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SOME FURTHER GROWTH AND DECAY RESULTS IN LINEAR THERMOELASTODYNAMICS

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Two time-weighted measures are introduced for studying the spatial behavior of the thermoelastic processes in an isotropic thermoelastic body in order to relax the hypotheses of positive definiteness on the thermoelastic coefficients. For the first measure we establish some spatial estimates describing the spatial behavior of solutions for the class of thermoelastic materials for which the Lamé moduli λ and μ range so that $\mu > 0$ and $\lambda > -\frac{4}{3}\mu$. The second measure allows us to establish spatial estimates describing the spatial behavior of the thermoelastic processes in an isotropic and homogeneous thermoelastic material for which the Lamé moduli range so that $\mu > 0$ and $-2\mu < \lambda < 0$.

Keywords isotropic thermoelastic body, nondefinite elasticity tensor, spatial behavior

Several studies on growth and decay spatial behavior for the solutions in the classical linear thermoelastodynamics have appeared in the literature (see, e.g., Boley [1], Chiriță [2–4], Chiriță and Ciarletta [5, 6], Chiriță and Dănescu [7], Ignaczak [8, 9], Quintanilla [10, 11]). Extensions to other generalized theories of thermoelasticity are also developed (see, e.g., Chiriță and Scalia [12], Pompei and Scalia [13], Quintanilla [14], Ciarletta and Scalia [15]). For a thorough review of the field on Saint–Venant’s principle and the spatial growth and decay behavior, up to 1996, the reader is referred to the survey articles of Horgan and Knowles [16] and Horgan [17, 18].

The lack of any domain of influence results in the classical linear thermoelasticity has been compensated by a number of spatial decay estimates for the

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thermoelastic processes. The two types of such estimates have been established [5] by using the positive definiteness conditions on the material constants: the first one is characterized by a time-dependent decay rate and is suitable for appropriate short values of time, while the second has a time-independent decay rate and is appropriate for large values of time. On the basis of appropriate time-weighted surface power functions, the spatial decay estimates of Saint-Venant-type for bounded bodies are established, while for unbounded bodies some alternatives of Phragmén–Lindelöf-type are obtained.

Quintanilla [10, 11] used comparison arguments to study the spatial behavior for a thermally insulated semiinfinite cylinder. For such a special problem the exponential decay of a polynomial of second order of a distance was established.

In this article we study the spatial behavior of solutions in classical linear thermoelastodynamics by relaxing the hypotheses of positive definiteness on the thermoelastic coefficients. Thus, we use some appropriate measures to study the spatial behavior of an isotropic and homogeneous thermoelastic body in which Lamé moduli λ and μ range over the intervals $\mu > 0$ and $\lambda > -\frac{4}{3}\mu$ or $\mu > 0$ and $\lambda > -2\mu$.

FORMULATION OF THE PROBLEM

Throughout this article we denote by B a bounded or unbounded regular region whose boundary surface is ∂B . We assume B to be filled by an anisotropic and homogeneous thermoelastic material. The fundamental system of field equations consists [19] of the strain–displacement relation

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \bar{B} \times [0, \infty) \quad (1)$$

the thermal gradient–temperature relation

$$g_i = \theta_{,i} \quad \text{in } \bar{B} \times [0, \infty) \quad (2)$$

the stress–strain–temperature relation

$$S_{ij} = C_{ijkl}e_{kl} + M_{ij}\theta \quad \text{in } \bar{B} \times [0, \infty) \quad (3)$$

the heat conduction equation

$$q_i = -K_{ij}g_j \quad \text{in } \bar{B} \times [0, \infty) \quad (4)$$

the equations of motion

$$S_{ji,j} + b_i = \rho \ddot{u}_i \quad \text{in } B \times [0, \infty) \quad (5)$$

and the energy equation

$$-q_{i,i} + \theta_0 M_{ij} \dot{e}_{ij} + h = c \dot{\theta} \quad \text{in } B \times [0, \infty) \quad (6)$$

where u_i are the components of the displacement vector, θ is the temperature variation from the uniform reference temperature θ_0 , e_{ij} are the components of the strain tensor, g_i are the components of the thermal gradient vector, q_i are the components of the heat-flux vector, b_i are the components of the body force vector, and h is the heat supply. Furthermore, ϱ is the constant density, C_{ijkl} are the components of the constant elasticity tensor, M_{ij} are the components of the constant stress-temperature tensor, c is the constant specific heat, and K_{ij} are the components of the constant conductivity tensor.

We denote by $\mathcal{P}$ the initial-boundary value problem defined by relations (1)–(6), the initial conditions

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}) \quad \dot{u}_i(\mathbf{x}, 0) = \dot{u}_i^0(\mathbf{x}) \quad \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}) \quad \mathbf{x} \in \bar{B} \quad (7)$$

and the following boundary conditions

$$\begin{aligned} u_i(\mathbf{x}, t) &= \tilde{u}_i(\mathbf{x}, t) && \text{on } \sum_1 \times [0, \infty) \\ S_{ji}(\mathbf{x}, t)n_j(\mathbf{x}) &= \tilde{s}_i(\mathbf{x}, t) && \text{on } \sum_2 \times [0, \infty) \\ \theta(\mathbf{x}, t) &= \tilde{\theta}(\mathbf{x}, t) && \text{on } \sum_3 \times [0, \infty) \\ q_i(\mathbf{x}, t)n_i(\mathbf{x}) &= \tilde{q}(\mathbf{x}, t) && \text{on } \sum_4 \times [0, \infty) \end{aligned} \quad (8)$$

where u_i^0 , $\dot{u}_i^0$, θ^0 , $\tilde{u}_i$, $\tilde{s}_i$, $\tilde{\theta}$, and $\tilde{q}$ are prescribed functions; $\sum_i$ ($i = 1, 2, 3, 4$) are parts of the boundary surface ∂B such that $\sum_1 \cup \overline{\sum_2} = \sum_3 \cup \overline{\sum_4} = \partial B$, $\sum_1 \cap \sum_2 = \sum_3 \cap \sum_4 = \emptyset$; and n_i are the components of the outward unit normal to the boundary surface ∂B .

