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ON SOME GROWTH-DECAY RESULTS IN THERMOELASTICITY OF POROUS MEDIA

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In this paper we study the spatial behaviour for a large class of isotropic and homogeneous porous thermoelastic materials for which the constitutive coefficients are supposed to satisfy some relaxed positive definiteness conditions. By using some appropriate measures, we are able to establish results describing the spatial behaviour of transient and steady-state solutions in these enlarged classes of thermoelastic porous materials.

Keywords: Spatial behaviour; Steady-state; Thermoelastic porous materials; Transient solutions

INTRODUCTION

The theory of elastic porous materials has been developed by Goodman and Cowin [1], Nunziato and Cowin [2] and Cowin and Nunziato [3] for describing the deformation of continuum with voids in which the matrix material is elastic and the interstices are void of material. The basic concept underlying this theory is that of a material for which the bulk density is written as the product of two fields, the density field of the matrix material and the volume fraction field. This representation was employed by Nunziato and Cowin [2] to develop a nonlinear theory of elastic materials with voids. The linear theory of elastic materials with voids has been developed by Cowin and Nunziato [3]. The intended applications of the theory of elastic materials with voids are to geological materials such as rock and soils and to manufactured porous materials.

The first investigations in the theory of thermoelastic materials with voids are due to Nunziato and Cowin [2] and Ieșan [4]. The linear theory of thermoelastic materials with voids was presented in [4] (see also [5]).

The spatial behaviour of solutions in elastostatics of cylinders made of a porous elastic material was studied by Chiriță [6] and Ieșan and Quintanilla [7]. The spatial behaviour of the transient and steady-state solutions in linear theory of thermoelastic materials with voids has been studied by Chiriță and Scalia [8] and

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Scalia, Pompei and Chiriță [9], respectively, under the assumption concerning the positive definiteness of the constitutive coefficients.

Recently, there have been prepared novel foam structures with negative Poisson's ratios and their mechanical behaviour and structure evaluated (see, e.g., Lakes [10] and Caddock and Evans [11]). Such auxetic or anti-rubber materials expand laterally when stretched, in contrast to ordinary materials. Some anisotropic polymer foams have been prepared which exhibit a Poisson's ratio exceeding 1 (see Lee and Lakes [12]). Materials of the above sorts are expected to have interesting mechanical properties, such as high energy absorption and fracture resistance, which may be useful in applications. Possible applications of such materials in prevention of pressure sores or ulcers are outlined by Wang and Lakes [13]. Saint-Venant end effects for materials with negative Poisson's ratio are studied by Lakes [14].

In this paper, we describe some methods, relevant to the study of spatial behaviour for the steady-state and transient solutions in a linear isotropic homogeneous thermoelastic body with voids, under mild conditions concerning the positive definiteness of the constitutive coefficients, including the class of auxetic materials. Concerning the transient solutions we establish some spatial estimates describing the exponential decay of appropriate measures with respect to the distance to the support of the given data. For the case of an unbounded body we consider a measure for which we are able to prove that the decay rate is quadratic in terms of the distance to the support of the given data.

For the study of the time-harmonic vibrations, we consider a right cylinder made of an isotropic and homogeneous thermoelastic material with voids and introduce two appropriate cross-sectional measures. The treatment of this problem leads to a second-order differential inequality furnishing the information concerning the spatial behaviour of the amplitude of the vibration, provided the frequency of the harmonic vibration is lower than a certain critical frequency. Such results are established for the same enlarged classes of materials with voids as for the transient solutions.

FORMULATION OF THE PROBLEM

Let B be a bounded or unbounded regular region of the physical space $\mathbb{R}^3$ with the piecewise smooth boundary surface ∂B . We designate by $\mathbf{n}$ the outward unit normal vector to the boundary. The region B is filled with an isotropic and homogeneous thermoelastic porous material. We select a rectangular system of coordinates and note that vectors and tensors will have components denoted by Latin subscripts ranging over 1, 2, 3. Summation over repeated subscripts and other typical conventions for differential operations are implied such as a superposed dot or a comma followed by a subscript to denote partial derivative with respect to time or the corresponding cartesian coordinate, respectively. Further, we suppress the dependence upon the spatial variable when no confusion may occur.

In the context of linear theory of thermoelasticity theory for a porous solid developed by Ieșan [4] (see also [5]), the equations of motion are

$$\begin{aligned} t_{ji,j} + \rho f_i &= \rho \ddot{u}_i \\ h_{j,j} + g + \rho l &= \rho \kappa \ddot{\phi}, \\ \rho T_0 \dot{\eta} &= q_{j,j} + \rho S \quad \text{in } B \times (0, \infty). \end{aligned} \tag{1}$$

In the above equations we have used the following notation: u_i are the components of the displacement vector field, ϕ is the change in volume fraction from the reference volume fraction, T_0 is the constant reference temperature, t_{ij} are the components of the stress tensor, h_i are the components of the equilibrated stress vector, g is the intrinsic equilibrated force, η is the entropy, q_i are the components of the heat flux vector, f_i are the components of the body force vector, l is the extrinsic equilibrated body force and S is the heat supply. Moreover, ρ and $\varkappa$ are the bulk mass density and the equilibrated inertia in the reference state.

The constitutive equations for the linear theory of isotropic and homogeneous thermoelastic porous media are [4, 5]

$$\begin{aligned} t_{ij} &= \lambda e_{rr} \delta_{ij} + 2\mu e_{ij} + b\phi \delta_{ij} - \beta\theta \delta_{ij} & h_j &= \alpha\phi_{,j} \\ g &= -be_{rr} - \xi\phi + m\theta \\ \rho\eta &= \beta e_{rr} + a\theta + m\phi & q_j &= k\theta_{,j} \quad \text{in } \bar{B} \times [0, \infty) \end{aligned} \quad (2)$$

where θ is the temperature variation from the reference temperature T_0 , λ , μ , a , b , k , m , α , β , and ξ are constant constitutive coefficients, δ_{ij} is the Kronecker's delta and

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (3)$$

The set $\bar{B}$ represents the closure of B .

The internal energy density $\mathcal{E}$ associated with the kinematic fields u_i , ϕ is defined by

$$\mathcal{E} = \frac{1}{2}\lambda e_{rr}e_{ss} + \mu e_{ij}e_{ij} + be_{rr}\phi + \frac{1}{2}\xi\phi^2 + \frac{1}{2}\alpha\phi_{,i}\phi_{,i} \quad (4)$$

and it is positive definite if and only if (see, e.g., Cowin and Nunziato [3])

$$\mu > 0 \quad \alpha > 0 \quad \xi > 0 \quad 3\lambda + 2\mu > 0 \quad (3\lambda + 2\mu)\xi > 3b^2 \quad (5)$$

As we will see later, the hypotheses of positive definiteness on the constitutive coefficients defined by (5) will be relaxed. To this end, we consider an isotropic, homogeneous thermoelastic porous solid so that the basic Eqs. (1) to (3) of the linear dynamic theory reduce to [4, 5]

$$\begin{aligned} \mu u_{j,rr} + (\lambda + \mu)u_{r,rj} + b\phi_{,j} - \beta\theta_{,j} + \rho f_j &= \rho \ddot{u}_j \\ \alpha\phi_{,rr} - bu_{r,r} - \xi\phi + m\theta + \rho l &= \rho \varkappa \ddot{\phi} \\ k\theta_{,rr} - \beta T_0 \dot{u}_{j,j} - aT_0 \dot{\theta} - mT_0 \dot{\phi} &= -\rho S \end{aligned} \quad (6)$$

The initial conditions are

$$u_j = u_j^0 \quad \dot{u}_j = \dot{u}_j^0 \quad \phi = \phi^0 \quad \dot{\phi} = \dot{\phi}^0 \quad \theta = \theta^0 \quad \text{on } \bar{B} \times \{0\} \quad (7)$$

where u_j^0 , $\dot{u}_j^0$, ϕ^0 , $\dot{\phi}^0$, and θ^0 are prescribed continuous functions.

In what follows, for convenience, we will consider the following boundary conditions

$$u_j = 0 \quad \phi = 0 \quad \theta = 0 \quad \text{on } \Sigma_1 \times [0, \infty) \quad (8)$$

$$t_{jk}n_j = \tilde{t}_k \quad h_j n_j = \tilde{h} \quad q_j n_j = \tilde{q} \quad \text{on } \Sigma_2 \times [0, \infty) \quad (9)$$

where n_j are the components of the outward unit vector normal to the boundary surface ∂B , Σ_1 and Σ_2 are subsets of ∂B so that $\overline{\Sigma_1} \cup \Sigma_2 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\tilde{t}_j$, $\tilde{h}$ and $\tilde{q}$ are prescribed functions on $\Sigma_2 \times [0, \infty)$.

