Structural Stability in Porous Elasticity

Author(s): S. Chiriță, M. Ciarletta and B. Straughan

Source: *Proceedings: Mathematical, Physical and Engineering Sciences*, Vol. 462, No. 2073 (Sep. 8, 2006), pp. 2593-2605

Published by: [The Royal Society](#)

Stable URL: <http://www.jstor.org/stable/20209027>

Accessed: 18/06/2014 17:24

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
<http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

The Royal Society is collaborating with JSTOR to digitize, preserve and extend access to *Proceedings: Mathematical, Physical and Engineering Sciences*.

<http://www.jstor.org>

Structural stability in porous elasticity

BY S. CHIRIȚĂ^{1,*}, M. CIARLETTA² AND B. STRAUGHAN³

¹*Faculty of Mathematics, Al. I. Cuza University, 700506 Iași, Romania*

²*Department of Engineering Information and Applied Mathematics,
University of Salerno, 84084 Fisciano (SA), Italy*

³*Department of Mathematical Sciences, Durham University,
Durham DH1 3LE, UK*

We consider the linearized system of equations for an elastic body with voids as derived by Cowin & Nunziato. We demonstrate that the solution depends continuously on changes in the coefficients, which couple the equations of elastic deformation and of voids. It is also shown that the solution to the coupled system converges, in an appropriate measure, to the solutions of the uncoupled systems as the coupling coefficients tend to zero.

Keywords: porous materials; energy bounds; structural stability; convergence

1. Introduction

Porous materials have applications in almost all fields of engineering, e.g. soil mechanics, petroleum industry, material science as well as in biomechanics. A review of the historical development of the porous media theories as well as reference to various contributions may be found in the monographs by Bowen (1976) and Ieșan (2004) and in the articles by Bedford & Drumheller (1983) and De Boer (1998). The theory of elastic materials with voids is one of the simple extensions of the classical theory of elasticity for the treatment of porous solids in which the matrix material is elastic and the interstices are void of material. Such theory seems to be an adequate tool to describe the behaviour of granular materials like rock, soils and manufactured porous bodies. The theory of elastic materials with voids has been developed by Nunziato & Cowin (1979) and Cowin & Nunziato (1983) and has received considerable interest in recent years. In fact, Goodman & Cowin (1972) have developed a continuum theory of granular materials with interstitial voids. The basic concept underlying this theory is that of a material for which the bulk density is written as the product of two fields, the density field of the matrix material and the volume fraction field. Such a representation was used by Nunziato & Cowin (1979) in order to develop a nonlinear theory of elastic materials with voids. The intended applications of the theory are to elastic bodies with small voids or vacuum pores which are distributed throughout the material. The linear theory of elastic materials with voids has been established by Cowin & Nunziato (1983).

* Author for correspondence (schirita@uaic.ro).

Received 18 October 2005
Accepted 15 February 2006

2593

© 2006 The Royal Society

In the present paper, we study the structural stability of the mathematical model of the linear elastic material with voids. One of the most important tasks in the study of the structural stability is to prove that the solutions of problems depend continuously on the constitutive quantities, which may be subjected to error or perturbations in the mathematical modelling process. Concerning the structural stability, we emphasize the continuous dependence on changes in the model itself rather than on the initial and given data. That means changes in coefficients in the partial differential equations and changes in the equations and may be reflected physically by changes in constitutive parameters (as, for example, the coefficients obtained in a small deformation superposed to a finite one). Estimates of continuous dependence play a central role in obtaining numerical approximations to these kinds of problems.

Since many physical phenomena can be modelled by idealized approaches, the derivation of continuous dependence inequalities on various types of data has attracted considerable attention. The structural stability was a subject of great interest in recent years. In this connection, we point out that many studies of this type have been inspired by the fundamental paper of Knops & Payne (1969), where such investigations were initiated in elasticity, cf. also Knops & Wilkes (1973, sections 73, 74) and Knops & Payne (1988). We also recall structural stability analyses in the book by Ames & Straughan (1997) and in a series of papers by Ames & Payne (1995), Franchi & Straughan (1996), Payne & Straughan (1996, 1998), Payne & Song (1997) and Quintanilla (2003).

Throughout this paper we restrict our attention to the case of a centrosymmetric inhomogeneous linear elastic material with voids and discuss the continuous dependence of solution with respect to the coupling coefficients of the model. In §2, we formulate the corresponding initial-boundary value problem in question and present some constitutive assumptions. Section 3 is devoted to obtain *a priori* estimates for some auxiliary static problems. While §4 gives some *a priori* bounds for the solution of the dynamic problem. The continuous dependence of the solution with respect to the coupling coefficients in question is established in §5. In §6, we investigate how the solution of the basic initial-boundary value problem behaves as the coupling coefficients tend to zero.

2. Equations of motion

Throughout this paper we will consider, for convenience, a centrosymmetric elastic material with voids. Consequently, the general linear equations, from Ciarletta & Ieșan (1993; eqns (7.3.7), (7.3.12), (7.3.13)) are (see also Cowin (1985))

$$\varrho \ddot{u}_i = (a_{ijkh} u_{k,h})_{,j} + (b_{ij} \phi)_{,j}, \quad (2.1)$$

$$\varrho \chi \ddot{\phi} = (a_{ij} \phi_{,j})_{,i} - \tau \dot{\phi} - b_{ij} u_{i,j} - \xi \phi, \quad (2.2)$$

in $\mathcal{Q} \times (0, T)$, with boundary condition on Γ

$$u_i = g_i(\mathbf{x}, t), \quad \phi = h(\mathbf{x}, t), \quad (2.3)$$

and initial data

$$\left. \begin{aligned} u_i(\mathbf{x}, 0) &= u_i^0(\mathbf{x}), & \phi(\mathbf{x}, 0) &= \phi^0(\mathbf{x}), \\ \dot{u}_i(\mathbf{x}, 0) &= v_i^0(\mathbf{x}), & \dot{\phi}(\mathbf{x}, 0) &= \zeta^0(\mathbf{x}). \end{aligned} \right\} \quad (2.4)$$

