\epsilon} \right) A^* + \frac{\delta \alpha}{h} \right] \frac{h}{\alpha} (\theta(t) + \alpha \dot{\theta}(t))^2 \right\} \, da, \end{aligned} \quad (3.11)$$

where γ and δ are arbitrary positive constants.

Now, we equate the coefficients of the various energetic terms in the integral in (3.11), that is we set

$$\gamma = \frac{1}{\gamma \rho} (1 + \epsilon) \mu^* = \frac{1}{\gamma \rho} \left(1 + \frac{1}{\epsilon} \right) A^* + \frac{\delta \alpha}{h} = \frac{k^*}{\delta}. \quad (3.12)$$

Therefore, we choose

$$\gamma = \sqrt{\frac{(1 + \epsilon) \mu^*}{\rho}}, \quad \delta = k^* \sqrt{\frac{\rho}{(1 + \epsilon) \mu^*}}, \quad (3.13)$$

where ϵ satisfies the algebraic equation (3.6).

With these choices and the notation (3.5), the inequality (3.11) and the relation (3.9) imply

$$c^{-1} \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial z} \leq 0, \quad (3.14)$$

and

$$-c^{-1} \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial z} \leq 0. \quad (3.15)$$

Let us fix $(z_0, t_0) \in [0, L] \times [0, \infty)$. If we set $t = t_0 + c^{-1}(z - z_0)$ in (3.14) we deduce that

$$\Phi(z, t_0 + c^{-1}(z - z_0)) \text{ is a non-increasing function of } z \text{ on } [0, L]; \quad (3.16)$$

while if we set $t = t_0 - c^{-1}(z - z_0)$ in (3.15) it results that

$$\Phi(z, t_0 - c^{-1}(z - z_0)) \text{ is a non-increasing function of } z \text{ on } [0, L]. \quad (3.17)$$

Finally, we note that the relation (3.9) shows that

$$\Phi(z, t) \text{ is a non-increasing function of } z \text{ on } [0, L] \text{ for all fixed } t \geq 0. \quad (3.18)$$

We have thus all the preliminary material requested in order to prove the relations (3.3) and (3.4).

We first consider that $0 \leq z \leq z_0$ and $\max[0, t_0 + c^{-1}(z - z_0)] \leq t \leq t_0$. Then we have $z \leq z_1 \equiv z_0 + c(t - t_0) \leq z_0$ so that the relation (3.18) implies

$$\Phi(z, t) \geq \Phi(z_1, t), \quad (3.19)$$

while the relation (3.16) implies

$$\Phi(z_1, t) \geq \Phi(z_0, t_0). \quad (3.20)$$

Thus, the relations (3.19) and (3.20) lead to

$$\Phi(z, t) \geq \Phi(z_0, t_0). \quad (3.21)$$

Let us now consider that $0 \leq z \leq z_0$ and $t_0 \leq t \leq t_0 - c^{-1}(z - z_0)$. Then we have $z \leq z_2 \equiv z_0 - c(t - t_0) \leq z_0$ so that the relation (3.18) implies

$$\Phi(z, t) \geq \Phi(z_2, t), \quad (3.22)$$

while the relation (3.17) implies

$$\Phi(z_2, t) \geq \Phi(z_0, t_0). \quad (3.23)$$

On the basis of the relations (3.22) and (3.23) we conclude that the relation (3.21) again holds for the present case.

Further, we see from the boundary conditions that

$$\Phi(L, t) = 0 \quad \text{for all } t \geq 0, \quad (3.24)$$

so that the relations (2.27) and (3.8) give

$$\Phi(z, t) = E_\pi(z, t). \quad (3.25)$$

Therefore, the relations (3.21) and (3.25) lead to the relation (3.3).