Under the positive definiteness conditions (5) for the potential energy density $\mathcal{E}$ and by assuming $\rho > 0$, $\kappa > 0$, the spatial behaviour of the transient solutions of the initial-boundary value problem $\mathcal{P}$ defined by the relations (6)–(9), has been established by Chiriță and Scalia [8]. Throughout this paper we will consider classical solutions of the initial-boundary value problem $\mathcal{P}$, that is we will consider the triplet of functions $\{u_j, \phi, \theta\}$ twice continuous differentiable with respect to the spatial and time variables, satisfying the relations (6)–(9). We have to outline that our method of proof can be used also for appropriate classes of weak solutions.

SPATIAL BEHAVIOUR OF THE TRANSIENT SOLUTIONS

Throughout this section we will establish the spatial behaviour of the transient solutions of the initial-boundary value problem $\mathcal{P}$ under hypotheses on the constitutive coefficients different from those used in Chiriță and Scalia [8] and Pompei and Scalia [15]. In fact, we will establish our results for appropriate complementary classes of porous thermoelastic materials by using the method developed by Chiriță and Ciarletta [16]. Clearly, the union of these classes is essentially larger than the class of porous thermoelastic materials described by the relation (5). This task is possible because we introduce some new measures associated with the solutions, essentially different from those used by Chiriță and Scalia [8] and Pompei and Scalia [15]. Our analysis is motivated by the existence of the novel foam structures for which the internal energy density $\mathcal{E}$ is not always a positive definite quadratic form.

For fixed $T > 0$, we consider the support $\widehat{D}_T$ of the initial and boundary data and the body supplies on the time interval $[0, T]$ and further, for convenience, we assume that it is a bounded regular region. Of course, $\widehat{D}_T$ is the union of the set of all $\mathbf{x} \in B$ for which

$$\begin{aligned} u_i^0(\mathbf{x}) \neq 0 \quad \text{or} \quad \dot{u}_i^0(\mathbf{x}) \neq 0 \quad \text{or} \quad \phi^0(\mathbf{x}) \neq 0 \quad \text{or} \quad \dot{\phi}^0(\mathbf{x}) \neq 0 \quad \text{or} \quad \theta^0(\mathbf{x}) \neq 0 \\ \text{or} \quad b_i(\mathbf{x}, \tau) \neq 0 \quad \text{or} \quad l(\mathbf{x}, \tau) \neq 0 \quad \text{or} \quad S(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T] \end{aligned}$$

and the set of all $\mathbf{x} \in \Sigma_2 \times [0, \infty)$ for which

$$\tilde{t}_i(\mathbf{x}, \tau) \neq 0 \quad \text{or} \quad \tilde{h}(\mathbf{x}, \tau) \neq 0 \quad \text{or} \quad \tilde{q}(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T].$$

Further, we introduce the following sets

$$\begin{aligned} D_r &= \{\mathbf{x} \in \overline{B} : \overline{S(\mathbf{x}, r)} \cap \widehat{D}_T \neq \emptyset\} \\ B_r &= B \setminus D_r \quad r \geq 0 \quad B(r_1, r_2) = B_{r_2} \setminus B_{r_1} \quad r_2 < r_1 \end{aligned}$$

where $\overline{S(\mathbf{x}, r)}$ is the closure of the ball with radius r and center at $\mathbf{x}$. Finally, we denote by S_r the subsurface of ∂B_r contained into inside of B and whose outward unit normal vector is forwarded to the exterior of D_r . We can observe that the initial and boundary data and the body forces are null on B_r , S_r .

First Measure and Related Estimates

Throughout this subsection we will assume that

$$\rho > 0 \quad \kappa > 0 \quad \alpha > 0 \quad \mu > 0 \quad a > 0 \quad k > 0 \quad \xi > 0 \quad 4\mu + 3\lambda > 0 \quad (4\mu + 3\lambda)\xi > 3b^2 \quad (10)$$

Corresponding to a solution $\mathbf{U} = \{u_j, \phi, \theta\}$ of the initial-boundary value problem $\mathcal{P}$ we introduce the following function

$$\Lambda(r, t) = - \int_0^t \int_{S_r} e^{-\sigma s} \left[\dot{u}_i(s) s_{ji}(s) + \dot{\phi}(s) h_j(s) + \frac{\theta(s)}{T_0} q_j(s) \right] n_j da ds \quad (11)$$

where σ is a positive parameter at hand and

$$s_{ji} = S_{ji} - \beta \theta \delta_{ij} \quad S_{ji} = \mu u_{i,j} + (\lambda + \mu) u_{r,r} \delta_{ij} + b \phi \delta_{ij} \quad (12)$$

This function is defined on $I \times [0, T]$, where

I is the interval $[0, \infty)$ if B is an unbounded body,

I is the interval $[0, L]$, where $L = \max_{\mathbf{x} \in \overline{B}} (\min_{\mathbf{y} \in \widehat{D}_T} |\mathbf{y} - \mathbf{x}|^{\frac{1}{2}})$, if B is a bounded body.

In this last case, from the relation (11) and the definition of S_r and D_r , we can conclude that $\Lambda(L, t) = 0$ for all $t \in [0, T]$.

By using Eq. (6), we get

$$\left[\dot{u}_i s_{ji} + \dot{\phi} h_j + \frac{1}{T_0} \theta q_j \right]_{,j} = \frac{1}{2} \frac{\partial}{\partial t} (K + W + a\theta^2) + \frac{k}{T_0} \theta_{,j} \theta_{,j} - \rho f_i \dot{u}_i - \rho l \dot{\phi} - \frac{\rho}{T_0} S \theta \quad (13)$$

where

$$K = \rho \dot{u}_i \dot{u}_i + \rho \kappa \dot{\phi}^2 \quad (14)$$

$$W(u, \phi) = W_1(u, \phi) + W_2(u) + W_3(\phi) \quad (15)$$

$$\begin{aligned} W_1(u, \phi) &= (\lambda + 2\mu)(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2) + 2(\lambda + \mu)(u_{1,1}u_{2,2} + u_{2,2}u_{3,3} + u_{3,3}u_{1,1}) \\ &\quad + \xi \phi^2 + 2b\phi(u_{1,1} + u_{2,2} + u_{3,3}) \\ W_2(u) &= \mu(u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{3,1}^2) \\ W_3(\phi) &= \alpha \phi_{,j} \phi_{,j} \end{aligned} \quad (16)$$

The definition for S_r , B_r , the divergence theorem and the relations (11), (13) lead to

$$\begin{aligned} \Lambda(r_1, t) - \Lambda(r_2, t) &= - \int_0^t \int_{\partial B(r_1, r_2)} e^{-\sigma s} \left[\dot{u}_i(s) s_{ji}(s) + \dot{\phi}(s) h_j(s) + \frac{\theta(s)}{T_0} q_j(s) \right] n_j da ds \\ &= - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} \left\{ \frac{1}{2} \frac{\partial}{\partial s} [K(s) + W(s) + a\theta^2(s)] + \frac{k}{T_0} \theta_{,j}(s) \theta_{,j}(s) \right\} dv ds \end{aligned} \quad (17)$$

for $r_2 < r_1$, $0 \leq t \leq T$. Taking into account (17) and the definition for S_r and B_r , it is a simple matter to obtain

$$\begin{aligned} \frac{\partial \Lambda}{\partial r}(r, t) &= - \frac{1}{2} \int_{S_r} e^{-\sigma t} [K(t) + W(t) + a\theta^2(t)] da \\ &\quad - \int_0^t \int_{S_r} e^{-\sigma s} \left\{ \frac{\sigma}{2} [K(s) + W(s) + a\theta^2(s)] + \frac{k}{T_0} \theta_{,j}(s) \theta_{,j}(s) \right\} da ds \end{aligned} \quad (18)$$

Obviously, if $\rho > 0$ and $\varkappa > 0$ then the quadratic form K in the variables $\{\dot{u}_1, \dot{u}_2, \dot{u}_3, \sqrt{\varkappa} \dot{\phi}\}$ is positive definite and if $\alpha > 0$ then the form W_3 in the variables $\{\phi_{,1}, \phi_{,2}, \phi_{,3}\}$ is positive definite. Moreover, W_2 is a positive definite quadratic form in the variables $\{u_{1,2}, u_{2,1}, u_{2,3}, u_{3,2}, u_{1,3}, u_{3,1}\}$ if and only if $\mu > 0$. On the other hand, the quadratic form W_1 in the variables $\{u_{1,1}, u_{2,2}, u_{3,3}, \phi\}$ is positive definite if and only if $\mu > 0$, $\xi > 0$, $4\mu + 3\lambda > 0$, $(4\mu + 3\lambda)\xi > 3b^2$. The eigenvalues of the matrix $\mathcal{A}$ of the quadratic form W_1 are

$$\begin{aligned} \chi_1 &= \chi_2 = \mu \\ \chi_3 &= \frac{1}{2} \left\{ \xi + 4\mu + 3\lambda + \sqrt{[\xi - (4\mu + 3\lambda)]^2 + 12b^2} \right\} \\ \chi_4 &= \frac{1}{2} \left\{ \xi + 4\mu + 3\lambda - \sqrt{[\xi - (4\mu + 3\lambda)]^2 + 12b^2} \right\} \end{aligned}$$

Consequently, the relations (10) and (18) imply that $\Lambda(r, t)$ is a non-increasing function with respect to r , for all $t \in [0, T]$. Since $\Lambda(L, t) = 0$ for all $t \in [0, T]$, it follows that for a bounded body we have $\Lambda(r, t) \geq 0$ for $(r, t) \in I \times [0, T]$.