The mass density ρ and the equilibrated inertia κ and the constitutive coefficients a_{ijkh} , b_{ij} , a_{ij} , τ and ξ are prescribed functions of $\mathbf{x}$, supposed to be as smooth as required in our subsequent analysis. Moreover, we require

$$a_{ijkh} = a_{khij}, \quad a_{ij} = a_{ji}, \quad b_{ij} = b_{ji},$$

$$\varrho, \kappa > 0, \quad \tau \geq 0, \quad (2.5)$$

$$a_{ijkh} \xi_{ij} \xi_{kh} \geq a_1 |\xi|^2, \quad a_1 > 0, \quad (2.6)$$

$$a_{ij} \xi_i \xi_j \geq a_2 |\xi|^2, \quad a_2 > 0. \quad (2.7)$$

We also assume that the energy density E is a positive definite quadratic form, i.e.

$$E \equiv \frac{1}{2} a_{ijkh} u_{i,j} u_{k,h} + \frac{1}{2} \xi \phi^2 + \frac{1}{2} a_{ij} \phi_{,i} \phi_{,j} + b_{ij} u_{i,j} \phi \geq \frac{a_0}{2} (|\nabla \mathbf{u}|^2 + |\phi|^2) + \frac{a_2}{2} |\nabla \phi|^2, \quad a_0 > 0. \quad (2.8)$$

In the above relations a_0 , a_1 and a_2 are appropriate constants.

The surface of the region is supposed to be as smooth as required, and smooth solutions are envisaged throughout.

3. Rellich identities

Before proceeding to derive an *a priori* estimate for a solution to equations (2.1)–(2.4), we need some bounds for a solution to certain auxiliary problems. These are achieved via the use of Rellich-like identities, cf. Payne & Weinberger (1958) and Bramble & Payne (1961, 1963).

Let H be a solution to the problem

$$\left. \begin{aligned} (a_{ij} H_{j,i})_{,i} &= 0, & \text{in } \Omega, \\ H &= q, & \text{on } \Gamma. \end{aligned} \right\} \quad (3.1)$$

Under appropriate regularity assumptions, the existence of the solution H is assured by means of the existence theory presented by Fichera (1972).

We form the identity

$$0 = \int_{\Omega} x_k H_{,k} (a_{ij} H_{j,i})_{,i} dx,$$

and then integrate by parts in succession to find

$$\begin{aligned}
 0 &= -\int_{\Omega} x_{k,i} H_{,k} a_{ij} H_{,j} dx - \int_{\Omega} x_k H_{,ki} a_{ij} H_{,j} dx + \oint_{\Gamma} n_i x_k H_{,k} a_{ij} H_{,j} dA \\
 &= -\int_{\Omega} a_{ij} H_{,i} H_{,j} dx - \frac{1}{2} \int_{\Omega} x_k a_{ij} (H_{,i} H_{,j})_{,k} dx + \oint_{\Gamma} n_i x_k H_{,k} a_{ij} H_{,j} dA \\
 &= \frac{1}{2} \int_{\Omega} a_{ij} H_{,i} H_{,j} dx + \frac{1}{2} \int_{\Omega} x_k a_{ij,k} H_{,i} H_{,j} dx + \oint_{\Gamma} n_i x_k H_{,k} a_{ij} H_{,j} dA \\
 &\quad - \frac{1}{2} \oint_{\Gamma} x_k n_k a_{ij} H_{,i} H_{,j} dA.
 \end{aligned} \tag{3.2}$$

We now write $H_{,i}$ on the surface Γ as $H_{,i} = n_i \partial H / \partial n + s_i \nabla_s H$, where n_i denotes the unit normal to Γ and s_i is the tangential vector. In addition, $\partial / \partial n$ is the normal derivative and ∇_s is the tangential derivative, i.e. $\nabla_s H = x_{i,\alpha} a^{\alpha\beta} H_{,\beta}$, where $a^{\alpha\beta}$ are the coefficients of the first fundamental form of the surface Γ , and the subscript ' $_{,\alpha}$ ' denotes differentiation with respect to the surface variables θ^α .

Suppose now also that Γ is star shaped with respect to an origin so that

$$x_k n_k \geq h_0 > 0, \quad \text{on } \Gamma, \tag{3.3}$$

and

$$|x_k s_k| \leq \delta_0, \quad |x_k a_{ij,k}| \leq a_s < \mu a_2,$$

for $0 < \mu < 1$, where δ_0 , a_s and μ are constants. Obviously (3.3) places a restriction on the size of the gradients of a_{ij} , but this is not a severe restriction. Then, from (3.2), we may find

$$\begin{aligned}
 &\frac{1}{2} (1 - \mu) \int_{\Omega} a_{ij} H_{,i} H_{,j} dx + \frac{h_0 a_2}{2} \oint_{\Gamma} \left(\frac{\partial H}{\partial n} \right)^2 dA \\
 &\leq \oint_{\Gamma} \left(\frac{1}{2} x_k n_k s_i s_j - n_i s_j s_k x_k \right) a_{ij} |\nabla_s q|^2 dA \\
 &\quad + \frac{\delta_0 \alpha}{2} \oint_{\Gamma} a_{ij} n_i n_j \left(\frac{\partial H}{\partial n} \right)^2 dA + \frac{\delta_0}{2\alpha} \oint_{\Gamma} a_{ij} n_i n_j |\nabla_s q|^2 dA,
 \end{aligned}$$

for $\alpha > 0$ at our disposal.