We proceed now to prove the relation (3.4). In this aim we consider first that $z_0 \leq z \leq L$ and $\max[0, t_0 - c^{-1}(z - z_0)] \leq t \leq t_0$. Then we have $z \geq z_3 \equiv z_0 - c(t - t_0) \geq z_0$ so that the relation (3.18) implies

$$\Phi(z, t) \leq \Phi(z_3, t), \quad (3.26)$$

while the relation (3.17) implies

$$\Phi(z_3, t) \leq \Phi(z_0, t_0). \quad (3.27)$$

Thus, the relations (3.26) and (3.27) give

$$\Phi(z, t) \leq \Phi(z_0, t_0). \quad (3.28)$$

Let us now consider $z_0 \leq z \leq L$ and $t_0 \leq t \leq t_0 + c^{-1}(z - z_0)$. Then we have $z \geq z_4 \equiv z_0 + c(t - t_0) \geq z_0$ and hence the relation (3.18) gives

$$\Phi(z, t) \leq \Phi(z_4, t), \quad (3.29)$$

and the relation (3.16) gives

$$\Phi(z_4, t) \leq \Phi(z_0, t_0). \quad (3.30)$$

On the basis of the relations (3.29) and (3.30) we conclude that the relation (3.28) remains valid in this case. Further, the relations (3.25) and (3.28) lead to the inequality (3.4) and the proof of the theorem is complete.

THEOREM 3.2. Assume $\pi = \{u_i, \theta\}$ satisfies the equations (2.2)–(2.8), the initial conditions (3.1) and the boundary conditions (3.2). Then we have

$$E_\pi(z, t) = 0 \quad \text{for } ct \leq z \leq L, \quad t \geq 0, \quad (3.31)$$

and

$$\varepsilon_\pi(z, t) \leq \left(1 - \frac{z}{ct}\right) \varepsilon_\pi(0, t) \quad \text{for } 0 \leq z \leq ct \leq L, \quad t \geq 0, \quad (3.32)$$

where

$$\varepsilon_\pi(z, t) = \int_0^t E_\pi(z, s) \, ds. \quad (3.33)$$

PROOF. If we set $z_0 = 0$ and $t_0 = 0$ within the Theorem 3.1 and we take into account the initial conditions (3.1) we deduce

$$E_\pi(z, t) \leq E_\pi(0, 0) = 0 \quad \text{for } 0 \leq ct \leq z. \quad (3.34)$$

Thus, from (3.34) we get the relation (3.31).

Further, on the basis of the relations (3.31) and (3.33), for $z \leq ct$, we get

$$\varepsilon_\pi(z, t) = \int_{z/c}^t E_\pi(z, s) \, ds. \quad (3.35)$$

If we set

$$s = \left(1 - \frac{z}{ct}\right) \tau + \frac{z}{c}, \quad (3.36)$$

into the integral on the right-hand side of (3.35), we obtain

$$\varepsilon_\pi(z, t) = \left(1 - \frac{z}{ct}\right) \int_0^t E_\pi\left(z, \left(1 - \frac{z}{ct}\right) \tau + \frac{z}{c}\right) d\tau. \quad (3.37)$$

Now, we set $z_0 = z$ and $t_0 = \left(1 - \frac{z}{ct}\right) \tau + \frac{z}{c}$ within the Theorem 3.1 and notice that $t_0 - \frac{z}{c} \leq \tau \leq t_0$. Then, we see that (i) implies

$$E_\pi(0, \tau) \geq E_\pi\left(z, \left(1 - \frac{z}{ct}\right) \tau + \frac{z}{c}\right). \quad (3.38)$$

Thus, the relations (3.37) and (3.38) yield

$$\varepsilon_\pi(z, t) \leq \left(1 - \frac{z}{ct}\right) \int_0^t E_\pi(0, \tau) d\tau. \quad (3.39)$$

Clearly, the estimate (3.32) follows now from the relations (3.33) and (3.39) and the proof is complete.

In the remainder of this section we proceed to obtain an appropriate bound for $\varepsilon_\pi(0, t)$ in terms of the end boundary data. This will make the spatial decay estimate (3.32) more explicit.

With this in mind, we complete now the initial-boundary value problem for R by specifying the boundary conditions on the end face S_0 in the form

$$s_i = t_i, \quad q = p \quad \text{on } S_0 \times (0, \infty), \quad (3.40)$$

where t_i and p are prescribed continuous differentiable functions on $S_0 \times (0, \infty)$. Further, we assume that traction is “self-equilibrated” at each moment, that is

$$\int_{S_0} t_i da = 0, \quad \int_{S_0} \epsilon_{ijk} x_j t_k da = 0. \quad (3.41)$$