On the other hand, by means of the Schwarz inequality and the arithmetic-geometric mean inequalities, we have

$$|\dot{u}_i s_{ji} n_j + \dot{\phi} h_j n_j| \leq \frac{1}{2} \left[\varepsilon_1 (\rho \dot{u}_i \dot{u}_i + \rho \varkappa \dot{\phi}^2) + \frac{1}{\varepsilon_1 \rho} \left(s_{ij} s_{ij} + \frac{1}{\varkappa} h_j h_j \right) \right] \quad (19)$$

for any arbitrary positive constant ε_1 . Furthermore, from the relation (12) we get

$$s_{ji} s_{ji} \leq (1 + \varepsilon) S_{ji} S_{ji} + 3\beta^2 \left(1 + \frac{1}{\varepsilon} \right) \theta^2 \quad \text{for all } \varepsilon > 0 \quad (20)$$

As was shown by Ciarletta, Chiriță and Passarella [17], from the relations (10), (12) and (16) we obtain

$$S_{ji}S_{ji} \leq \chi_M(W_1 + W_2) \quad (21)$$

where

$$\chi_M = \max \left\{ \mu, \frac{1}{2} \left[\xi + 4\mu + 3\lambda + \sqrt{[\xi - (4\mu + 3\lambda)]^2 + 12b^2} \right] \right\} \quad (22)$$

By using the relations (20) and (21) in (19), we obtain the following estimate

$$|\dot{u}_i s_{ji} n_j + \dot{\phi} h_j n_j| \leq \frac{\varepsilon_1}{2} K + \frac{1}{2\varepsilon_1 \rho} \left[\chi_M(1 + \varepsilon) + \frac{\alpha}{\varkappa} \right] W + \frac{3\beta^2}{2\varepsilon_1 \rho} \left(1 + \frac{1}{\varepsilon} \right) \theta^2 \quad (23)$$

On the other hand, from the relation (2) we get

$$\left| \frac{\theta}{T_0} q_j n_j \right| \leq \frac{1}{2T_0} \left(\varepsilon_2 a \theta^2 + \frac{1}{\varepsilon_2 a} q_j q_j \right) \leq \frac{\varepsilon_2 a}{2T_0} \theta^2 + \frac{k}{2\varepsilon_2 a} \frac{k}{T_0} \theta_{,j} \theta_{,j} \quad \text{for all } \varepsilon_2 > 0 \quad (24)$$

Concluding, the relations (23) and (24) give

$$\begin{aligned} \left| \dot{u}_i s_{ji} n_j + \dot{\phi} h_j n_j + \frac{\theta}{T_0} q_j n_j \right| &\leq \varepsilon_1 \frac{1}{2} K + \frac{1}{\varepsilon_1 \rho} \left[\chi_M(1 + \varepsilon) + \frac{\alpha}{\varkappa} \right] \frac{1}{2} W \\ &\quad + \left[\frac{3\beta^2}{\varepsilon_1 \rho a} \left(1 + \frac{1}{\varepsilon} \right) + \frac{\varepsilon_2}{T_0} \right] \frac{1}{2} a \theta^2 + \frac{k}{2\varepsilon_2 a} \frac{k}{T_0} \theta_{,j} \theta_{,j} \end{aligned} \quad (25)$$

for all strictly positive constants $\varepsilon, \varepsilon_1, \varepsilon_2$.

At this stage we request the equalities

$$\frac{\varepsilon_1}{\sigma} = \frac{1}{\sigma \varepsilon_1} \left[\frac{\chi_M}{\rho} (1 + \varepsilon) + \frac{\alpha}{\rho \varkappa} \right] = \frac{3\beta^2}{\varepsilon_1 \rho a \sigma} \left(1 + \frac{1}{\varepsilon} \right) + \frac{\varepsilon_2}{T_0 \sigma} = \frac{k}{2\varepsilon_2 a} \quad (26)$$

That is, we set

$$\varepsilon_1 = c_1 = \sqrt{\frac{\chi_M}{\rho} (1 + \varepsilon) + \frac{\alpha}{\rho \varkappa}} \quad \varepsilon_2 = \frac{k\sigma}{2a} \left[\frac{\chi_M}{\rho} (1 + \varepsilon) + \frac{\alpha}{\rho \varkappa} \right]^{-\frac{1}{2}} \quad (27)$$

where ε is the positive root of the following algebraic equation

$$\varepsilon^2 + \left(1 + \frac{\alpha}{\varkappa \chi_M} - \frac{\rho k \sigma}{2a T_0 \chi_M} - \frac{3\beta^2}{a \chi_M} \right) \varepsilon - \frac{3\beta^2}{a \chi_M} = 0 \quad (28)$$

Therefore, with the above choices, relations (11), (18), and (25) furnish the following first-order differential inequality

$$|\Lambda(r, t)| + \frac{c_1}{\sigma} \frac{\partial \Lambda}{\partial r}(r, t) \leq 0 \quad \text{for all } (r, t) \in I \times [0, T] \quad (29)$$

As in [16], we can exploit the differential inequality (29) in order to prove for a bounded body the following estimate

$$\Lambda(r, t) \leq \Lambda(0, t) \exp\left(-\frac{\sigma}{c_1} r\right) \quad \text{for all } (r, t) \in I \times [0, T] \quad (30)$$

Let us now study the case of an unbounded body. We can conclude that either $\Lambda(r, t) \geq 0$ for all $(r, t) \in I \times [0, T]$ and then a spatial decay estimate of the type described by (30) holds true, or there exists a value $r_t^0 \geq 0$ such that $\Lambda(r_t^0, t) < 0$ and then $\Lambda(r, t) < 0$ for all $r > r_t^0$ and $t \in [0, T]$ and, moreover,

$$-\Lambda(r, t) \geq -\Lambda(r_t^0, t) \exp\left[\frac{\sigma}{c_1}(r - r_t^0)\right] \quad r > r_t^0, \quad t \in [0, T] \quad (31)$$

Let us consider further that we have $\Lambda(r, t) \geq 0$ for all $(r, t) \in I \times [0, T]$. In view of the spatial decay (30) we can introduce the new measure

$$\begin{aligned} \Lambda_1(r, t) &= \int_r^\infty \Lambda(\eta, t) d\eta \\ &= - \int_0^t \left\{ \int_r^\infty d\eta \int_{S_\eta} e^{-\sigma s} \left[\dot{u}_i(s) s_{ji}(s) + \dot{\phi}(s) h_j(s) + \frac{\theta(s)}{T_0} q_j(s) \right] n_j da \right\} ds \end{aligned} \quad (32)$$

By means of the relations (2) and (30) and the definition of S_r , we have

$$\begin{aligned} - \int_r^\infty d\eta \int_{S_\eta} e^{-\sigma s} \frac{\theta(s)}{T_0} q_j(s) n_j da &= - \int_r^\infty d\eta \int_{S_\eta} e^{-\sigma s} \frac{k}{T_0} \theta(s) \frac{\partial \theta(s)}{\partial n} da \\ &= - \int_r^\infty d\eta \int_{S_\eta} e^{-\sigma s} \frac{\partial}{\partial n} \left(\frac{k}{2T_0} \theta^2(s) \right) da \\ &= - \int_r^\infty \frac{\partial}{\partial \eta} \int_{S_\eta} e^{-\sigma s} \frac{k}{2T_0} \theta^2(s) da d\eta \\ &= \int_{S_r} e^{-\sigma s} \frac{k}{2T_0} \theta^2(s) da \end{aligned} \quad (33)$$

and hence the measure $\Lambda_1(r, t)$ becomes

$$\begin{aligned} \Lambda_1(r, t) &= - \int_0^t \left\{ \int_r^\infty d\eta \int_{S_\eta} e^{-\sigma s} [\dot{u}_i(s) s_{ji}(s) + \dot{\phi}(s) h_j(s)] n_j da \right\} ds \\ &\quad + \int_0^t \int_{S_r} e^{-\sigma s} \frac{k}{2T_0} \theta^2(s) da ds \end{aligned} \quad (34)$$