If we now let $a_3 = \max_{\Gamma} |n_i n_j a_{ij}|$, then we pick α such that

$$h_0 a_2 > \delta_0 \alpha a_3, \quad \text{e.g. } \alpha = \frac{h_0 a_2}{2\delta_0 a_3}.$$

Then we find

$$\begin{aligned} & \frac{1-\mu}{2} \int_{\Omega} a_{ij} H_{,i} H_{,j} dx + \frac{h_0 a_2}{4} \oint_{\Gamma} \left(\frac{\partial H}{\partial n} \right)^2 dA \\ & \leq \oint_{\Gamma} \left[\left(\frac{1}{2} x_k n_k s_i s_j - n_i s_j s_k x_k \right) + \frac{\delta_0^2 a_3}{h_0 a_2} n_i n_j \right] a_{ij} |\nabla_s q|^2 dA. \end{aligned} \quad (3.4)$$

We deduce from the Poincaré inequality that

$$\lambda_1 \int_{\Omega} H^2 dx \leq \int_{\Omega} H_{,i} H_{,i} dx + C_1 \oint_{\Gamma} H^2 dA. \quad (3.5)$$

Together, inequalities (3.4) and (3.5) furnish *a priori* bounds for $\int_{\Omega} a_{ij} H_{,i} H_{,j} dx$, $\|H\|^2$ and $\oint_{\Gamma} (\partial H / \partial n)^2 dA$ in terms of the data function q .

We also need an equivalent bound for a vector version of (3.4). So, let G_i be a solution to the problem

$$\left. \begin{aligned} (a_{ijkh} G_{k,h})_{,j} &= 0 \quad \text{in } \Omega, \\ G_i &= \dot{g}_i \quad \text{on } \Gamma. \end{aligned} \right\} \quad (3.6)$$

We remark that existence of the function G_i (as that of H in equation (3.1)) follows, for example, from the work of Fichera (1972).

We commence with the identity

$$0 = \int_{\Omega} (a_{ijkh} G_{k,h})_{,j} x_r G_{i,r} dx.$$

After integration by parts, we find

$$\begin{aligned} 0 &= \frac{1}{2} \int_{\Omega} a_{ijkh} G_{i,j} G_{k,h} dx + \frac{1}{2} \int_{\Omega} x_r a_{ijkh,r} G_{i,j} G_{k,h} dx + \oint_{\Gamma} n_j x_r G_{i,r} a_{ijkh} G_{k,h} dA \\ &\quad - \frac{1}{2} \oint_{\Gamma} x_r n_r a_{ijkh} G_{i,j} G_{k,h} dA. \end{aligned} \quad (3.7)$$

We now write $G_{i,j}$ on Γ as $G_{i,j} = n_j \partial G_i / \partial n + s_j \nabla_s G_i$ and then from (3.7), we deduce that

$$\begin{aligned} 0 &= \frac{1}{2} \int_{\Omega} a_{ijkh} G_{i,j} G_{k,h} dx + \frac{1}{2} \int_{\Omega} x_r a_{ijkh,r} G_{i,j} G_{k,h} dx \\ &\quad + \frac{1}{2} \oint_{\Gamma} x_r n_r a_{ijkh} n_j \frac{\partial G_i}{\partial n} n_h \frac{\partial G_k}{\partial n} dA + \oint_{\Gamma} x_r s_r a_{ijkh} n_j \nabla_s G_i n_h \frac{\partial G_k}{\partial n} dA \\ &\quad - \frac{1}{2} \oint_{\Gamma} x_r n_r a_{ijkh} s_j \nabla_s \dot{g}_i s_h \nabla_s \dot{g}_k dA + \oint_{\Gamma} x_r s_r a_{ijkh} n_j \nabla_s \dot{g}_i s_h \nabla_s \dot{g}_k dA. \end{aligned} \quad (3.8)$$

If we now suppose $|x_r a_{ijkh,r}| \leq A_0 < a_1$, $|a_{ijkh}| \leq A_1$ for constants A_0, A_1 , then from (3.8) we may derive

$$\begin{aligned} & \frac{a_1 - A_0}{2} \int_{\Omega} G_{i,j} G_{i,j} dx + \frac{a_1 h_0}{4} \oint_{\Gamma} \frac{\partial G_i}{\partial n} \frac{\partial G_i}{\partial n} dA \leq \frac{(\delta_0 A_1)^2}{a_1 h_0} \oint_{\Gamma} \nabla_s \dot{g}_i \nabla_s \dot{g}_i dA \\ & \quad + \frac{1}{2} \oint_{\Gamma} x_r n_r a_{ijkh} s_j \nabla_s \dot{g}_i s_h \nabla_s \dot{g}_k dA - \oint_{\Gamma} x_r s_r a_{ijkh} n_j \nabla_s \dot{g}_i s_h \nabla_s \dot{g}_k dA \equiv D_1. \end{aligned} \quad (3.9)$$

The right-hand side D_1 is a data term and then equation (3.9) represents an *a priori* estimate for $\int_{\Omega} a_{ijkh} G_{i,j} G_{k,h} dx, \oint_{\Gamma} (\partial G_i / \partial n) (\partial G_i / \partial n) dA$. Upon using the Poincaré inequality, we have

$$\lambda_1 \int_{\Omega} G_i G_i dx \leq \int_{\Omega} G_{i,j} G_{i,j} dx + C_1 \oint_{\Gamma} \dot{g}_i \dot{g}_i dA. \quad (3.10)$$

In this way, we may use equation (3.9) to also obtain an *a priori* estimate for $\|\mathbf{G}\|^2$.

4. *A priori* estimates

Let u_i and ϕ satisfy the system

$$\left. \begin{aligned} -\varrho \ddot{u}_i + (a_{ijkh} u_{k,h})_{,j} + (\gamma_{ij} \phi)_{,j} &= 0, \\ -\varrho \kappa \ddot{\phi} - \tau \dot{\phi} - \xi \phi - \gamma_{ij} u_{i,j} + (a_{ij} \phi_{,j})_{,i} &= 0, \end{aligned} \right\} \quad (4.1)$$

in $\Omega \times (0, T)$ with $u_i = g_i$, $\phi = h$ on Γ , u_i , $\dot{u}_i$, ϕ , $\dot{\phi}$ given at $t=0$ as in equation (2.4). In order to derive structural stability theorems, we first need to produce *a priori* estimates for $\int_0^t \|\nabla \phi\|^2 ds$ and $\int_0^t \|\nabla \mathbf{u}\|^2 ds$.