THEOREM 3.3. *Assume $\pi = \{u_i, \theta\}$ satisfies the equations (2.2)–(2.8), the initial conditions (3.1) and the boundary conditions (3.2) and (3.40). Then we have the following estimate*

$$\begin{aligned} \varepsilon_\pi(0, t) \leq & 16b^{-1} \left[\int_0^t \left(\int_0^s \int_{S_0} \dot{t}_i(\tau) \dot{t}_i(\tau) da d\tau \right)^{1/2} ds \right]^2 \\ & + \frac{4C}{\theta_0^2} \left[\alpha \left(\int_0^t \int_{S_0} p^2(s) da ds \right)^{1/2} + \int_0^t \left(\int_0^s \int_{S_0} [p(\tau) - \alpha \dot{p}(\tau)]^2 da d\tau \right)^{1/2} ds \right]^2, \end{aligned} \quad (3.42)$$

where b and C are positive constants depending on R , S_0 , θ_0 and the thermoelastic moduli.

PROOF. On the basis of the relations (2.2), (3.1), (3.2), (3.40) and (3.41), we deduce that

$$\int_R u_i dv = 0, \quad \int_R \epsilon_{ijk} x_j u_k dv = 0. \quad (3.43)$$

Thus, we have [14]

$$\int_{S_0} u_i u_i da \leq b^{-1} \int_R k_{ijmn} e_{ij} e_{mn} dv, \quad (3.44)$$

where b depends only on R , S_0 and the elasticity coefficients k_{ijmn} . Moreover, as in [15], we can deduce

$$\int_{\partial R} \theta^2 dS \leq C \int_R \left[\left(d - \frac{h}{\alpha}\right) \theta^2 + \alpha k_{ij} \theta_{,i} \theta_{,j} \right] dv, \quad (3.45)$$

where C is a positive constant depending only on R and the thermoelastic coefficients.

In view of the inequalities (3.44) and (3.45) and by using the relation (2.27) and (3.33), it follows that

$$\int_0^t \int_{S_0} u_i(s) u_i(s) da ds \leq 2b^{-1} \varepsilon_\pi(0, t), \quad (3.46)$$

$$\int_0^t \int_{S_0} \theta^2(s) da ds \leq 2C \varepsilon_\pi(0, t). \quad (3.47)$$

We return to the identity (3.8) and note that the end conditions (3.40) and the relations (2.27), (3.1), (3.7) and (3.33) and an integration by parts yield

$$\begin{aligned} \varepsilon_\pi(0, t) = & \int_0^t \int_{S_0} \left\{ -u_i(s)t_i(s) + \frac{\alpha}{\theta_0} \theta(s)p(s) \right. \\ & \left. + \int_0^s \left[u_i(\tau)\dot{t}_i(\tau) + \frac{1}{\theta_0} \theta(\tau)(p(\tau) - \alpha\dot{p}(\tau)) \right] d\tau \right\} da \, ds. \end{aligned} \quad (3.48)$$

Further, by means of the Schwarz's inequality, from (3.48) it results

$$\begin{aligned} \varepsilon_\pi(0, t) \leq & \left(\int_0^t \int_{S_0} u_i(s)u_i(s) \, da \, ds \right)^{1/2} \left[\left(\int_0^t \int_{S_0} t_i(s)t_i(s) \, da \, ds \right)^{1/2} + \int_0^t \left(\int_0^s \int_{S_0} \dot{t}_i(\tau)\dot{t}_i(\tau) \, da \, d\tau \right)^{1/2} ds \right] \\ & + \frac{1}{\theta_0} \left(\int_0^t \int_{S_0} \theta^2(s) \, da \, ds \right)^{1/2} \left[\alpha \left(\int_0^t \int_{S_0} p^2(s) \, da \, ds \right)^{1/2} \right. \\ & \left. + \int_0^t \left(\int_0^s \int_{S_0} (p(\tau) - \alpha\dot{p}(\tau))^2 \, da \, d\tau \right)^{1/2} ds \right]. \end{aligned} \quad (3.49)$$

On the other hand, since $t_i(0) = 0$, we have

$$\begin{aligned} \frac{d}{dt} \left(\int_0^t \int_{S_0} t_i(s)t_i(s) \, da \, ds \right) &= 2 \int_0^t \int_{S_0} \dot{t}_i(s)t_i(s) \, da \, ds \\ &\leq 2 \left(\int_0^t \int_{S_0} t_i(s)t_i(s) \, da \, ds \right)^{1/2} \left(\int_0^t \int_{S_0} \dot{t}_i(s)\dot{t}_i(s) \, da \, ds \right)^{1/2}, \end{aligned} \quad (3.50)$$

which after an integration gives

$$\left(\int_0^t \int_{S_0} t_i(s)t_i(s) \, da \, ds \right)^{1/2} \leq \int_0^t \left(\int_0^s \int_{S_0} \dot{t}_i(\tau)\dot{t}_i(\tau) \, da \, d\tau \right)^{1/2} ds. \quad (3.51)$$

In view of the estimates (3.46), (3.47) and (3.51), from (3.49) we deduce the estimate (3.42) and the proof is complete.