Obviously, from (18), (32), and (34), we obtain

$$\frac{\partial \Lambda_1}{\partial t}(r, t) = - \int_r^\infty d\eta \int_{S_\eta} e^{-\sigma t} [\dot{u}_i(t) s_{ji}(t) + \dot{\phi}(t) h_j(t)] n_j da + \int_{S_r} e^{-\sigma t} \frac{k}{2T_0} \theta^2(t) da \quad (35)$$

and

$$\begin{aligned} \frac{\partial \Lambda_1}{\partial r}(r, t) = & - \int_{B_r} e^{-\sigma t} \frac{1}{2} [K(t) + W(t) + a\theta^2(t)] dv \\ & - \int_0^t \int_{B_r} e^{-\sigma s} \left\{ \frac{\sigma}{2} [K(s) + W(s) + a\theta^2(s)] + \frac{1}{T_0} k \theta_{,j}(s) \theta_{,j}(s) \right\} dv ds \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{\partial^2 \Lambda_1}{\partial r^2}(r, t) = & \int_{S_r} e^{-\sigma t} \frac{1}{2} [K(t) + W(t) + a\theta^2(t)] da \\ & + \int_0^t \int_{S_r} e^{-\sigma s} \left\{ \frac{\sigma}{2} [K(s) + W(s) + a\theta^2(s)] + \frac{1}{T_0} k \theta_{,j}(s) \theta_{,j}(s) \right\} da ds \end{aligned} \quad (37)$$

Furthermore, by using the estimate (23) with

$$\varepsilon_1 = c_2 = \sqrt{\frac{1}{\rho} \left[\chi_M (1 + \varepsilon_0) + \frac{\alpha}{\varkappa} \right]} \quad \varepsilon_0^2 + \left(1 + \frac{\alpha}{\varkappa \chi_M} - \frac{3\beta^2}{a \chi_M} \right) \varepsilon_0 - \frac{3\beta^2}{a \chi_M} = 0 \quad (38)$$

in (35), we get

$$\frac{\partial \Lambda_1}{\partial t}(r, t) \leq \frac{c_2}{2} \int_r^\infty d\eta \int_{S_\eta} e^{-\sigma t} [K(t) + W(t) + a\theta^2(t)] da + \frac{k}{aT_0} \int_{S_r} e^{-\sigma t} \frac{1}{2} a\theta^2(t) da \quad (39)$$

Concluding, from the relations (36), (37), and (39) we obtain the following differential inequality

$$\frac{\partial \Lambda_1}{\partial t}(r, t) \leq -c_2 \frac{\partial \Lambda_1}{\partial r}(r, t) + \frac{k}{aT_0} \frac{\partial^2 \Lambda_1}{\partial r^2}(r, t) \quad r \in [0, \infty), \quad t \in [0, T] \quad (40)$$

By using the change of function

$$\Lambda_1(r, t) = \exp \left(-\frac{aT_0 c_2^2}{4k} t \right) \exp \left(\frac{aT_0 c_2}{2k} r \right) \mathcal{L}_1(r, t) \quad (41)$$

we can write the differential inequality (40) in the following form

$$\frac{k}{aT_0} \frac{\partial^2 \mathcal{L}_1}{\partial r^2}(r, t) - \frac{\partial \mathcal{L}_1}{\partial t}(r, t) \geq 0 \quad (42)$$

We further note that

$$\begin{aligned} \mathcal{L}_1(r, 0) = 0 \quad \text{for } r > 0, \quad \mathcal{L}_1(0, t) = e^{\frac{aT_0 c_2^2}{4k} t} \Lambda_1(0, t) \quad \text{for } t \geq 0 \\ \mathcal{L}_1(r, t) \rightarrow 0 \quad (\text{uniformly in } t) \quad \text{as } r \rightarrow \infty \end{aligned} \quad (43)$$

By means of the maximum principle for parabolic differential inequalities, we have

$$\mathcal{L}_1(r, t) \leq \mathcal{F}(r, t) \quad r \geq 0 \quad t \geq 0 \quad (44)$$

where

$$\begin{aligned}\frac{k}{aT_0} \frac{\partial^2 \mathcal{T}}{\partial r^2}(r, t) - \frac{\partial \mathcal{T}}{\partial t}(r, t) &= 0 \quad r > 0, \quad t > 0 \\ \mathcal{T}(r, 0) &= \mathcal{L}_1(r, 0) = 0 \quad r \geq 0 \\ \mathcal{T}(0, t) &= \mathcal{L}_1(0, t) \quad t \geq 0 \\ \mathcal{T}(r, t) - \mathcal{L}_1(r, t) &\rightarrow 0 \text{ (uniformly in } t \text{ as } r \rightarrow \infty\end{aligned}\tag{45}$$

The solution of the above problem is given by

$$\mathcal{T}(r, t) = \sqrt{\frac{aT_0}{k}} \frac{1}{2\sqrt{\pi}} \int_0^t \frac{r}{(t-\tau)^{\frac{3}{2}}} e^{-\frac{aT_0}{4k(t-\tau)}r^2} e^{\frac{aT_0 c_2^2}{4k}\tau} \Lambda_1(0, \tau) d\tau\tag{46}$$

Therefore, the relations (44) and (45) give

$$\mathcal{L}_1(r, t) \leq e^{\frac{aT_0 c_2^2}{2k}r} \mathcal{L}_1(0, t) \mathcal{G}(r, t)\tag{47}$$

where

$$\mathcal{G}(r, t) = \sqrt{\frac{aT_0}{k}} \frac{1}{2\sqrt{\pi}} \int_0^t rs^{-\frac{3}{2}} e^{-\left(\frac{aT_0}{4ks}r^2 + \frac{aT_0 c_2^2}{4k}s\right)} ds\tag{48}$$

The function $\mathcal{G}(r, t)$ can be estimated as in Horgan, Payne, and Wheeler [18] to obtain

$$\mathcal{G}(r, t) \leq \sqrt{\frac{kt}{aT_0\pi}} \frac{2r}{r^2 - c_2^2 t^2} e^{-\frac{aT_0 c_2^2}{4k}t} e^{-\frac{aT_0}{4kt}r^2} \quad \text{for } r > c_2 t\tag{49}$$

Thus, from the relations (41), (47), and (49) we get the following estimate

$$\Lambda_1(r, t) \leq \Lambda_1(0, t) \sqrt{\frac{kt}{aT_0\pi}} \frac{2r}{r^2 - c_2^2 t^2} e^{-\frac{aT_0 c_2^2}{4k}t} e^{\frac{aT_0 c_2^2}{2k}r - \frac{aT_0}{4kt}r^2} \quad \text{for } r > c_2 t, \quad t \in [0, T]\tag{50}$$

Second Measure and Related Estimates

Throughout this subsection we will assume the following hypotheses

$$\begin{aligned}\rho > 0 \quad \varkappa > 0 \quad \alpha > 0 \quad \mu > 0 \quad \xi > 0 \quad -2\mu < \lambda < 0 \\ (\lambda + 2\mu)\xi > 3b^2 \quad a > 0 \quad k > 0.\end{aligned}\tag{51}$$

We now introduce the following time-weighted function on $I \times [0, T]$:

$$\Pi(r, t) = - \int_0^t \int_{s_r} e^{-\sigma s} \left[\dot{u}_i(s) r_{ji}(s) + \dot{\phi}(s) h_j(s) + \frac{\theta(s)}{T_0} q_j(s) \right] n_j da ds\tag{52}$$

where

$$r_{ji} = T_{ji} - \beta\theta\delta_{ij} \quad T_{ji} = \mu u_{i,j} + (\lambda + \mu)u_{j,i} + b\phi\delta_{ij} \quad (53)$$