Let now H be the function which solves equation (3.1) with $q = \dot{h}$ on Γ . We commence with the identity

$$\int_0^t ((a_{ij} \phi_{,j})_{,i} - \gamma_{ij} u_{i,j} - \tau \dot{\phi} - \xi \phi - \varrho \kappa \ddot{\phi}, H - \dot{\phi}) ds = 0.$$

After some integrations by parts in space and in time, we may show that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \xi \phi^2(t) dx + \frac{1}{2} \int_{\Omega} \varrho \kappa \dot{\phi}^2(t) dx + \int_0^t \int_{\Omega} \tau \dot{\phi}^2(s) dx ds \\ & + \frac{1}{2} \int_{\Omega} a_{ij} \phi_{,i} \phi_{,j} dx + \int_0^t \int_{\Omega} \gamma_{ij} u_{i,j} \dot{\phi} dx ds \\ & = \frac{1}{2} \int_{\Omega} a_{ij} \phi_i^0 \phi_j^0 dx + \int_0^t \oint_{\Gamma} n_i n_j a_{ij} h \frac{\partial H}{\partial n} dA ds + \int_0^t \oint_{\Gamma} n_j s_i a_{ij} h \nabla_s \dot{h} dA ds \\ & + \int_0^t (\xi \phi, H) ds + \frac{1}{2} (\xi \phi^0, \phi^0) + \int_0^t (\gamma_{ij} u_{i,j}, H) ds + (\varrho \kappa \dot{\phi}, H) \\ & - (\varrho \kappa \zeta^0, H(0)) - \int_0^t (\varrho \kappa \dot{\phi}, \dot{H}) ds + \int_0^t (\tau \dot{\phi}, H) ds + \frac{1}{2} (\varrho \kappa \zeta^0, \zeta^0). \end{aligned} \quad (4.2)$$

The idea is now to use the arithmetic–geometric mean inequality on the right-hand side of equation (4.2) and bound the ϕ , $\dot{\phi}$, $\phi_{,i}$ terms using the left-hand side and leave only data terms and terms involving H on the right. The H terms may then be bounded by using the Rellich identity estimates of §3. With

$$\gamma_M = \max_{\overline{\Omega}} |\gamma_{ij}|, \quad \varrho_M = \max_{\overline{\Omega}} |\varrho|, \quad \kappa_M = \max_{\overline{\Omega}} |\kappa|, \quad \xi_M = \max_{\overline{\Omega}} \xi, \quad \tau_M = \max_{\overline{\Omega}} \tau,$$

we use the arithmetic–geometric mean inequality for $\mu_1, \dots, \mu_5 > 0$ arbitrary constants at our disposal to derive from equation (4.2),

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \xi \phi^2(t) dx + \frac{1}{2} \int_{\Omega} \varrho \kappa \dot{\phi}^2(t) dx + \int_0^t \int_{\Omega} \tau \dot{\phi}^2(s) dx ds + \frac{1}{2} \int_{\Omega} a_{ij} \phi_{,i} \phi_{,j} dx \\
& + \int_0^t \int_{\Omega} \gamma_{ij} u_{i,j} \dot{\phi} dx ds \leq \frac{\mu_1}{2} \int_0^t \|\phi\|^2 ds + \frac{\xi_M^2}{2\mu_1} \int_0^t \|H\|^2 ds + \frac{\mu_2}{2} \int_0^t \|u_{i,j}\|^2 ds \\
& + \frac{\gamma_M^2}{2\mu_2} \int_0^t \|H\|^2 ds + \frac{\mu_3}{2} (\varrho \kappa \dot{\phi}, \dot{\phi}) + \frac{\varrho_M \kappa_M}{2\mu_3} \|H\|^2 + \frac{\mu_4}{2} \int_0^t \|\dot{\phi}\|^2 ds \\
& + \frac{(\varrho_M \kappa_M)^2}{2\mu_4} \int_0^t \|\dot{H}\|^2 ds + \frac{\mu_5}{2} \int_0^t \|\dot{\phi}\|^2 ds + \frac{\tau_M^2}{2\mu_5} \int_0^t \|H\|^2 ds + \frac{1}{2} \int_{\Omega} a_{ij} \phi_{,i}^0 \phi_{,j}^0 dx \\
& + \int_0^t \oint_{\Gamma} n_i n_j a_{ij} h \frac{\partial H}{\partial n} dA ds + \int_0^t \oint_{\Gamma} n_j s_i a_{ij} h \nabla_s h dA ds + \frac{1}{2} (\xi \phi^0, \phi^0) \\
& - (\varrho \kappa \zeta^0, H(0)) + \frac{1}{2} (\varrho \kappa \zeta^0, \zeta^0). \tag{4.3}
\end{aligned}$$

To handle the u_i equation, we commence with the identity

$$\int_0^t ((a_{ijkh} u_{k,h})_{,j} + (\gamma_{ij} \phi)_{,j} - \varrho \ddot{u}_i, G_i - \dot{u}_i) ds = 0.$$

After some integration by parts, we may show that

$$\begin{aligned}
& \frac{1}{2} \int_0^t \frac{d}{ds} \left[\int_{\Omega} \rho \dot{u}_i \dot{u}_i dx + \int_{\Omega} a_{ijkh} u_{i,j} u_{k,h} dx \right] ds + \int_0^t \int_{\Omega} \gamma_{ij} \phi \dot{u}_{i,j} dx ds \\
& = \int_0^t \oint_{\Gamma} n_h a_{ijkh} g_k G_{i,j} dA ds + \int_0^t \int_{\Omega} \gamma_{ij} \phi G_{i,j} dx ds + \int_{\Omega} \varrho G_i \dot{u}_i dx \\
& - \int_{\Omega} \varrho G_i(0) v_i^0 dx - \int_0^t \int_{\Omega} \varrho \dot{G}_i \dot{u}_i dx ds. \tag{4.4}
\end{aligned}$$