Let us consider a given time $T > 0$. We then define the support of the given data $\hat{D}_T$ to be the set of all $\mathbf{x} \in \bar{B}$ such that (i) if $\mathbf{x} \in B$, then

$$u_i^0(\mathbf{x}) \neq 0 \quad \text{or} \quad \dot{u}_i^0(\mathbf{x}) \neq 0 \quad \text{or} \quad \theta^0(\mathbf{x}) \neq 0 \quad (9)$$

or

$$b_i(\mathbf{x}, \tau) \neq 0 \quad \text{or} \quad h(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T] \quad (10)$$

or (ii) if $\mathbf{x} \in \sum_1$ then

$$\tilde{u}_i(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T] \quad (11)$$

or if $\mathbf{x} \in \sum_2$ then

$$\tilde{s}_i(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T] \quad (12)$$

or if $\mathbf{x} \in \sum_3$ then

$$\tilde{\theta}(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T] \quad (13)$$

or if $\mathbf{x} \in \sum_4$ then

$$\tilde{q}(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T] \quad (14)$$

For convenience, in what follows we shall assume that $\hat{D}_T$ is a bounded set. Furthermore, on this basis, we define the set $\hat{D}_T^*$ so that $\hat{D}_T \subset \hat{D}_T^* \subset \bar{B}$, $\hat{D}_T^* \neq \emptyset$ and

- i. if $\hat{D}_T \cap B \neq \emptyset$ then we choose $\hat{D}_T^*$ to be the smallest bounded regular region in $\bar{B}$ that includes $\hat{D}_T$; in particular, we set $\hat{D}_T^* = \hat{D}_T$ if $\hat{D}_T$ also happens to be a regular region;
- ii. if $\emptyset \neq \hat{D}_T \subset \partial B$, then we choose $\hat{D}_T^*$ to be the smallest regular subsurface of ∂B that includes $\hat{D}_T$; in particular, we set $\hat{D}_T^* = \hat{D}_T$ if $\hat{D}_T$ is a regular subsurface of ∂B ;
- iii. if $\hat{D}_T = \emptyset$, then we choose $\hat{D}_T^*$ to be an arbitrary nonempty regular subsurface of ∂B .

Furthermore, we introduce the set D_r , $r \geq 0$, by

$$D_r = \{\mathbf{x} \in \bar{B} : \hat{D}_T^* \cap \overline{\sum(\mathbf{x}, r)} \neq \emptyset\} \quad (15)$$

where $\sum(\mathbf{x}, r)$ is an open ball with radius r and center at $\mathbf{x}$. Moreover, we shall use the notation B_r for the part of B contained in $B \setminus D_r$ and we set $B(r_1, r_2) = B_{r_2} \setminus B_{r_1}$, $r_1 > r_2$. We further denote by S_r the subsurface of ∂B_r contained inside of B and whose outward unit normal vector is forwarded to the exterior of D_r .

With the solution (u_i, θ) of the problem $\mathcal{P}$ we associate the following time-weighted surface power functions

$$Q(r, t) = - \int_0^t \int_{S_r} e^{-\sigma s} \left[S_{ji}(s) n_j \dot{u}_i(s) - \frac{1}{\theta_0} q_i(s) n_i \theta(s) \right] da ds$$

$$r \geq 0 \quad t \in [0, T] \quad (16)$$

$$\tilde{Q}(r, t) = \int_0^t Q(r, s) ds \quad (17)$$

where σ is a prescribed positive parameter.

By supposing that the elasticity tensor C_{ijkl} is positive definite, the conductivity tensor K_{ij} is positive definite, $c > 0$ and $q > 0$ in [5] we have established the following first-order differential inequalities:

$$\frac{\sigma}{\kappa} |Q(r, t)| + \frac{\partial}{\partial r} Q(r, t) \leq 0 \quad r \geq 0 \quad t \in [0, T] \quad (18)$$

$$t\chi(t) \frac{\partial}{\partial r} \tilde{Q}(r, t) + |\tilde{Q}(r, t)| \leq 0 \quad r \geq 0 \quad t \in [0, T] \quad (19)$$

where

$$\kappa = \sqrt{\frac{(1 + \varepsilon_0)\mu_M}{\varrho}} \quad (20)$$

ε_0 is the positive root of the algebraic equation

$$\varepsilon^2 + \varepsilon \left(1 - \frac{m^2 \theta_0}{c\mu_M} - \frac{\sigma \varrho k_M}{2c\mu_M} \right) - \frac{m^2 \theta_0}{c\mu_M} = 0 \quad (21)$$