The relations (6) and (53) give

$$\left[\dot{u}_i r_{ji} + \dot{\phi} h_j + \frac{1}{T_0} \theta q_j \right]_{,j} = \frac{1}{2} \frac{\partial}{\partial t} (K + W^* + a\theta^2) + \frac{k}{T_0} \theta_{,j} \theta_{,j} - \rho f_i \dot{u}_i - \rho l \dot{\phi} - \frac{\rho}{T_0} S \theta \quad (54)$$

in which

$$W^*(u, \phi) = W_1^*(u, \phi) + W_2^*(u) + W_3(\phi) \quad (55)$$

$$\begin{aligned} W_1^*(u, \phi) &= (\lambda + 2\mu)(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2) + 2b\phi(u_{1,1} + u_{2,2} + u_{3,3}) + \xi\phi^2 \\ W_2^*(u) &= \mu(u_{1,2}^2 + u_{2,1}^2 + u_{2,3}^2 + u_{3,2}^2 + u_{1,3}^2 + u_{3,1}^2) \\ &\quad + 2(\lambda + \mu)(u_{1,2}u_{2,1} + u_{2,3}u_{3,2} + u_{3,1}u_{1,3}) \end{aligned} \quad (56)$$

Clearly, K is positive definite if and only if $\rho > 0$, $\kappa > 0$, while W_3 is positive definite if and only if $\alpha > 0$. Moreover, the quadratic form W_2^* in the variables $\{u_{1,2}, u_{2,1}, u_{2,3}, u_{3,2}, u_{3,1}, u_{1,3}\}$ is positive definite if and only if $\lambda < 0$, $\lambda + 2\mu > 0$. The eigenvalues of the matrix $\mathcal{B}$ of this quadratic form are

$$\check{\chi}_1 = \check{\chi}_2 = \check{\chi}_3 = -\lambda \quad \check{\chi}_4 = \check{\chi}_5 = \check{\chi}_6 = \lambda + 2\mu. \quad (57)$$

Further, the quadratic form W_1^* in the variables $\{u_{1,1}, u_{2,2}, u_{3,3}, \phi\}$ is positive definite if and only if $\xi > 0$, $\lambda + 2\mu > 0$, $(\lambda + 2\mu)\xi > 3b^2$. The matrix $\mathcal{C}$ of the quadratic form W_1^* has the following eigenvalues

$$\hat{\chi}_1 = \hat{\chi}_2 = \lambda + 2\mu \quad \hat{\chi}_{3,4} = \frac{1}{2} \left\{ \xi + \lambda + 2\mu \pm \sqrt{[\xi - (\lambda + 2\mu)]^2 + 12b^2} \right\} \quad (58)$$

In view of the results established in [17], we can conclude that

$$T_{ij}T_{ij} \leq \check{\chi}_M W_2^* + \hat{\chi}_M W_1^* \quad (59)$$

where

$$\begin{aligned} \check{\chi}_M &= \max\{-\lambda, \lambda + 2\mu\} \\ \hat{\chi}_M &= \max \left\{ \lambda + 2\mu, \frac{1}{2} \left[\xi + \lambda + 2\mu + \sqrt{[\xi - (\lambda + 2\mu)]^2 + 12b^2} \right] \right\} \end{aligned} \quad (60)$$

Taking into account the definition for S_r , B_r and the divergence theorem, then the relations (52) and (53) imply that

$$\begin{aligned} &\Pi(r_1, t) - \Pi(r_2, t) \\ &= - \int_0^t \int_{B(r_1, r_2)} e^{-\sigma s} \left\{ \frac{1}{2} \frac{\partial}{\partial s} [K(s) + W^*(s) + a\theta^2(s)] + \frac{k}{T_0} \theta_{,j}(s) \theta_{,j}(s) \right\} dv ds \end{aligned} \quad (61)$$

and therefore, we get

$$\begin{aligned} \frac{\partial \Pi}{\partial r}(r, t) = & -\frac{1}{2} \int_{S_r} e^{-\sigma t} [K(t) + W^*(t) + a\theta^2(t)] da \\ & - \int_0^t \int_{S_r} e^{-\sigma s} \left\{ \frac{\sigma}{2} [K(s) + W^*(s) + a\theta^2(s)] + \frac{k}{T_0} \theta_{,j}(s) \theta_{,j}(s) \right\} da ds \end{aligned} \quad (62)$$

Moreover, from (52) we have

$$\frac{\partial \Pi}{\partial t}(r, t) = - \int_{S_r} e^{-\sigma t} \left[\dot{u}_i(t) r_{ji}(t) + \dot{\phi}(t) h_j(t) + \frac{\theta(t)}{T_0} q_j(t) \right] n_j da \quad (63)$$

In view of a procedure similar to that in the above subsection, we can obtain the following differential inequality

$$|\Pi(r, t)| + \frac{\tilde{c}_1}{\sigma} \frac{\partial \Pi}{\partial r}(r, t) \leq 0 \quad \text{for all } (r, t) \in I \times [0, T] \quad (64)$$

where

$$\begin{aligned} \tilde{c}_1 &= \sqrt{\frac{\tilde{\chi}_M}{\rho} (1 + \tilde{\varepsilon}) + \frac{\alpha}{\rho \varkappa}} \\ \tilde{\chi}_M &= \max \left\{ \frac{\alpha}{\varkappa}, -\lambda, \lambda + 2\mu, \frac{1}{2} \left(\xi + \lambda + 2\mu + \sqrt{[\xi - (\lambda + 2\mu)]^2 + 12b^2} \right) \right\} \end{aligned} \quad (65)$$

and $\tilde{\varepsilon}$ is the positive root of the algebraic equation

$$\tilde{\varepsilon}^2 + \left(1 + \frac{\alpha}{\varkappa \tilde{\chi}_M} - \frac{\rho k \sigma}{2a T_0 \tilde{\chi}_M} - \frac{3\beta^2}{a \tilde{\chi}_M} \right) \tilde{\varepsilon} - \frac{3\beta^2}{a \tilde{\chi}_M} = 0 \quad (66)$$

Thus, from (64) we obtain the following estimate

$$\Pi(r, t) \leq \Pi(0, t) \exp \left(-\frac{\sigma}{\tilde{c}_1} r \right) \quad \text{for all } (r, t) \in I \times [0, T] \quad (67)$$

for a bounded body. While for an unbounded body we can conclude that either $\Pi(r, t) \geq 0$ for all $(r, t) \in I \times [0, T]$ and then a spatial decay estimate of the type described by (67) holds true, or there exists a value $\tilde{r}_t^0 \geq 0$ such that $\Pi(\tilde{r}_t^0, t) < 0$ and then we have

$$-\Pi(r, t) \geq -\Pi(\tilde{r}_t^0, t) \exp \left[\frac{\sigma}{\tilde{c}_1} (r - \tilde{r}_t^0) \right] \geq 0 \quad \text{for all } (r, t) \in [\tilde{r}_t^0, \infty) \times [0, T] \quad (68)$$

Furthermore, by assuming that $\Pi(r, t) \geq 0$ for all $(r, t) \in I \times [0, T]$, we can introduce the new measure

$$\Pi_1(r, t) = \int_r^\infty \Pi(\eta, t) d\eta \quad (69)$$

so that we are lead to the counterpart of (50) in the form

$$\Pi_1(r, t) \leq \Pi_1(0, t) \sqrt{\frac{kt}{aT_0\pi}} \frac{2r}{r^2 - \tilde{c}_2^2 t^2} e^{-\frac{aT_0\tilde{c}_2^2}{4k}t} e^{\frac{aT_0\tilde{c}_2}{2k}r - \frac{aT_0}{4kt}r^2} \quad \text{for } r > \tilde{c}_2 t, \quad t \in [0, T] \quad (70)$$

where

$$\tilde{c}_2 = \sqrt{\frac{1}{\rho} \left[\tilde{\chi}_M (1 + \tilde{\varepsilon}_0) + \frac{\alpha}{\varkappa} \right]} \quad \tilde{\varepsilon}_0^2 + \left(1 + \frac{\alpha}{\varkappa \tilde{\chi}_M} - \frac{3\beta^2}{a\tilde{\chi}_M} \right) \tilde{\varepsilon}_0 - \frac{3\beta^2}{a\tilde{\chi}_M} = 0 \quad (71)$$