We next use the arithmetic–geometric mean inequality on the right-hand side of equation (4.4) to find for $\alpha_1, \alpha_2, \alpha_3 > 0$ at our disposal,

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \varrho \dot{u}_i \dot{u}_i dx + \frac{1}{2} \int_{\Omega} a_{ijkh} u_{i,j} u_{k,h} dx + \int_0^t \int_{\Omega} \gamma_{ij} \phi \dot{u}_{i,j} dx ds \\
& \leq \frac{1}{2} \int_{\Omega} \varrho v_i^0 v_i^0 dx + \frac{1}{2} \int_{\Omega} a_{ijkh} u_{i,j}^0 u_{k,h}^0 dx + \frac{\alpha_1}{2} \int_0^t \|\phi\|^2 dx + \frac{\gamma_M^2}{2\alpha_1} \int_0^t \|G_{i,j}\|^2 ds \\
& + \int_0^t \oint_{\Gamma} n_h a_{ijkh} g_k \left(n_j \frac{\partial G_i}{\partial n} + s_j \nabla_s g_i \right) dA ds + \frac{\alpha_2}{2} \int_{\Omega} \varrho \dot{u}_i \dot{u}_i dx \\
& + \frac{1}{2\alpha_2} \int_{\Omega} \varrho G_i G_i dx - \int_{\Omega} \varrho G_i(0) v_i^0 dx + \frac{\alpha_3}{2} \int_0^t \int_{\Omega} \varrho \dot{u}_i \dot{u}_i dx ds \\
& + \frac{1}{2\alpha_3} \int_0^t \int_{\Omega} \varrho \dot{G}_i \dot{G}_i dx ds. \tag{4.5}
\end{aligned}$$

The next step is to add equations (4.3) and (4.5) to see that

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} \varrho \dot{u}_i \dot{u}_i dx + \frac{1}{2} \int_{\Omega} a_{ijkh} u_{i,j} u_{k,h} dx + \frac{1}{2} (\xi \phi, \phi) + \frac{1}{2} (\varrho \mathbf{x} \dot{\phi}, \dot{\phi}) + \int_0^t (\tau \dot{\phi}, \dot{\phi}) ds \\
 & + \frac{1}{2} \int_{\Omega} a_{ij} \phi_{,i} \phi_{,j} dx + \int_{\Omega} \gamma_{ij} u_{i,j} \phi dx \leq \frac{\mu_1}{2} \int_0^t \|\phi\|^2 ds + \frac{\mu_2}{2} \int_0^t \|u_{i,j}\|^2 ds \\
 & + \frac{\mu_3}{2} (\varrho \mathbf{x} \dot{\phi}, \dot{\phi}) + \frac{\mu_4 + \mu_5}{2} \int_0^t \|\dot{\phi}\|^2 ds + \frac{\alpha_1}{2} \int_0^t \|\phi\|^2 ds + \frac{\alpha_2}{2} \int_{\Omega} \varrho \dot{u}_i \dot{u}_i dx \\
 & + \frac{\alpha_3}{2} \int_0^t \int_{\Omega} \varrho \dot{u}_i \dot{u}_i dx ds + F_1,
 \end{aligned} \tag{4.6}$$

where F_1 is a term involving data or terms which are easily estimated in terms of data using the bounds in §3. In fact,

$$\begin{aligned}
 F_1 = & \int_{\Omega} \gamma_{ij} u_{i,j}^0 \phi^0 dx + \frac{1}{2} \int_{\Omega} \varrho v_i^0 v_i^0 dx + \frac{1}{2} \int_{\Omega} a_{ijkh} u_{i,j}^0 u_{k,h}^0 dx + \frac{\gamma_M^2}{2\alpha_1} \int_0^t \|G_{i,j}\|^2 ds \\
 & + \int_0^t \oint_{\Gamma} n_h a_{ijkh} g_k s_j \nabla_s \dot{g}_i dA ds + \int_0^t \oint_{\Gamma} n_j s_i a_{ij} h \nabla_s \dot{h} dA ds \\
 & + \frac{1}{2} \int_0^t \oint_{\Gamma} |a_{ijkh}| |\mathbf{g}|^2 dA ds + \frac{1}{2} \int_0^t \oint_{\Gamma} a_M \frac{\partial G_i}{\partial n} \frac{\partial G_i}{\partial n} dA ds + \frac{1}{2\alpha_2} \int_{\Omega} \varrho G_i G_i dx \\
 & - \int_{\Omega} \varrho G_i(0) v_i^0 dx + \frac{1}{2\alpha_3} \int_0^t \int_{\Omega} \rho \dot{G}_i \dot{G}_i dx ds + \frac{\xi_M^2}{2\mu_1} \int_0^t \|H\|^2 ds + \frac{\gamma_M^2}{2\mu_2} \int_0^t \|H\|^2 ds \\
 & + \frac{\varrho_M \mathbf{x}_M}{2\mu_3} \|H\|^2 + \frac{(\varrho_M \mathbf{x}_M)^2}{2\mu_4} \int_0^t \|\dot{H}\|^2 ds + \frac{\tau_M^2}{2\mu_5} \int_0^t \|H\|^2 ds + \frac{1}{2} \int_{\Omega} a_{ij} \phi_{,i}^0 \phi_{,j}^0 dx \\
 & + \frac{1}{2} \int_0^t \oint_{\Gamma} |a_{ij}| h^2 dA ds + \frac{1}{2} \int_0^t \oint_{\Gamma} |a_{ij}| \left(\frac{\partial H}{\partial n} \right)^2 dA ds + \frac{1}{2} (\xi \phi^0, \phi^0) \\
 & - (\varrho \mathbf{x} \zeta^0, H(0)) + \frac{1}{2} (\varrho \mathbf{x} \zeta^0, \zeta^0),
 \end{aligned} \tag{4.7}$$