μ_M is the maximum elastic modulus for C_{ijkl} , $m = (M_{ij}M_{ij})^{\frac{1}{2}}$, k_M is the largest eigenvalue of K_{ij} and

$$\chi(t) = \sqrt{\frac{[1 + \delta_0(t)]\mu_M}{\varrho}} \quad (22)$$

where $\delta_0(t)$ is the positive root of the algebraic equation

$$\delta^2 + \delta \left(1 - \frac{m^2 \theta_0}{c\mu_M} - \frac{\varrho k_M}{2tc\mu_M} \right) - \frac{m^2 \theta_0}{c\mu_M} = 0 \quad (23)$$

Furthermore, from relations (18) and (19), it follows that $Q(r, t) \geq 0$, $\tilde{Q}(r, t) \geq 0$, $r \geq 0$, $t \in [0, T]$ for a bounded body (that means r ranges on $[0, L]$) and, moreover, we have

$$Q(r, t) \leq Q(0, t) \exp\left(-\frac{\sigma}{\kappa} r\right) \quad r \in [0, L] \quad t \in [0, T] \quad (24)$$

$$\tilde{Q}(r, t) \leq \tilde{Q}(0, t) \exp\left(-\frac{1}{t\chi(t)} r\right) \quad r \in [0, L] \quad t \in [0, T] \quad (25)$$

while for an unbounded body the following result of Phragmén–Lindelöf-type is established. For each fixed $t \in [0, T]$ we have either (i) $Q(r, t) \geq 0$ for all $r \geq 0$ and then

$$Q(r, t) \leq Q(0, t) \exp\left(-\frac{\sigma}{\kappa} r\right) \quad r \geq 0 \quad (26)$$

or (ii) there exists a value $r_t \geq 0$ so that $Q(r_t, t) < 0$ and then $Q(r, t) < 0$ for all $r \geq r_t$ and

$$-Q(r, t) \geq -Q(r_t, t) \exp\left[\frac{\sigma}{\kappa} (r - r_t)\right] \quad r \geq r_t \quad (27)$$

A statement entirely similar to that of Eqs. (26) and (27) can be written in terms of $\tilde{Q}(r, t)$.

From the preceding general results we can deduce the spatial behavior of thermoelastic processes for a homogeneous and isotropic thermoelastic material. In fact, for such a material we have

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad M_{ij} = M \delta_{ij} \quad K_{ij} = K \delta_{ij} \quad (28)$$

where λ and μ are the Lamé constant moduli, M is the stress–temperature modulus, and K is the positive constant conductivity. As it is known [19, 20] C_{ijkl} is positive definite if and only if

$$\mu > 0 \quad \lambda > -\frac{2}{3}\mu \quad (29)$$

and the maximum and minimum elastic moduli are $\mu_M, \mu_m \in \{2\mu, 3\lambda + 2\mu\}$. It follows then for $\lambda \geq 0$ that we have $\mu_M = 3\lambda + 2\mu$, while for $-\frac{2}{3}\mu < \lambda \leq 0$ we have $\mu_M = 2\mu$. Furthermore, K_{ij} is positive definite when $K > 0$ and, therefore, $k_M = K$ and $m = \sqrt{3}|M|$. Thus, Eq. (21) becomes

$$\begin{aligned} \varepsilon^2 + \varepsilon \left[1 - \frac{3M^2\theta_0}{c \max(2\mu, 3\lambda + 2\mu)} - \frac{\sigma_0 K}{2c \max(2\mu, 3\lambda + 2\mu)} \right] \\ - \frac{3M^2\theta_0}{c \max(2\mu, 3\lambda + 2\mu)} = 0 \end{aligned} \quad (30)$$

and Eq. (20) becomes

$$\kappa = \sqrt{\frac{(1 + \varepsilon_0) \max(2\mu, 3\lambda + 2\mu)}{\varrho}} \quad (31)$$

Furthermore, relation (23) furnishes

$$\begin{aligned} \delta^2 + \delta \left[1 - \frac{3M^2\theta_0}{c \max(2\mu, 3\lambda + 2\mu)} - \frac{\varrho K}{2tc \max(2\mu, 3\lambda + 2\mu)} \right] \\ - \frac{3M^2\theta_0}{c \max(2\mu, 3\lambda + 2\mu)} = 0 \end{aligned} \quad (32)$$

and relation (22) gives

$$\chi(t) = \sqrt{\frac{[1 + \delta_0(t)] \max(2\mu, 3\lambda + 2\mu)}{\varrho}} \quad (33)$$

The main aim of this article is to prove that the positive definiteness of the thermoelastic coefficients could be relaxed and, therefore, conclusions (24)–(27) can be established for an extended class of thermoelastic materials, provided some

appropriate measures are used. In this connection we consider an isotropic and homogeneous thermoelastic solid so that the basic equations reduce to

$$\begin{aligned} \mu u_{i,jj} + (\lambda + \mu) u_{r,ri} + M \theta_{,i} + b_i &= \varrho \ddot{u}_i \\ K \theta_{,jj} + M \theta_0 \dot{u}_{r,r} + h &= c \dot{\theta} \quad \text{in } B \times (0, \infty) \end{aligned} \quad (34)$$

FIRST EXTENSION AND RELATED ESTIMATES

In this section we establish some spatial estimates describing the spatial behavior of solutions for the class of thermoelastic materials for which $\mu > 0$ and $\lambda > -\frac{4}{3}\mu$. To this end, we write the system of equations (34) in the following form:

$$\begin{aligned} T_{ji,j} + b_i &= \varrho \ddot{u}_i \\ K \theta_{,jj} + M \theta_0 \dot{u}_{r,r} + h &= c \dot{\theta} \quad \text{in } B \times (0, \infty) \end{aligned} \quad (35)$$

where

$$T_{ji} = \mu u_{i,j} + (\lambda + \mu) u_{r,r} \delta_{ij} + M \theta \delta_{ij} \quad (36)$$