Thus, the estimate (3.42) when it is used in the spatial decay estimate (3.32) leads to an explicit result describing the spatial decay of end effects in linear generalized thermoelasticity.

4. SPATIAL DECAY ESTIMATES FOR SOLUTIONS DESCRIBING HARMONIC VIBRATIONS IN A THERMOELASTIC CYLINDER

For the remainder of this paper we consider an isotropic and homogeneous thermoelastic body occupying the right cylinder $R \equiv S_0 \times [0, L]$. Thus, the constitutive equations (2.5)–(2.7) are now assumed in the form

$$t_{rs} = \lambda e_{mm} \delta_{rs} + 2\mu e_{rs} - \beta(\theta + \alpha\dot{\theta})\delta_{rs}, \quad (4.1)$$

$$q_r = -\theta_0 k \theta_{,r}, \quad (4.2)$$

$$\rho\eta = a + d\theta + h\dot{\theta} + \beta e_{rr}, \quad (4.3)$$

where λ , μ , $\alpha_i = \beta/(3\lambda + 2\mu)$ and k are the Lamé moduli, coefficient of thermal expansion and the conductivity, respectively.

If we substitute the relations (4.1)–(4.3) into the relations (2.2)–(2.4) and (2.8), we deduce

$$\mu u_{r,ss} + (\lambda + \mu) u_{s,ss} - \beta(\theta + \alpha\dot{\theta})_{,r} = \rho \ddot{u}_r, \quad (4.4)$$

$$k \theta_{,rr} - d\dot{\theta} - h\ddot{\theta} - \beta \dot{u}_{r,r} = 0. \quad (4.5)$$

In this section we study the harmonic vibrations of the thermoelastic medium, that is we consider solutions $\{u_i, \theta\}$ of (4.4), (4.5) in the form

$$u_r(\mathbf{x}, t) = v_r(\mathbf{x})\exp(i\omega t), \quad \theta(\mathbf{x}, t) = T(\mathbf{x})\exp(i\omega t), \quad (4.6)$$

where ω is a positive constant. Then, the amplitude $p = \{v_r, T\}$ of the steady state vibration satisfies the equations

$$\mu v_{r,ss} + (\lambda + \mu)v_{s,sr} - \beta(1 + \alpha i\omega)T_r + \rho\omega^2 v_r = 0, \quad (4.7)$$

$$kT_{,rr} - (di\omega - h\omega^2)T - \beta i\omega v_{r,r} = 0. \quad (4.8)$$

Throughout this section we assume that the thermoelastic cylinder R is subject to non-zero harmonic boundary conditions only on the end S_0 . Therefore, the amplitude of the steady state vibration satisfies the equations (4.7) and (4.8) and the lateral boundary conditions

$$v_r = 0, \quad T = 0 \quad \text{on} \quad \partial R \setminus S_0. \quad (4.9)$$

Therefore, in what follows we shall discuss decay estimates describing the behaviour of the solution $\{v_r, T\}$ of the boundary value problem defined by the equations (4.7), (4.8), the boundary conditions (4.9) and appropriate boundary conditions on the end S_0 . In this aim we introduce the following cross-sectional measure $I_p(z)$ defined by

$$\begin{aligned} I_p(z) = \int_{S_z} \{ & \mu(v_r \bar{v}_{r,3} + \bar{v}_r v_{r,3}) + (\lambda + \mu)(v_3 \bar{v}_{r,r} + \bar{v}_3 v_{r,r}) \\ & - \beta(1 - \alpha i\omega)v_3 \bar{T} - \beta(1 + \alpha i\omega)\bar{v}_3 T + \epsilon k(T \bar{T}_{,3} + \bar{T} T_{,3}) \} da, \end{aligned} \quad (4.10)$$

where the superposed bar denotes complex conjugate, and ϵ is a positive constant to be chosen later.