where $a_M = \max_{\Gamma} |a_{ijkh}|$. On the basis of our assumptions (2.5) and (2.8), it follows that the left-hand side of equation (4.6) is a positive definite measure. We may choose $\mu_3 = 1/2$, $\alpha_2 = 1/2$. Then, we select as our energy measure

$$\begin{aligned}
 \mathcal{E}(t) = & \frac{1}{2} \int_{\Omega} (\varrho \dot{u}_i \dot{u}_i + \varrho \mathbf{x} \dot{\phi}^2) dx + \frac{1}{2} \int_{\Omega} a_{ijkh} u_{i,j} u_{k,h} dx \\
 & + \int_{\Omega} \gamma_{ij} u_{i,j} \phi dx + \frac{1}{2} (\xi \phi(t), \phi(t)) + \frac{1}{2} \int_{\Omega} a_{ij} \phi_{,i} \phi_{,j} dx.
 \end{aligned}$$

By using the bounds (3.4), (3.5), (3.9) and (3.10), we can estimate F_1 , as given by equation (4.7), in terms of a data term, generically denoted by $\mathcal{F}$, to see that for

a computable constant $K > 0$, we may obtain from equation (4.6)

$$\mathcal{E}(t) \leq K \int_0^t \mathcal{E}(s) ds + \mathcal{F}. \quad (4.8)$$

We multiply the inequality (4.8) by e^{-Kt} and then integrate the result over $[0, t]$ to obtain the *a priori* bound

$$\int_0^t \mathcal{E}(s) ds \leq \frac{e^{KT}}{K} \mathcal{F}. \quad (4.9)$$

Inequality (4.9) leads directly to *a priori* estimates for $\int_0^t \|\nabla \mathbf{u}\|^2 ds$ and $\int_0^t \|\nabla \phi\|^2 ds$.

5. Continuous dependence on the coupling coefficient b_{ij}

To study continuous dependence on the coefficient b_{ij} in equations (2.1)–(2.4), we let (u_i, ϕ) and (v_i, ψ) be solutions to these equations for the same boundary and initial data, but for different coupling coefficients γ_{ij} and δ_{ij} , respectively. The coefficients ϱ , κ , a_{ijkh} , τ , ξ and a_{ij} are the same for both systems.

Define the difference variables w_i , θ , β_{ij} as

$$w_i = u_i - v_i, \quad \theta = \phi - \psi, \quad \beta_{ij} = \gamma_{ij} - \delta_{ij}. \quad (5.1)$$

One finds (w_i, θ) satisfies the system

$$(a_{ijkh} w_{k,h})_{,j} + (\delta_{ij} \theta)_{,j} - \varrho \ddot{w}_i = -(\beta_{ij} \phi)_{,j}, \quad (5.2)$$

$$(a_{ij} \theta_{,j})_{,i} - \xi \theta - \tau \dot{\theta} - \delta_{ij} w_{i,j} - \varrho \kappa \ddot{\theta} = \beta_{ij} u_{i,j}, \quad (5.3)$$

in $\Omega \times (0, T)$. The boundary and initial conditions are:

$$w_i = 0, \quad \theta = 0, \quad \text{on } \Gamma \times (0, T), \quad (5.4)$$

$$\left. \begin{aligned} w_i(\mathbf{x}, 0) &= 0, & \dot{w}_i(\mathbf{x}, 0) &= 0, \\ \theta(\mathbf{x}, 0) &= 0, & \dot{\theta}(\mathbf{x}, 0) &= 0. \end{aligned} \right\} \quad (5.5)$$

To establish continuous dependence, we multiply equation (5.2) by $\dot{w}_i$ and integrate over Ω , then multiply equation (5.3) by $\dot{\theta}$ and similarly integrate over Ω , to find

$$\frac{d}{dt} \frac{1}{2} \left[\int_{\Omega} \varrho \dot{w}_i \dot{w}_i dx + \int_{\Omega} a_{ijkh} w_{i,j} w_{k,h} dx \right] + \int_{\Omega} \delta_{ij} \theta \dot{w}_{i,j} dx = \int_{\Omega} (\beta_{ij} \phi)_{,j} \dot{w}_i dx, \quad (5.6)$$

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \left[(\varrho \kappa \dot{\theta}, \dot{\theta}) + \int_{\Omega} a_{ij} \theta_{,i} \theta_{,j} dx + (\xi \theta, \theta) \right] + (\tau \dot{\theta}, \dot{\theta}) + \int_{\Omega} \delta_{ij} w_{i,j} \dot{\theta} dx \\ &= - \int_{\Omega} \beta_{ij} \dot{\theta} u_{i,j} dx. \end{aligned} \quad (5.7)$$

Upon addition of these equations, we may see that

$$\begin{aligned}
 & \frac{d}{dt} \left\{ \frac{1}{2} \left[\int_{\Omega} \varrho \dot{w}_i \dot{w}_i dx + \int_{\Omega} a_{ijkh} w_{i,j} w_{k,h} dx + (\varrho \mathbf{x} \dot{\theta}, \dot{\theta}) + (\xi \theta, \theta) \right. \right. \\
 & \quad \left. \left. + \int_{\Omega} a_{ij} \theta_{,i} \theta_{,j} dx \right] + \int_{\Omega} \delta_{ij} w_{i,j} \theta dx \right\} + (\tau \dot{\theta}, \dot{\theta}) \\
 & = \int_{\Omega} (\beta_{ij} \phi)_{,j} \dot{w}_i dx - \int_{\Omega} \beta_{ij} \dot{\theta} u_{i,j} dx \\
 & \leq \frac{1}{2\varrho_m} \int_{\Omega} \varrho \dot{w}_i \dot{w}_i dx + \frac{1}{2} \int_{\Omega} \beta^2 \phi_{,r} \phi_{,r} dx + \frac{1}{2\varrho_m} \int_{\Omega} \varrho \dot{w}_i \dot{w}_i dx \\
 & \quad + \frac{1}{2} \int_{\Omega} \hat{\beta}^2 \phi^2 dx + \int_{\Omega} \frac{\beta^2}{2a_1} a_{ijkh} u_{i,j} u_{k,h} dx + \frac{1}{2} \|\dot{\theta}\|^2, \tag{5.8}
 \end{aligned}$$