We introduce the initial-boundary value problem $\mathcal{P}_1$ defined by the relations (35) and (36), the initial conditions (7), and the following boundary conditions:

$$\begin{aligned} u_i(\mathbf{x}, t) &= \tilde{u}_i(\mathbf{x}, t) & \text{on } \sum_1 \times [0, \infty) \\ T_{ji}(\mathbf{x}, t) n_j(\mathbf{x}) &= \tilde{t}_i(\mathbf{x}, t) & \text{on } \sum_2 \times [0, \infty) \\ \theta(\mathbf{x}, t) &= \tilde{\theta}(\mathbf{x}, t) & \text{on } \sum_3 \times [0, \infty) \\ q_i(\mathbf{x}, t) n_i(\mathbf{x}) &= \tilde{q}(\mathbf{x}, t) & \text{on } \sum_4 \times [0, \infty) \end{aligned} \quad (37)$$

where $\tilde{t}_i$ is a prescribed function. For such a problem we introduce the notations D_r , B_r , S_r , and $B(r_1, r_2)$ in the same manner as in the preceding section and then we define the following time-weighted function:

$$\begin{aligned} U(r, t) &= - \int_0^t \int_{S_r} e^{-\sigma s} \left[T_{ji}(s) n_j \dot{u}_i(s) + \frac{K}{\theta_0} \theta_{,i}(s) n_i \theta(s) \right] da ds, \\ r &\geq 0 \quad t \in [0, T] \end{aligned} \quad (38)$$

where σ is a prescribed positive parameter.

It follows then from the boundary conditions (37) and the definitions for S_r and $B(r_1, r_2)$ that for all $t \in [0, T]$ and $0 < r_2 < r_1$, we have

$$\begin{aligned} U(r_1, t) - U(r_2, t) &= - \int_0^t \int_{\partial B(r_1, r_2)} e^{-\sigma s} \left[T_{ji}(s) n_j \dot{u}_i(s) \right. \\ &\quad \left. + \frac{K}{\theta_0} \theta_{,i}(s) n_i \theta(s) \right] da ds \quad r \geq 0 \quad t \in [0, T] \end{aligned} \quad (39)$$

so that, by means of the divergence theorem and by using Eq. (35), we deduce that

$$\begin{aligned} U(r_1, t) - U(r_2, t) = & - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} \left[\varrho \dot{u}_i(s) \ddot{u}_i(s) + T_{ji}(s) \dot{u}_{i,j}(s) \right. \\ & - M \dot{u}_{r,r}(s) \theta(s) + \frac{c}{\theta_0} \theta(s) \dot{\theta}(s) \\ & \left. + \frac{K}{\theta_0} \theta_{,i}(s) \theta_{,i}(s) \right] dv ds \quad 0 < r_2 < r_1 \quad t \in [0, T] \end{aligned} \quad (40)$$

By substituting relation (36) into Eq. (40), using an integration by parts with respect to s , and taking into account the initial conditions (7) and the definition of $B(r_1, r_2)$, we get

$$\begin{aligned} U(r_1, t) - U(r_2, t) = & - \int_{B(r_1, r_2)} e^{-\sigma t} \frac{1}{2} \left[\varrho \dot{u}_i \dot{u}_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s} + \frac{c}{\theta_0} \theta^2 \right] dv \\ & - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} \left\{ \frac{\sigma}{2} [\varrho \dot{u}_i(s) \dot{u}_i(s) + \mu u_{i,j}(s) u_{i,j}(s) \right. \\ & \left. + (\lambda + \mu) u_{r,r}(s) u_{s,s}(s) + \frac{c}{\theta_0} \theta^2(s)] + \frac{K}{\theta_0} \theta_{,i}(s) \theta_{,i}(s) \right\} dv ds \\ & 0 < r_2 < r_1 \quad t \in [0, T] \end{aligned} \quad (41)$$

Thus, we obtain

$$\begin{aligned} \frac{\partial U}{\partial r}(r, t) = & - \int_{S_r} e^{-\sigma t} \frac{1}{2} \left[\varrho \dot{u}_i \dot{u}_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s} + \frac{c}{\theta_0} \theta^2 \right] da \\ & - \int_0^t \int_{S_r} e^{-\sigma s} \left\{ \frac{\sigma}{2} \left[\varrho \dot{u}_i(s) \dot{u}_i(s) + \mu u_{i,j}(s) u_{i,j}(s) \right. \right. \\ & \left. \left. + (\lambda + \mu) u_{r,r}(s) u_{s,s}(s) + \frac{c}{\theta_0} \theta^2(s) \right] + \frac{K}{\theta_0} \theta_{,i}(s) \theta_{,i}(s) \right\} da ds \\ & r \geq 0 \quad t \in [0, T] \end{aligned} \quad (42)$$

Now we proceed to obtain an estimate for $U(r, t)$. To this end we write

$$T_{ji} = t_{ji} + M\theta\delta_{ij} \quad t_{ji} = \mu u_{i,j} + (\lambda + \mu) u_{r,r} \delta_{ij} \quad (43)$$

and note that

$$T_{ij} T_{ij} \leq (1 + \varepsilon) t_{ij} t_{ij} + \left(1 + \frac{1}{\varepsilon}\right) 3M^2 \theta^2 \quad \forall \varepsilon > 0 \quad (44)$$