SPATIAL BEHAVIOUR OF THE STEADY-STATE SOLUTIONS

Here we discuss the problem of spatial behaviour of the steady vibrations. For this purpose we stipulate that the region B from here on refers to the interior of a prismatic cylinder with parallel plane ends. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that one end of the cylinder lies in the $x_1 O x_2$ plane and contains the origin. We suppose that the length of the cylinder is L and that D_{x_3} represents the bounded cross-section at distance x_3 from the $x_1 O x_2$ plane. The boundary ∂D of each cross-section is assumed to be a piecewise smooth simple closed curve. The Greek subscripts range over 1,2. In what follows we shall assume that the external body supplies are absent. Moreover, we assume the following boundary conditions for all $t \in [0, T]$

$$u_j = 0 \quad \phi = 0 \quad \theta = 0 \quad \text{on } (\partial D \times [0, L]) \cup D_L \quad t \in [0, T] \quad (72)$$

$$\begin{aligned} \mu(u_{3,j} + u_{j,3}) + (\lambda u_{r,r} + b\phi - \beta\theta)\delta_{j3} &= \tilde{t}_j e^{i\omega t} \\ \alpha\phi_{,3} &= \tilde{h} e^{i\omega t} \quad k\theta_{,3} = \tilde{q} e^{i\omega t} \quad \text{for all } (x_1, x_2, x_3) \in D \times \{0\} \quad t \in [0, T] \end{aligned} \quad (73)$$

where $\tilde{t}_j$, $\tilde{h}$, and $\tilde{q}$ are prescribed functions on $(x_1, x_2) \in D$ and ω is a positive prescribed parameter. Other boundary conditions can be viewed but essentially for our considerations is to conserve the boundary condition (72). In this section we discuss the problem of steady-state vibrations assuming that

$$u_j = \Re[e[v_j(\mathbf{x}; \omega)e^{i\omega t}]] \quad \phi = \Re[\psi(\mathbf{x}; \omega)e^{i\omega t}] \quad \theta = \Re[\vartheta(\mathbf{x}; \omega)e^{i\omega t}] \quad (74)$$

where $\Re[f]$ represents the real part of f . Then the equations of motion (6) reduce to

$$\mu v_{j,rr} + (\lambda + \mu)v_{r,rj} + b\psi_{,j} - \beta\vartheta_{,j} + \rho\omega^2 v_j = 0 \quad (75)$$

$$\alpha\psi_{,rr} - b v_{r,r} - \xi\psi + m\vartheta + \rho\varkappa\omega^2 \psi = 0 \quad (76)$$

$$k\vartheta_{,rr} - i\omega\beta T_0 v_{r,r} - i\omega a T_0 \vartheta - i\omega m T_0 \psi = 0 \quad (77)$$

while the boundary conditions (72) and (73) reduce to

$$v_j = 0 \quad \psi = 0 \quad \vartheta = 0 \quad \text{on } (\partial D \times [0, L]) \cup D_L \quad (78)$$

$$\begin{aligned} \mu(v_{3,j} + v_{j,3}) + (\lambda v_{r,r} + b\psi - \beta\vartheta)\delta_{j3} &= \tilde{t}_j \\ \alpha\psi_{,3} &= \tilde{h} \quad k\vartheta_{,3} = \tilde{q} \quad \text{for all } (x_1, x_2, x_3) \in D \times \{0\} \end{aligned} \quad (79)$$

In what follows, we will study the spatial behaviour of the amplitude (v_j, ψ, ϑ) of the harmonic vibration described by the relation (74). To this end, we note that (v_j, ψ, ϑ) is the solution of the boundary value problem $\mathcal{P}_0$ defined by the Eqs. (75)–(77) and by the boundary conditions (78) and (79).

First Measure

Throughout this subsection we will assume that the hypotheses described by the relation (10) hold true. Then we associate with the solution (v_j, ψ, ϑ) of the problem $\mathcal{P}_0$ the function $\mathcal{J}$ on $[0, L]$ defined by

$$\begin{aligned} \mathcal{J}(x_3) = & \int_{D_{x_3}} \left[\mu v_r \bar{v}_r + (\lambda + \mu) v_3 \bar{v}_3 + \alpha \psi \bar{\psi} + \frac{\delta}{T_0} k \vartheta \bar{\vartheta} \right] da \\ & - \int_{B_{x_3}} [-(\lambda + \mu)(v_\rho \bar{v}_{3,\rho} + \bar{v}_\rho v_{3,\rho}) + b(\psi \bar{v}_3 + \bar{\psi} v_3) - \beta(v_3 \bar{\vartheta} + \bar{v}_3 \vartheta)] dv \\ & x_3 \in [0, L], \end{aligned} \quad (80)$$

where a superposed bar denotes the complex conjugate and δ is a prescribed parameter at our hand and $B_{x_3} = D \times [x_3, L]$. We have to note that the boundary condition (78) implies that

$$\mathcal{J}(L) = 0 \quad (81)$$

In what follows we will prove that this function is convex on $[0, L]$, for appropriate values of the parameters δ and ω . To this end, we note that

$$\begin{aligned} \mathcal{J}'(x_3) = & \int_{D_{x_3}} \left[\mu(\bar{v}_r v_{r,3} + v_r \bar{v}_{r,3}) + (\lambda + \mu)(v_3 \bar{v}_{r,r} + \bar{v}_3 v_{r,r}) + b(v_3 \bar{\psi} + \bar{v}_3 \psi) \right. \\ & \left. - \beta(v_3 \bar{\vartheta} + \bar{v}_3 \vartheta) + \alpha(\psi \bar{\psi}_{,3} + \bar{\psi}_{,3} \psi) + \frac{\delta}{T_0} k(\vartheta \bar{\vartheta}_{,3} + \bar{\vartheta}_{,3} \vartheta) \right] da \\ & x_3 \in [0, L] \end{aligned} \quad (82)$$

and, by the relation (78), we have

$$\mathcal{J}'(L) = 0 \quad (83)$$

Furthermore, on the basis of Eqs. (75) to (77) and by using the boundary conditions (78), we get

$$\begin{aligned} \mathcal{J}''(x_3) = & \int_{D_{x_3}} \left\{ 2w + 2\alpha \psi_{,r} \bar{\psi}_{,r} + \frac{2}{T_0} \delta k \vartheta_{,r} \bar{\vartheta}_{,r} - 2\omega^2(\rho v_r \bar{v}_r + \rho \varkappa \psi \bar{\psi}) \right. \\ & - \beta[(1 - i\omega\delta)v_{r,r} \bar{\vartheta} + (1 + i\omega\delta)\bar{v}_{r,r} \vartheta] \\ & \left. - m[(1 - i\omega\delta)\psi \bar{\vartheta} + (1 + i\omega\delta)\bar{\psi} \vartheta] \right\} da \end{aligned} \quad (84)$$

where

$$w = w_1 + w_2 \quad (85)$$

$$\begin{aligned} w_1 = & (\lambda + 2\mu)(v_{1,1}\bar{v}_{1,1} + v_{2,2}\bar{v}_{2,2} + v_{3,3}\bar{v}_{3,3}) \\ & + (\lambda + \mu)(v_{1,1}\bar{v}_{2,2} + \bar{v}_{1,1}v_{2,2} + v_{2,2}\bar{v}_{3,3} + \bar{v}_{2,2}v_{3,3} + v_{3,3}\bar{v}_{1,1} + \bar{v}_{3,3}v_{1,1}) \\ & + \xi\psi\bar{\psi} + b[\bar{\psi}(v_{1,1} + v_{2,2} + v_{3,3}) + \psi(\bar{v}_{1,1} + \bar{v}_{2,2} + \bar{v}_{3,3})] \end{aligned} \quad (86)$$

$$w_2 = \mu(v_{1,2}\bar{v}_{1,2} + v_{2,1}\bar{v}_{2,1} + v_{2,3}\bar{v}_{2,3} + v_{3,2}\bar{v}_{3,2} + v_{3,1}\bar{v}_{3,1} + v_{1,3}\bar{v}_{1,3}) \quad (87)$$

On the basis of the assumption (10) we deduce that

$$w_1 \geq \chi_m(v_{1,1}\bar{v}_{1,1} + v_{2,2}\bar{v}_{2,2} + v_{3,3}\bar{v}_{3,3} + \psi\bar{\psi}) \quad (88)$$

where

$$\chi_m = \min \left\{ \mu, \frac{1}{2}[\xi + 4\mu + 3\lambda - \sqrt{[\xi - (4\mu + 3\lambda)]^2 + 12b^2}] \right\} \quad (89)$$

Moreover, we deduce that

$$w \geq \chi_m(v_{j,k}\bar{v}_{j,k} + \psi\bar{\psi}) \quad (90)$$

In view of the boundary conditions (78), we have

$$\begin{aligned} \lambda_1 \int_{D_{x_3}} v_r \bar{v}_r da & \leq \int_{D_{x_3}} v_{r,\alpha} \bar{v}_{r,\alpha} da, \\ \lambda_1 \int_{D_{x_3}} \vartheta \bar{\vartheta} da & \leq \int_{D_{x_3}} \vartheta_{,\alpha} \bar{\vartheta}_{,\alpha} da, \quad \lambda_1 \int_{D_{x_3}} \psi \bar{\psi} da \leq \int_{D_{x_3}} \psi_{,\alpha} \bar{\psi}_{,\alpha} da \end{aligned} \quad (91)$$

where λ_1 is the first eigenvalue corresponding to the membrane problem for the section D_{x_3} .