where $\beta^2 = \beta_{ij} \beta_{ij}$, $\hat{\beta}^2 = \beta_{ir,r} \beta_{is,s}$ and $\varrho_m = \min_{\Omega} \varrho$. If we put $\beta_M^2 = \max_{\bar{\Omega}} \beta^2$, $\hat{\beta}_M^2 = \max_{\bar{\Omega}} \hat{\beta}^2$, then define

$$\begin{aligned}
 \mathcal{E}_1(t) &= \frac{1}{2} \left[\int_{\Omega} \varrho \dot{w}_i \dot{w}_i dx + \int_{\Omega} a_{ijkh} w_{i,j} w_{k,h} dx + (\varrho \mathbf{x} \dot{\theta}, \dot{\theta}) + (\xi \theta, \theta) + \int_{\Omega} a_{ij} \theta_{,i} \theta_{,j} dx \right] \\
 & \quad + \int_{\Omega} \delta_{ij} w_{i,j} \theta dx,
 \end{aligned}$$

and then for $C = \max\{2/\varrho_m, 1/\varrho_m \mathbf{x}_m\}$, we find

$$\frac{d\mathcal{E}_1}{dt} \leq C\mathcal{E}_1 + \beta_M^2 \left(\int_{\Omega} \frac{1}{2a_1} a_{ijkh} u_{i,j} u_{k,h} dx + \frac{1}{2} \int_{\Omega} \phi_{,i} \phi_{,i} dx \right) + \frac{1}{2} \hat{\beta}_M^2 \|\phi\|^2.$$

This inequality is integrated to see that

$$\begin{aligned}
 \mathcal{E}_1(t) &\leq C \int_0^t \mathcal{E}_1(s) ds + \beta_M^2 \left[\int_0^t \int_{\Omega} \frac{1}{2a_1} a_{ijkh} u_{i,j} u_{k,h} dx ds + \frac{1}{2} \int_0^t \int_{\Omega} \phi_{,i} \phi_{,i} dx ds \right] \\
 & \quad + \frac{1}{2} \hat{\beta}_M^2 \int_0^t \|\phi\|^2 ds. \tag{5.9}
 \end{aligned}$$

The last term of equation (5.9) is bounded using the *a priori* estimate (4.9) to find

$$\mathcal{E}_1(t) \leq C \int_0^t \mathcal{E}_1(s) ds + \beta_M^2 \mathcal{F}_1 + \hat{\beta}_M^2 \mathcal{F}_2, \tag{5.10}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are some data terms. Upon integration of equation (5.10), we may arrive at

$$\int_0^t \mathcal{E}_1(s) ds \leq (\beta_M^2 \mathcal{F}_1 + \hat{\beta}_M^2 \mathcal{F}_2) \frac{1}{C} (e^{CT} - 1). \tag{5.11}$$

Inequality (5.11) is truly *a priori* and demonstrates the solution to (2.1)–(2.4) depends continuously on changes in the coupling coefficient b_{ij} , in the measure

$\int_0^t \mathcal{E}_1(s) ds$. By combining equations (5.10) and (5.11), we may also show that

$$\mathcal{E}_1(t) \leq (\beta_M^2 \mathcal{F}_1 + \hat{\beta}_M^2 \mathcal{F}_2) e^{CT}. \quad (5.12)$$

Thus, continuous dependence on β_{ij} in the measure $\mathcal{E}_1(t)$ also follows.

6. Convergence

We now investigate how the solution to (2.1)–(2.4) behaves as the coefficient $b_{ij} \rightarrow 0$. Thus, let (u_i, ϕ) be a solution to (2.1)–(2.4), where now we assume b_{ij} are constants. Let also (v_i, ψ) be a solution to (2.1)–(2.4) with $b_{ij} \equiv 0$ (so the equations are uncoupled). The difference solution $w_i = u_i - v_i$, $\theta = \phi - \psi$ is seen to satisfy the boundary-initial value problem,

$$\varrho \ddot{w}_i = (a_{ijkh} w_{k,h})_{,j} + (b_{ij} \phi)_{,j}, \quad (6.1)$$

$$\varrho \mathbf{x} \ddot{\theta} = (a_{ij} \theta_{,j})_{,i} - \tau \dot{\theta} - \xi \theta - b_{ij} u_{i,j}, \quad (6.2)$$

in $\Omega \times (0, T)$, with

$$w_i = 0, \quad \theta = 0, \quad \text{on } \Gamma \times (0, T), \quad (6.3)$$

and

$$w_i = 0, \quad \dot{w}_i = 0, \quad \theta = 0, \quad \dot{\theta} = 0, \quad \text{at } t = 0. \quad (6.4)$$

Multiply equation (6.1) by $\dot{w}_i$ and integrate over Ω , and likewise multiply equation (6.2) by $\dot{\theta}$ and integrate over Ω to find

$$\frac{d}{dt} \frac{1}{2} \left[\int_{\Omega} \varrho \dot{w}_i \dot{w}_i dx + \int_{\Omega} a_{ijkh} w_{i,j} w_{k,h} dx \right] = \int_{\Omega} \dot{w}_i (b_{ij} \phi)_{,j} dx, \quad (6.5)$$

$$\frac{d}{dt} \frac{1}{2} \left[(\varrho \mathbf{x} \dot{\theta}, \dot{\theta}) + \int_{\Omega} a_{ij} \theta_{,i} \theta_{,j} dx + (\xi \theta, \theta) \right] + \frac{d}{dt} \int_0^t (\tau \dot{\theta}, \dot{\theta}) ds = - \int_{\Omega} b_{ij} u_{i,j} \dot{\theta} dx. \quad (6.6)$$