Furthermore, we consider the following quadratic form

$$\mathcal{F}(\xi) = \mu \xi_{ij} \xi_{ij} + (\lambda + \mu) \xi_{rr} \xi_{ss} \quad \forall \xi_{ij} \quad (45)$$

and note that it is positive definite if and only if

$$\mu > 0 \quad \lambda > -\frac{4}{3}\mu \quad (46)$$

and the eigenvalues of its matrix are μ and $3\lambda + 2\mu$. Therefore, we have

$$\mathcal{F}(\xi) \leq \mu_M^* \xi_{ij} \xi_{ij} \quad \forall \xi_{ij} \quad (47)$$

where

$$\mu_M^* = \max(\mu, 3\lambda + 4\mu) \quad (48)$$

In what follows, we shall assume that relation (46) is true. Then, in view of relation (43), we have

$$\begin{aligned} t_{ij} t_{ij} &= \mu u_{i,j} t_{ij} + (\lambda + \mu) u_{r,r} t_{ss} \\ &\leq [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s}]^{\frac{1}{2}} [\mu t_{ij} t_{ij} + (\lambda + \mu) t_{rr} t_{ss}]^{\frac{1}{2}} \\ &\leq [\mu_M^* t_{ij} t_{ij}]^{\frac{1}{2}} [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s}]^{\frac{1}{2}} \end{aligned} \quad (49)$$

so that we get

$$t_{ij} t_{ij} \leq \mu_M^* [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s}] \quad (50)$$

By substituting relation (50) into relation (44), we obtain

$$\begin{aligned} T_{ij} T_{ij} &\leq (1 + \varepsilon) \mu_M^* [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{r,r} u_{s,s}] \\ &\quad + 3M^2 \left(1 + \frac{1}{\varepsilon}\right) \theta^2 \quad \forall \varepsilon > 0 \end{aligned} \quad (51)$$

On the basis of relations (38) and (51), we have

$$\begin{aligned} |U(r, t)| &\leq \frac{1}{2} \int_0^t \int_{S_r} e^{-\sigma s} \left[\frac{\varepsilon_1}{\varrho} T_{ji}(s) T_{ji}(s) + \frac{1}{\varepsilon_1} \varrho \dot{u}_i(s) \dot{u}_i(s) \right. \\ &\quad \left. + \frac{\varepsilon_2}{c\theta_0} K^2 \theta_{,i}(s) \theta_{,i}(s) + \frac{1}{\varepsilon_2 \theta_0} c \theta^2(s) \right] da ds \\ &\leq \int_0^t \int_{S_r} e^{-\sigma s} \left\{ \frac{1}{\sigma \varepsilon_1} \left[\frac{\sigma}{2} \varrho \dot{u}_i(s) \dot{u}_i(s) \right] + \frac{\varepsilon_1 (1 + \varepsilon) \mu_M^*}{\sigma \varrho} \right. \\ &\quad \times \left[\frac{\sigma}{2} (\mu u_{i,j}(s) u_{i,j}(s) + (\lambda + \mu) u_{r,r}(s) u_{s,s}(s)) \right] \\ &\quad + \left[\frac{3M^2 \varepsilon_1 \theta_0}{\sigma \varrho c} \left(1 + \frac{1}{\varepsilon}\right) + \frac{1}{\sigma \varepsilon_2} \right] \left[\frac{\sigma}{2} \frac{c}{\theta_0} \theta^2(s) \right] \\ &\quad \left. + \frac{\varepsilon_2 K}{2c} \left[\frac{K}{\theta_0} \theta_{,i}(s) \theta_{,i}(s) \right] \right\} da ds \quad r \geq 0 \quad t \in [0, T] \quad \forall \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \end{aligned} \quad (52)$$

By equating the coefficients of the various energetic terms in the last integral in relation (52), we get

$$\frac{1}{\sigma \varepsilon_1} = \frac{\varepsilon_1(1 + \varepsilon)\mu_M^*}{\sigma \varrho} = \frac{3M^2 \varepsilon_1 \theta_0}{\sigma \varrho c} \left(1 + \frac{1}{\varepsilon}\right) + \frac{1}{\sigma \varepsilon_2} = \frac{\varepsilon_2 K}{2c} \quad (53)$$

that is,

$$\varepsilon_1 = \frac{1}{\kappa^*} \quad \varepsilon_2 = \frac{2c\kappa^*}{\sigma K} \quad \kappa^* = \sqrt{\frac{(1 + \varepsilon_0^*)\mu_M^*}{\varrho}} \quad (54)$$

where ε_0^* is the positive root of the algebraic equation

$$\varepsilon^2 + \varepsilon \left(1 - \frac{3M^2 \theta_0}{c\mu_M^*} - \frac{\sigma \varrho K}{2c\mu_M^*}\right) - \frac{3M^2 \theta_0}{c\mu_M^*} = 0 \quad (55)$$

With these choices, from relations (42) and (52) we deduce that

$$\frac{\sigma}{\kappa^*} |U(r, t)| + \frac{\partial U}{\partial r}(r, t) \leq 0 \quad r \geq 0 \quad t \in [0, T] \quad (56)$$