By a differentiation of the relation (4.10), we get

$$\begin{aligned} I'_p(z) = \int_{S_z} \{ & 2\mu v_{r,3} \bar{v}_{r,3} + 2\epsilon k T_{,3} \bar{T}_{,3} + (\lambda + \mu)(v_{r,r} \bar{v}_{3,3} + \bar{v}_{r,r} v_{3,3}) \\ & - \beta[(1 + \alpha i\omega)(\bar{v}_3 T_{,3} + v_{3,3} \bar{T}) + (1 - \alpha i\omega)(v_{3,3} \bar{T} + v_3 \bar{T}_{,3})] \} da \\ & + \int_{S_z} \{ \mu(v_r \bar{v}_{r,33} + v_{r,33} \bar{v}_r) + (\lambda + \mu)(v_{r,r3} \bar{v}_3 + v_3 \bar{v}_{r,r3}) + \epsilon k(T \bar{T}_{,33} + \bar{T} T_{,33}) \} da. \end{aligned} \quad (4.11)$$

Further, by using the equations (4.7) and (4.8) and the boundary conditions (4.9) in the second integral of (4.11), we deduce

$$\begin{aligned} I'_p(z) = \int_{S_z} \{ & 2\mu v_{r,s} \bar{v}_{r,s} + 2(\lambda + \mu)v_{r,r} \bar{v}_{s,s} + 2\epsilon k T_{,r} \bar{T}_{,r} - 2\rho\omega^2 v_r \bar{v}_r \\ & - 2\epsilon h\omega^2 T \bar{T} - \beta[1 + (\epsilon + \alpha)i\omega] \bar{v}_{s,s} T - \beta[1 - (\epsilon + \alpha)i\omega] v_{s,s} \bar{T} \} da. \end{aligned} \quad (4.12)$$

Moreover, we can write

$$\begin{aligned} I'_p(z) = \int_{S_z} \{ & 2\mu v_{r,s} \bar{v}_{r,s} + (2 - \gamma)(\lambda + \mu)v_{r,r} \bar{v}_{s,s} - 2\rho\omega^2 v_r \bar{v}_r \\ & + \gamma(\lambda + \mu) \left[v_{r,r} - \frac{\beta}{\gamma(\lambda + \mu)} (1 + (\epsilon + \alpha)i\omega) T \right] \left[\bar{v}_{s,s} - \frac{\beta}{\gamma(\lambda + \mu)} (1 - (\epsilon + \alpha)i\omega) \bar{T} \right] \\ & + 2\epsilon k T_{,r} \bar{T}_{,r} - \left[2\epsilon h\omega^2 + \frac{\beta^2}{\gamma(\lambda + \mu)} (1 + (\epsilon + \alpha)^2 \omega^2) \right] T \bar{T} \} da, \end{aligned} \quad (4.13)$$

where γ is a positive constant to be chosen later.

Now, on the basis of the boundary conditions (4.9), the following inequalities hold

$$\int_{S_z} v_{r,\rho} \bar{v}_{r,\rho} da \geq \lambda_1 \int_{S_z} v_r \bar{v}_r da, \quad (4.14)$$

$$\int_{S_z} T_{,\rho} \bar{T}_{,\rho} da \geq \lambda_1 \int_{S_z} T \bar{T} da, \quad (4.15)$$

where λ_1 is the lowest eigenvalue in the two-dimensional clamped membrane problem for the cross-section S_z .

Thus, by means of inequalities (4.14) and (4.15), from (4.12), we get

$$\begin{aligned} I'_p(z) &\geq 2(\mu - \lambda_1^{-1} \rho w^2) \int_{S_z} v_{r,s} \bar{v}_{r,s} da + (2 - \gamma)(\lambda + \mu) \int_{S_z} v_{r,r} \bar{v}_{s,s} da \\ &\quad + \left\{ 2\epsilon k - \lambda_1^{-1} \left[2\epsilon h w^2 + \frac{\beta^2}{\gamma(\lambda + \mu)} (1 + (\epsilon + \alpha)^2 w^2) \right] \right\} \int_{S_z} T_{,r} \bar{T}_{,r} da. \end{aligned} \quad (4.16)$$