Then, the relations (90) and (91) imply

$$\begin{aligned} \mathcal{J}''(x_3) \geq & \int_{D_{x_3}} \left\{ 2\chi_m v_{r,k} \bar{v}_{r,k} + 2(\chi_m + \alpha\lambda_1)\psi\bar{\psi} + \frac{2}{T_0}\delta k \lambda_1 \vartheta \bar{\vartheta} \right. \\ & - \beta[(1 - i\omega\delta)v_{r,r}\bar{\vartheta} + (1 + i\omega\delta)\bar{v}_{r,r}\vartheta] \\ & \left. - m[(1 - i\omega\delta)\psi\bar{\vartheta} + (1 + i\omega\delta)\bar{\psi}\vartheta] - 2\omega^2 \left(\frac{\rho}{\lambda_1} v_{r,k} \bar{v}_{r,k} + \rho \varkappa \psi \bar{\psi} \right) \right\} da \end{aligned} \quad (92)$$

Furthermore, from (92) we get

$$\begin{aligned} \mathcal{J}''(x_3) \geq & \int_{D_{x_3}} \left\{ \chi_m \left[v_{r,k} - \frac{\beta}{\chi_m} (1 + i\omega\delta) \vartheta \delta_{rk} \right] \left[\bar{v}_{r,k} - \frac{\beta}{\chi_m} (1 - i\omega\delta) \bar{\vartheta} \delta_{rk} \right] \right. \\ & + (\chi_m + \alpha\lambda_1) \left[\psi - \frac{m}{\chi_m + \alpha\lambda_1} (1 + i\omega\delta) \vartheta \right] \left[\bar{\psi} - \frac{m}{\chi_m + \alpha\lambda_1} (1 - i\omega\delta) \bar{\vartheta} \right] \\ & \left. + M_1 v_{r,k} \bar{v}_{r,k} + M_2 \psi \bar{\psi} + M_3 \vartheta \bar{\vartheta} \right\} da \end{aligned} \quad (93)$$

where

$$\begin{aligned} M_1 &= \chi_m - \frac{2\rho}{\lambda_1} \omega^2 & M_2 &= \chi_m + \alpha\lambda_1 - 2\rho\kappa\omega^2 \\ M_3 &= \frac{2}{T_0} \delta k \lambda_1 - \frac{3\beta^2}{\chi_m} - \frac{m^2}{\chi_m + \alpha\lambda_1} - \omega^2 \delta^2 \left(\frac{3\beta^2}{\chi_m} + \frac{m^2}{\chi_m + \alpha\lambda_1} \right) \end{aligned} \quad (94)$$

Concluding from (93), we can see that

$$\mathcal{F}''(x_3) \geq 0 \quad \text{for all } x_3 \in [0, L] \quad (95)$$

if we choose the parameter δ so that

$$\delta \geq \frac{T_0}{2k\lambda_1} \left(\frac{3\beta^2}{\chi_m} + \frac{m^2}{\chi_m + \alpha\lambda_1} \right) \quad (96)$$

and assume that the frequency ω of the given vibration satisfies

$$\omega < \omega_0 \quad (97)$$

where

$$\omega_0 = \min \left\{ \sqrt{\frac{\chi_m \lambda_1}{2\rho}}, \sqrt{\frac{\chi_m + \alpha\lambda_1}{2\rho\kappa}}, \sqrt{\frac{2k\lambda_1 \chi_m (\chi_m + \alpha\lambda_1)}{T_0 \delta [\chi_m (3\beta^2 + m^2) + 3\alpha\beta^2 \lambda_1]} - \frac{1}{\delta^2}} \right\} \quad (98)$$

Throughout this subsection we will assume the relations (96) and (97) hold true. Then the relation (95) proves that $\mathcal{F}'(x_3)$ is a non-decreasing function with respect to x_3 on $[0, L]$ so that, by using (83), we obtain

$$\mathcal{F}(x_3) \leq \mathcal{F}(L) = 0 \quad \text{for all } x_3 \in [0, L] \quad (99)$$

Thus, $\mathcal{F}(x_3)$ is a non-increasing function on $[0, L]$ and hence, by using (81), we get

$$\mathcal{F}(x_3) \geq 0 \quad \text{for all } x_3 \in [0, L] \quad (100)$$

On the other hand, by using the relations (83) and (93) we get

$$-\mathcal{F}'(x_3) \geq \int_{B_{x_3}} [M_1 v_{r,k} \bar{v}_{r,k} + M_2 \psi \bar{\psi} + M_3 \vartheta \bar{\vartheta}] dv \quad (101)$$

Now, by using the arithmetic–geometric inequality and the Schwarz inequality, from (80) we obtain

$$\begin{aligned} 0 \leq \mathcal{F}(x_3) &\leq \int_{D_{x_3}} \left[\max\{\mu, \lambda + 2\mu\} v_r \bar{v}_r + \alpha \psi \bar{\psi} + \frac{\delta}{T_0} k \vartheta \bar{\vartheta} \right] da \\ &\quad + \int_{B_{x_3}} \left[\frac{|\lambda + \mu|}{\sqrt{\lambda_1}} (\lambda_1 v_\rho \bar{v}_\rho + v_{3,\rho} \bar{v}_{3,\rho}) + \frac{|b|}{\sqrt{\lambda_1}} (\lambda_1 v_3 \bar{v}_3 + \psi \bar{\psi}) \right. \\ &\quad \left. + \frac{|\beta| T_0}{\sqrt{\lambda_1}} (\lambda_1 v_3 \bar{v}_3 + \frac{1}{T_0^2} \vartheta \bar{\vartheta}) \right] dv \end{aligned} \quad (102)$$

Further, we shall use the inequalities described in (91) to obtain

$$0 \leq \mathcal{F}(x_3) \leq \int_{D_{x_3}} \left(\frac{\max\{\mu, \lambda + 2\mu\}}{\lambda_1} v_{r,k} \bar{v}_{r,k} + \alpha \psi \bar{\psi} + \frac{\delta}{T_0} k \vartheta \bar{\vartheta} \right) da \\ + \int_{B_{x_3}} (m_1 v_{r,k} \bar{v}_{r,k} + m_2 \psi \bar{\psi} + m_3 \vartheta \bar{\vartheta}) dv \quad (103)$$

where

$$m_1 = \frac{1}{\sqrt{\lambda_1}} (|\lambda + \mu| + |b| + |\beta| T_0) \quad m_2 = \frac{|b|}{\sqrt{\lambda_1}} \quad m_3 = \frac{|\beta|}{T_0 \sqrt{\lambda_1}} \quad (104)$$

By combining the relations (93), (101), and (103) we get the following second order differential inequality

$$\mathcal{F}(x_3) + c_3 \mathcal{F}'(x_3) \leq c_4 \mathcal{F}''(x_3) \quad \text{for all } x_3 \in [0, L] \quad (105)$$

with

$$c_3 = \max \left\{ \frac{m_1}{M_1}, \frac{m_2}{M_2}, \frac{m_3}{M_3} \right\} \quad (106)$$

$$c_4 = \max \left\{ \frac{\max\{\mu, \lambda + 2\mu\}}{\lambda_1 M_1}, \frac{\alpha}{M_2}, \frac{\delta k}{T_0 M_3} \right\} \quad (107)$$

It follows from the inequality (105) and a well-known Comparison Principle (a generalisation of the curve under chord property for convex functions [19]) that $\mathcal{F}(x_3)$ is bounded above by $\mathcal{F}^*(x_3)$, the solution of the associated differential equation with the same boundary conditions, that is

$$\mathcal{F}^{*''}(x_3) - \frac{c_3}{c_4} \mathcal{F}^{*'}(x_3) - \frac{1}{c_4} \mathcal{F}^*(x_3) = 0 \\ \mathcal{F}^*(0) = \mathcal{F}(0) \quad \mathcal{F}^*(L) = 0 \quad (108)$$

On this basis we obtain

$$0 \leq \mathcal{F}(x_3) \leq \frac{1 - e^{(r_2 - r_1)(L - x_3)}}{1 - e^{(r_2 - r_1)L}} \mathcal{F}(0) e^{r_2 x_3} \quad \text{for all } x_3 \in [0, L] \quad (109)$$

where

$$r_1 = \frac{1}{2c_4} \left(c_3 + \sqrt{c_3^2 + 4c_4} \right) \quad r_2 = \frac{1}{2c_4} \left(c_3 - \sqrt{c_3^2 + 4c_4} \right) \quad (110)$$

This last result gives

$$0 \leq \mathcal{F}(x_3) \leq \mathcal{F}(0) e^{r_2 x_3} \quad \text{for all } x_3 \in [0, L] \quad (111)$$

provided the relations (96) and (97) hold true.