As in [5], we can exploit differential inequality (56) in order to prove for a bounded body that $U(r, t) \geq 0$, $r \geq 0$, $t \in [0, T]$, and, moreover,

$$U(r, t) \leq U(0, t) \exp\left(-\frac{\sigma}{\kappa^*} r\right) \quad r \in [0, L] \quad t \in [0, T] \quad (57)$$

Furthermore, for an unbounded body, we can conclude that either $U(r, t) \geq 0$ for all $r \geq 0$ and then

$$U(r, t) \leq U(0, t) \exp\left(-\frac{\sigma}{\kappa^*} r\right) \quad r \geq 0 \quad t \in [0, T] \quad (58)$$

or there exists a value $r_t^* \geq 0$ such that $U(r_t^*, t) < 0$, and then $U(r, t) < 0$ for all $r > r_t^*$ and

$$-U(r, t) \geq -U(r_t^*, t) \exp\left[\frac{\sigma}{\kappa^*} (r - r_t^*)\right] \quad r \geq r_t^* \quad (59)$$

A completely similar conclusion can be formulated in the terms of the measure

$$\tilde{U}(r, t) = \int_0^t U(r, s) ds \quad (60)$$

The reader will be able to establish further conclusions by means of the preceding procedure.

As a result, the spatial behavior obtained in the present section covers a larger class of thermoelastic materials than that corresponding to materials with positive definite elasticity tensor. However, these considerations have to be applied only for the case when the problems $\mathcal{P}$ and $\mathcal{P}_1$ coincide, that is, for the case when $\text{meas } \sum_2 = 0$ or in the case when $\text{meas } \sum_2 \neq 0$ and $(u_{j,i} - u_{l,l}\delta_{ij})n_j = 0$ on $\sum_2 \times [0, \infty)$; in what follows, this is the case. With regard to the decay rate, it can be shown that it becomes slower as soon as the common part of the two classes of thermoelastic materials is considered. However, the preceding estimates can be used in combination for obtaining a good description of the spatial behavior. That means we can use the description obtained for $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$, with that corresponding to the intervals $\mu > 0$ and $-\frac{4}{3}\mu < \lambda \leq -\frac{2}{3}\mu$.

THE SECOND EXTENSION AND RELATED ESTIMATES

In this section we introduce a measure that describes the spatial behavior for the class of materials for which $\mu > 0$ and $-2\mu < \lambda < 0$. To this end we write Eqs. (34) in the following form:

$$\begin{aligned} R_{ji,j} + b_i &= \rho \ddot{u}_i \\ K\theta_{,jj} + M\theta_0 \dot{u}_{r,r} + h &= c\dot{\theta} \quad \text{in } B \times (0, \infty) \end{aligned} \quad (61)$$

where

$$R_{ji} = \mu u_{i,j} + (\lambda + \mu)u_{j,i} + M\theta\delta_{ij} \quad (62)$$

We further consider the initial-boundary value problem $\mathcal{P}_2$ defined by relations (61) and (62), initial conditions (7), and the following boundary conditions:

$$\begin{aligned} u_i(x, t) &= \tilde{u}_i(x, t) & \text{on } \sum_1 \times [0, \infty) \\ R_{ji}(x, t)n_j(x) &= \tilde{r}_i(x, t) & \text{on } \sum_2 \times [0, \infty) \\ \theta(x, t) &= \tilde{\theta}(x, t) & \text{on } \sum_3 \times [0, \infty) \\ q_i(x, t)n_i(x) &= \tilde{q}(x, t) & \text{on } \sum_4 \times [0, \infty) \end{aligned} \quad (63)$$

where $\tilde{r}_i$ is a prescribed function. To obtain a spatial behavior for the solutions of the problem $\mathcal{P}_2$ we introduce, as before, the same sets S_r , $B(r_1, r_2)$ and define the following time-weighted function:

$$\begin{aligned} V(r, t) &= - \int_0^t \int_{S_r} e^{-\sigma s} [R_{ji}(s)n_j \dot{u}_i(s) + \frac{K}{\theta_0} \theta_{,i}(s)n_i \theta(s)] da ds \\ r &\geq 0 \quad t \in [0, T] \end{aligned} \quad (64)$$

where σ is a prescribed positive parameter. By using the Eq. (61), boundary conditions (63), and the definitions for S_r and $B(r_1, r_2)$, for all $t \in [0, T]$ and for all $0 < r_2 < r_1$, we deduce

$$\begin{aligned} V(r_1, t) - V(r_2, t) = & - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} \left[\varrho \dot{u}_i(s) \ddot{u}_i(s) + R_{ji}(s) \dot{u}_{i,j}(s) \right. \\ & \left. - M \dot{u}_{r,r}(s) \theta(s) + \frac{c}{\theta_0} \theta(s) \dot{\theta}(s) + \frac{K}{\theta_0} \theta_{,i}(s) \theta_{,i}(s) \right] dv ds \\ & 0 < r_2 < r_1 \quad t \in [0, T] \end{aligned} \quad (65)$$