At this time, we choose the arbitrary positive constants ϵ and γ so that the following inequalities hold

$$0 < \gamma < 2, \quad (4.17)$$

$$\epsilon > \frac{\beta^2}{2\lambda_1 k \gamma (\lambda + \mu)}. \quad (4.18)$$

Further, we introduce

$$w_m^2(\epsilon, \gamma) = \min \left\{ \frac{\lambda_1 \mu}{\rho}, \frac{2\epsilon \gamma \lambda_1 k (\lambda + \mu) - \beta^2}{2\epsilon h \gamma (\lambda + \mu) + \beta^2 (\epsilon + \alpha)^2} \right\}, \quad (4.19)$$

and choose w to satisfy the inequality

$$w^2 < w_m^2(\epsilon, \gamma). \quad (4.20)$$

Then, we have

$$I'_p(z) \geq \int_{S_z} (m_1 v_{r,s} \bar{v}_{r,s} + m_2 v_{r,r} \bar{v}_{s,s} + m_3 T_{,r} \bar{T}_{,r}) da, \quad (4.21)$$

where

$$m_1 = 2\mu \left(1 - \frac{w^2}{w_m^2} \right), \quad m_2 = (\lambda + \mu)(2 - \gamma), \quad m_3 = \left[2\epsilon k - \frac{\beta^2}{\lambda_1 \gamma (\lambda + \mu)} \right] \left(1 - \frac{w^2}{w_m^2} \right). \quad (4.22)$$

On the other hand, by using the arithmetic-geometric and the Schwarz inequalities, from (4.10), (4.14) and (4.15), one obtains

$$|I_p(z)| \leq \int_{S_z} (M_1 v_{r,s} \bar{v}_{r,s} + M_2 v_{r,r} \bar{v}_{s,s} + M_3 T_{,r} \bar{T}_{,r}) da, \quad (4.23)$$

where

$$M_1 = \frac{\lambda + 3\mu}{\sqrt{\lambda_1}}, \quad M_2 = \frac{\lambda + \mu}{\sqrt{\lambda_1}}, \quad M_3 = \frac{\beta^2(1 + \alpha^2 w^2)}{\mu \sqrt{\lambda_1^3}} + \frac{\epsilon k}{\sqrt{\lambda_1}}. \quad (4.24)$$

Clearly, the relations (4.21) and (4.23) lead to the following first order differential inequality

$$|I_p(z)| \leq v^{-1} I'_p(z), \quad z \in [0, L], \quad (4.25)$$

where

$$v^{-1} = \max \left(\frac{M_1}{m_1}, \frac{M_2}{m_2}, \frac{M_3}{m_3} \right). \quad (4.26)$$

In view of the relations (4.9) and (4.10) it results

$$I_p(L) = 0. \quad (4.27)$$

From the relation (4.21) it follows $I'_p(z) \geq 0$ so that $I_p(z)$ is a non-decreasing function on $[0, L]$. Thus, the relation (4.27) implies that $I_p(z) \leq 0$ for all $z \in [0, L]$. Therefore, the differential inequality (4.25) takes the form

$$vI_p(z) + I'_p(z) \geq 0, \quad z \in [0, L]. \quad (4.28)$$

An integration of the differential inequality (4.28) leads to the following result

THEOREM 4.1. *Let $I_p(z)$ be the measure defined by (4.10) in connection with the steady state amplitude $p = \{v_r, T\}$ determined by (4.7)–(4.9) and the appropriate end boundary conditions on S_0 . Then*

$$0 \leq -I_p(z) \leq -I_p(0)\exp(-vz), \quad z \in [0, L], \quad (4.29)$$

provided

$$w < w_m(\epsilon, \gamma), \quad (4.30)$$

where $w_m(\epsilon, \gamma)$ is defined by (4.19).

We remark that Theorem 4.1 gives spatial decay estimates away from a harmonically excited end provided that the excitation frequency is below a certain critical level.

5. COMMENTS

It is worthy to compare the above results with those described in [10] for classical thermoelasticity. Then it is seen that for the generalized approach of thermoelasticity we have obtained a more exactly description of the spatial behaviour of the transient solutions. While one would like to obtain a similar result for the classical thermoelasticity, it is not clear how this can be done: the crux of the matter is that one requires an analogue of (3.14) and (3.15) and the availability of such an analogue is not immediately apparent. This there seems to be due to the parabolic equations of the classical approach of thermoelasticity.

As regards the steady state vibrations, it can be seen that differences between the two approaches of thermoelasticity appear into the decay rate of end effects and the critical level of frequencies of vibrations. A further comparison in detail requires a considerable effort. We do not pursue this problem in the present paper.

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