Second Measure

In the remainder of this paper we shall assume that the hypotheses described by the relation (51) hold true. Then we introduce the following measure

$$\begin{aligned}\mathcal{K}(x_3) = & \int_{D_{x_3}} \left[\mu v_r \bar{v}_r + (\lambda + \mu) v_3 \bar{v}_3 + \alpha \psi \bar{\psi} + \frac{\delta}{T_0} k \vartheta \bar{\vartheta} \right] da \\ & - \int_{B_{x_3}} [(\lambda + \mu)(v_\rho \bar{v}_{3,\rho} + \bar{v}_\rho v_{3,\rho}) + b(v_3 \bar{\psi} + \bar{v}_3 \psi) - \beta(v_3 \bar{\vartheta} + \bar{v}_3 \vartheta)] dv \\ & x_3 \in [0, L]\end{aligned}\quad (112)$$

where δ is a positive parameter at our hand. Furthermore, we have

$$\begin{aligned}\mathcal{K}''(x_3) = & \int_{D_{x_3}} \left\{ 2w^* + 2\alpha \psi_{,r} \bar{\psi}_{,r} + \frac{2}{T_0} \delta k \vartheta_{,r} \bar{\vartheta}_{,r} - 2\omega^2(\rho v_r \bar{v}_r + \rho \varkappa \psi \bar{\psi}) \right. \\ & - \beta[(1 - i\omega\delta)v_{r,r} \bar{\vartheta} + (1 + i\omega\delta)\bar{v}_{r,r} \vartheta] \\ & \left. - m[(1 - i\omega\delta)\psi \bar{\vartheta} + (1 + i\omega\delta)\bar{\psi} \vartheta] \right\} da\end{aligned}\quad (113)$$

where

$$w^* = w_1^* + w_2^* \quad (114)$$

$$\begin{aligned}w_1^* = & (\lambda + 2\mu)(v_{1,1} \bar{v}_{1,1} + v_{2,2} \bar{v}_{2,2} + v_{3,3} \bar{v}_{3,3}) \\ & + b[\psi(\bar{v}_{1,1} + \bar{v}_{2,2} + \bar{v}_{3,3}) + \bar{\psi}(v_{1,1} + v_{2,2} + v_{3,3})] + \xi \psi \bar{\psi}\end{aligned}\quad (115)$$

$$\begin{aligned}w_2^* = & \mu(v_{1,2} \bar{v}_{1,2} + v_{2,1} \bar{v}_{2,1} + v_{2,3} \bar{v}_{2,3} + v_{3,2} \bar{v}_{3,2} + v_{3,1} \bar{v}_{3,1} + v_{1,3} \bar{v}_{1,3}) \\ & + (\lambda + \mu)(v_{1,2} \bar{v}_{2,1} + v_{2,1} \bar{v}_{1,2} + v_{2,3} \bar{v}_{3,2} + v_{3,2} \bar{v}_{2,3} + v_{3,1} \bar{v}_{1,3} + v_{1,3} \bar{v}_{3,1})\end{aligned}\quad (116)$$

On the basis of the assumption (51) we deduce that

$$\begin{aligned}w_1^* \geq & \check{\chi}_m(v_{1,1} \bar{v}_{1,1} + v_{2,2} \bar{v}_{2,2} + v_{3,3} \bar{v}_{3,3} + \psi \bar{\psi}) \\ w_2^* \geq & \hat{\chi}_m(v_{1,2} \bar{v}_{1,2} + v_{2,1} \bar{v}_{2,1} + v_{2,3} \bar{v}_{2,3} + v_{3,2} \bar{v}_{3,2} + v_{3,1} \bar{v}_{3,1} + v_{1,3} \bar{v}_{1,3})\end{aligned}\quad (117)$$

where

$$\begin{aligned}\check{\chi}_m = & \min\{-\lambda, \lambda + 2\mu\} \\ \hat{\chi}_m = & \min\left\{\lambda + 2\mu, \frac{1}{2}[\xi + 4\mu + 3\lambda - \sqrt{[\xi - (4\mu + 3\lambda)]^2 + 12b^2}]\right\}\end{aligned}\quad (118)$$

Thus, we deduce that

$$w \geq \tilde{\chi}_m(v_{j,k} \bar{v}_{j,k} + \psi \bar{\psi}) \quad \tilde{\chi}_m = \min\{\check{\chi}_m, \hat{\chi}_m\} \quad (119)$$

Furthermore, from (113) we get

$$\mathcal{K}''(x_3) \geq \int_{D_{x_3}} \{\tilde{M}_1 v_{j,k} \bar{v}_{j,k} + \tilde{M}_2 \psi \bar{\psi} + \tilde{M}_3 \vartheta \bar{\vartheta}\} da \quad (120)$$

where

$$\begin{aligned}\tilde{M}_1 &= \tilde{\chi}_m - \frac{2\rho}{\lambda_1} \omega^2 & \tilde{M}_2 &= \tilde{\chi}_m + \alpha\lambda_1 - 2\rho\kappa\omega^2 \\ \tilde{M}_3 &= \frac{2}{T_0} \delta k \lambda_1 - \frac{3\beta^2}{\tilde{\chi}_m} - \frac{m^2}{\tilde{\chi}_m + \alpha\lambda_1} - \omega^2 \delta^2 \left(\frac{3\beta^2}{\tilde{\chi}_m} + \frac{m^2}{\tilde{\chi}_m + \alpha\lambda_1} \right)\end{aligned}\quad (121)$$

Thus, we have

$$\mathcal{H}''(x_3) \geq 0 \quad \text{for all } x_3 \in [0, L] \quad (122)$$

provided that

$$\delta \geq \frac{T_0}{2k\lambda_1} \left(\frac{3\beta^2}{\tilde{\chi}_m} + \frac{m^2}{\tilde{\chi}_m + \alpha\lambda_1} \right) \quad (123)$$

and

$$\omega < \tilde{\omega}_0 \quad (124)$$

with

$$\tilde{\omega}_0 = \min \left\{ \sqrt{\frac{\tilde{\chi}_m \lambda_1}{2\rho}}, \sqrt{\frac{\tilde{\chi}_m + \alpha\lambda_1}{2\rho\kappa}}, \sqrt{\frac{2k\lambda_1 \tilde{\chi}_m (\tilde{\chi}_m + \alpha\lambda_1)}{T_0 \delta [\tilde{\chi}_m (3\beta^2 + m^2) + 3\alpha\beta^2 \lambda_1]} - \frac{1}{\delta^2}} \right\} \quad (125)$$

Therefore, as in the above subsection we can obtain the following spatial estimate

$$0 \leq \mathcal{H}(x_3) \leq \mathcal{H}(0) e^{\tilde{r}_2 x_3} \quad \text{for all } x_3 \in [0, L] \quad (126)$$

where

$$\begin{aligned}\tilde{r}_2 &= \frac{1}{2\tilde{c}_4} (\tilde{c}_3 - \sqrt{\tilde{c}_3^2 + 4\tilde{c}_4}) \\ \tilde{c}_3 &= \max \left\{ \frac{m_1}{\tilde{M}_1}, \frac{m_2}{\tilde{M}_2}, \frac{m_3}{\tilde{M}_3} \right\} & \tilde{c}_4 &= \max \left\{ \frac{\max\{\mu, \lambda + 2\mu\}}{\lambda_1 \tilde{M}_1}, \frac{\alpha}{\tilde{M}_2}, \frac{\delta k}{T_0 \tilde{M}_3} \right\}\end{aligned}\quad (127)$$

CONCLUDING REMARKS

In the present paper we introduced the new measures (11), (32), (52), (69), and (80), (112) in order to study the spatial behaviour of the transient and steady-state solutions, respectively. The measures defined by the relations (11), (32), and (80) are suitable for the class of porous elastic materials described by the relation (10); while the measures (52), (69), and (112) are useful in the class described by the relation (51). These classes of porous elastic materials are not coincident with the class defined by (5) and treated in [8, 9]. The method presented here is believed to be used successfully for many novel foam structures with negative Poisson's ratio, because the relations (10) and (51) can include such materials, not included in the class defined by (5).

On the other hand, we have to outline that we can combine our estimates (30), (50), (67), (70), (111), and (126) with those predicted in [8, 9] in order to obtain complete information about the spatial behaviour of the transient and steady-state solutions.

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