We now substitute relation (62) into relation (65) and use initial conditions (7) to get

$$\begin{aligned} V(r_1, t) - V(r_2, t) = & - \int_{B(r_1, r_2)} e^{-\sigma t} \frac{1}{2} \left[\varrho \dot{u}_i \dot{u}_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{i,j} u_{j,i} \right. \\ & \left. + \frac{c}{\theta_0} \theta^2 \right] dv - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} \left\{ \frac{\sigma}{2} \left[\varrho \dot{u}_i(s) \dot{u}_i(s) \right. \right. \\ & \left. \left. + \mu u_{i,j}(s) u_{i,j}(s) + (\lambda + \mu) u_{i,j}(s) u_{j,i}(s) + \frac{c}{\theta_0} \theta^2(s) \right] \right. \\ & \left. + \frac{K}{\theta_0} \theta_{,i}(s) \theta_{,i}(s) \right\} dv ds \quad 0 < r_2 < r_1 \quad t \in [0, T] \end{aligned} \quad (66)$$

so that we obtain

$$\begin{aligned} \frac{\partial V}{\partial r}(r, t) = & - \int_{S_r} e^{-\sigma t} \frac{1}{2} \left[\varrho \dot{u}_i \dot{u}_i + \mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{i,j} u_{j,i} \right. \\ & \left. + \frac{c}{\theta_0} \theta^2 \right] da - \int_0^t \int_{S_r} e^{-\sigma s} \left\{ \frac{\sigma}{2} \left[\varrho \dot{u}_i(s) \dot{u}_i(s) \right. \right. \\ & \left. \left. + \mu u_{i,j}(s) u_{i,j}(s) + (\lambda + \mu) u_{i,j}(s) u_{j,i}(s) + \frac{c}{\theta_0} \theta^2(s) \right] \right. \\ & \left. + \frac{K}{\theta_0} \theta_{,i}(s) \theta_{,i}(s) \right\} da ds \quad r \geq 0 \quad t \in [0, T] \end{aligned} \quad (67)$$

Proceeding as before, we consider the following quadratic form:

$$\mathfrak{G}(\xi) = \mu \xi_{ij} \xi_{ij} + (\lambda + \mu) \xi_{ij} \xi_{ji} \quad \forall \xi_{ij} \quad (68)$$

and we note that it is positive definite if and only if

$$\mu > 0 \quad -2\mu < \lambda < 0 \quad (69)$$

and the eigenvalues of its matrix are $\lambda + 2\mu$ and $-\lambda$. Thus, we have

$$\mathfrak{G}(\xi) \leq \max(\lambda + 2\mu, -\lambda) \xi_{ij} \xi_{ij} \quad \forall \xi_{ij} \quad (70)$$

Then we write

$$R_{ji} = r_{ji} + M\theta\delta_{ij} \quad r_{ji} = \mu u_{i,j} + (\lambda + \mu)u_{j,i} \quad (71)$$

and note that

$$R_{ij}R_{ij} \leq (1 + \varepsilon)r_{ij}r_{ij} + \left(1 + \frac{1}{\varepsilon}\right)3M^2\theta^2 \quad \forall \varepsilon > 0 \quad (72)$$

In view of relations (68), (70), and (71), we get

$$r_{ij}r_{ij} \leq \mu_M^{**}[\mu u_{i,j}u_{i,j} + (\lambda + \mu)u_{i,j}u_{j,i}] \quad (73)$$

where

$$\mu_M^{**} = \max(\lambda + 2\mu, -\lambda) \quad (74)$$

Thus, we have

$$\begin{aligned} R_{ij}R_{ij} &\leq (1 + \varepsilon)\mu_M^{**}[\mu u_{i,j}u_{i,j} + (\lambda + \mu)u_{i,j}u_{j,i}] \\ &\quad + 3M^2\left(1 + \frac{1}{\varepsilon}\right)\theta^2 \quad \forall \varepsilon > 0 \end{aligned} \quad (75)$$

By following the same procedure as that for obtaining an estimate for $U(r, t)$, we deduce the following differential inequality:

$$\frac{\sigma}{\kappa^{**}}|V(r, t)| + \frac{\partial V}{\partial r}(r, t) \leq 0 \quad r \geq 0 \quad t \in [0, T] \quad (76)$$

where

$$\kappa^{**} = \sqrt{\frac{(1 + \varepsilon_0^{**})\mu_M^{**}}{\varrho}} \quad (77)$$

and ε_0^{**} is the positive root of the algebraic equation

$$\varepsilon^2 + \varepsilon\left(1 - \frac{3M^2\theta_0}{c\mu_M^{**}} - \frac{\sigma\varrho K}{2c\mu_M^{**}}\right) - \frac{3M^2\theta_0}{c\mu_M^{**}} = 0 \quad (78)$$

As a consequence of differential inequality (76), we can conclude that, for a bounded body, we have $V(r, t) \geq 0$ and

$$V(r, t) \leq V(0, t) \exp\left(-\frac{\sigma}{\kappa^{**}}r\right) \quad r \in [0, L] \quad t \in [0, T] \quad (79)$$

Moreover, for an unbounded body, the result is that for each fixed $t \in [0, T]$ either $V(r, t) > 0$ for all $r \geq 0$ and then

$$V(r, t) \leq V(0, t) \exp\left(-\frac{\sigma}{\kappa^{**}} r\right) \quad r \geq 0 \quad (80)$$

or there is a value $r_t^{**} > 0$ so that $V(r_t^{**}, t) < 0$ and then we have $V(r, t) < 0$ for all $r > r_t^{**}$ and

$$-V(r, t) \geq -V(r_t^{**}, t) \exp\left(\frac{\sigma}{\kappa^{**}} (r - r_t^{**})\right) \quad r \geq 0 \quad t \in [0, T] \quad (81)$$

We note that similar results can be formulated in the terms of the measure

$$\tilde{V}(r, t) = \int_0^t V(r, s) ds \quad (82)$$

With reference to the case when $\text{meas } \Sigma_2 = 0$ or the case when $\text{meas } \Sigma_2 \neq 0$ and then $(u_{j,i} - u_{l,i} \delta_{ij}) n_j = 0$ on $\Sigma_2 \times [0, \infty)$, the preceding analysis establishes spatial estimates describing the spatial behavior of the thermoelastic processes in an isotropic and homogeneous thermoelastic material for which the Lamé moduli range so that $\mu > 0$ and $-2\mu < \lambda < 0$. To obtain a good description of the spatial behavior in the extended class of thermoelastic materials defined by $\mu > 0$ and $-2\mu < \lambda < \infty$, we have to combine the use of the measure $Q(r, t)$ for the set $\mu > 0$ and $\lambda > -\frac{2}{3}\mu$ with the use of the measure $U(r, t)$ for the set $\mu > 0$ and $-\frac{4}{3}\mu < \lambda < -\frac{2}{3}\mu$ and the use of the measure $V(r, t)$ for the set $\mu > 0$ and $-2\mu < \lambda < -\frac{4}{3}\mu$.

REFERENCES

1. B. A. Boley, Some Observations on Saint-Venant's Principle, in *3rd U.S. Natl. Cong. Appl. Mech.*, pp. 259–264, ASME, New York, 1958.
2. S. Chiriță, Saint-Venant's Principle in Linear Thermoelasticity, *J. Thermal Stresses*, vol. 18, pp. 485–496, 1995.
3. S. Chiriță, A Phragmén-Lindelöf principle in Dynamic Linear Thermoelasticity, *J. Thermal Stresses*, vol. 20, 505–516, 1997.
4. S. Chiriță, Spatial Behaviour in Thermoelasticity, in J. J. Skrzypek and R. B. Hetnarski (eds.), *Proceedings of the Third International Congress on Thermal Stresses, Thermal Stresses'99*, pp. 103–106, Krakow, Poland, 1999.
5. S. Chiriță and M. Ciarletta, Time-Weighted Surface Power Function Method for the Study of Spatial Behaviour in Dynamics of Continua, *Eur. J. Mech. A—Solids*, vol. 18, pp. 915–933, 1999.
6. S. Chiriță and M. Ciarletta, Time-Weighted Surface Power Function Method in Linear Thermoelasticity, in J. J. Skrzypek and R. B. Hetnarski (eds.), *Proceedings of the Third International Congress on Thermal Stresses, Thermal Stresses'99*, pp. 587–590, Krakow, Poland, 1999.
7. S. Chiriță and A. Dănescu, Spatial Decay and Growth Properties for the Incremental Thermoelasticity, *J. Appl. Math. Phys. (ZAMP)*, vol. 51, pp. 39–60, 2000.
8. J. Ignaczak, A Spatial Decay Estimate for Transient Thermoelastic Process in a Composite Semispace, *J. Thermal Stresses*, vol. 23, pp. 1–14, 2000.
9. J. Ignaczak, Saint-Venant's Principle for a Microperiodic Composite Thermoelastic Semispace: The Dynamical Refined Average Theory, *J. Thermal Stresses*, vol. 25, pp. 1065–1079, 2002.

10. R. Quintanilla, End Effects in Thermoelasticity, *Math. Method. Appl. Sci.*, vol. 24, pp. 93–102, 2001.
11. R. Quintanilla, Spatial Asymptotic Behaviour in Incremental Thermoelasticity, *Asymptotic Anal.*, vol. 27, pp. 265–279, 2001.
12. S. Chiriță and A. Scalia, On the Spatial and Temporal Behavior in Linear Thermoelasticity of Materials with Voids, *J. Thermal Stresses*, vol. 24, pp. 433–455, 2001.
13. A. Pompei and A. Scalia, On the Asymptotic Spatial Behavior in Linear Thermoelasticity of Materials with Voids, *J. of Thermal Stresses*, vol. 25, pp. 183–193, 2002.
14. R. Quintanilla, On the Spatial Decay for the Dynamical Problem of Thermo-Microstretch Elastic Solids, *Int. J. Engng. Sci.*, vol. 40, pp. 109–121, 2002.
15. M. Ciarletta and A. Scalia, Some Results in Linear Theory of Thermomicrostretch Elastic Solids, *Meccanica*, in press, 2003.
16. C. O. Horgan and J. K. Knowles, Recent Developments Concerning Saint-Venant's Principle, in J. W. Hutchinson and T. Y. Wu (eds.), *Advances in Applied Mechanics*, vol. 23, pp. 179–269, Academic Press, New York, 1983.
17. C. O. Horgan, Recent Developments Concerning Saint-Venant's Principle: An Update, *Appl. Mech. Rev.*, vol. 42, pp. 295–303, 1989.
18. C. O. Horgan, Recent Developments Concerning Saint-Venant's Principle: A Second Update, *Appl. Mech. Rev.*, vol. 49, pp. S101–S111, 1996.
19. D. E. Carlson, Linear Thermoelasticity, in C. Truesdell (ed.), *Encyclopedia of Physics*, vol. VIa/2, Mechanics of Solids II, pp. 297–389, Springer-Verlag, Berlin, 1972.
20. M. A. Gurtin, Linear Theory of Elasticity, in C. Truesdell (ed.), *Encyclopedia of Physics*, vol. VIa/2, Mechanics of Solids II, pp. 1–295, Springer-Verlag, Berlin, 1972.