We next add these equations and use the arithmetic–geometric mean inequality on the right-hand side of the resulting equation to arrive at

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \left[\int_{\Omega} \varrho \dot{w}_i \dot{w}_i dx + \int_{\Omega} a_{ijkh} w_{i,j} w_{k,h} dx + (\varrho \mathbf{x} \dot{\theta}, \dot{\theta}) + (\xi \theta, \theta) + \int_{\Omega} a_{ij} \theta_{,i} \theta_{,j} dx \right] \\ & + (\tau \dot{\theta}, \dot{\theta}) \leq \frac{1}{2} \|\dot{\theta}\|^2 + \frac{1}{2\varrho_m} \int_{\Omega} \varrho \dot{w}_i \dot{w}_i dx + \frac{b_M^2}{2} (\|\nabla \phi\|^2 + \|\nabla \mathbf{u}\|^2), \end{aligned} \quad (6.7)$$

where $b_M = \max_{\bar{\Omega}} |b_{ij}|$. If we denote by

$$\mathcal{E}_2(t) = \frac{1}{2} \left[\int_{\Omega} \varrho \dot{w}_i \dot{w}_i dx + \int_{\Omega} a_{ijkh} w_{i,j} w_{k,h} dx + (\varrho \mathbf{x} \dot{\theta}, \dot{\theta}) + (\xi \theta, \theta) + \int_{\Omega} a_{ij} \theta_{,i} \theta_{,j} dx \right],$$

then from equation (6.7), we deduce that for $\lambda = \max\{\varrho_m^{-1}\kappa_m^{-1}, \varrho_m^{-1}\}$,

$$\mathcal{E}_2(t) \leq \lambda \int_0^t \mathcal{E}_2(s) ds + \frac{b_M^2}{2} \int_0^t (\|\nabla \phi\|^2 + \|\nabla \mathbf{u}\|^2) ds. \quad (6.8)$$

Thanks to the *a priori* bound (4.9), $\int_0^t (\|\nabla \phi\|^2 + \|\nabla \mathbf{u}\|^2) ds$ is bounded in terms of a known data constant, $2k_1$ say. Thus from equation (6.8), we find that

$$\mathcal{E}_2(t) \leq \lambda \int_0^t \mathcal{E}_2(s) ds + k_1 b_M^2. \quad (6.9)$$

After integration, equation (6.9) yields

$$\int_0^t \mathcal{E}_2(s) ds \leq \frac{k_1}{\lambda} (e^{\lambda T} - 1) b_M^2. \quad (6.10)$$

Upon using equation (6.9), we may also show that

$$\mathcal{E}_2(t) \leq k_1 e^{\lambda T} b_M^2. \quad (6.11)$$

Estimates (6.10) and (6.11) demonstrate convergence in the measures $\mathcal{E}_2(t)$ and $\int_0^t \mathcal{E}_2(s) ds$, as $b_{ij} \rightarrow 0$.

Remark

We have only dealt with continuous dependence and convergence questions involving the coupling coefficients b_{ij} . We could also establish similar results for other coefficients, such as ξ and τ .

References

- Ames, K. A. & Payne, L. E. 1995 Continuous dependence on the initial time geometry for a thermoelastic system with sign-indefinite elasticities. *J. Math. Anal. Appl.* **189**, 693–717.
- Ames, K. A. & Straughan, B. 1997 *Non-standard and improperly posed problems*. New York: Academic Press.
- Bedford, A. & Drumheller, D. S. 1983 Theory of immiscible and structured mixtures. *Int. J. Eng. Sci.* **21**, 863–960.
- Bowen, R. M. 1976 In *Theory of mixtures* (ed. A. C. Eringen), *Continuum physics*, vol. 3. New York: Academic Press.
- Bramble, J. H. & Payne, L. E. 1961 *A priori* bounds in the first boundary value problem in elasticity. *J. Res. Natl Bur. Stand.* **65**, 269–276.
- Bramble, J. H. & Payne, L. E. 1963 Bounds for solutions of second order partial differential equations. *Contrib. Differ. Equations* **1**, 94–127.
- Ciarletta, M. & Ieșan, D. 1993 *Non-classical elastic solids*. London: Longman Scientific and Technical.
- Cowin, S. C. 1985 The viscoelastic behavior of linear elastic materials with voids. *J. Elasticity* **15**, 185–191.
- Cowin, S. C. & Nunziato, J. W. 1983 Linear elastic materials with voids. *J. Elasticity* **13**, 125–147.
- De Boer, R. 1998 Theory of porous media—past and present. *Z. Angew. Math. Mech.* **78**, 441–466.
- Fichera, G. 1972 In *Existence theorems in elasticity* (ed. S. Flügge), *Handbuch der Physik*, vol. VIa/2. Berlin: Springer.
- Franchi, F. & Straughan, B. 1996 Structural stability for the Brinkman equations of porous media. *Math. Meth. Appl. Sci.* **19**, 1335–1347.

- Goodman, M. A. & Cowin, S. C. 1972 A continuum theory for granular materials. *Arch. Ration. Mech. Anal.* **44**, 249–266.
- Ieşan, D. 2004 *Thermoelastic models of continua. Solid mechanics and its applications*, vol. 118. Dordrecht: Kluwer Academic Publishers.
- Knops, R. J. & Payne, L. E. 1969 Continuous data dependence for the equations of classical elastodynamics. *Proc. Camb. Phil. Soc.* **66**, 481–491.
- Knops, R. J. & Payne, L. E. 1988 Improved estimates for continuous data dependence in linear elastodynamics. *Math. Proc. Camb. Phil. Soc.* **103**, 535–559.
- Knops, R. J. & Wilkes, E. W. 1973 In *Theory of elastic stability* (ed. C. Truesdell), *Handbuch der Physik*, vol. VIa/3. Berlin: Springer.
- Nunziato, J. W. & Cowin, S. C. 1979 A nonlinear theory of elastic materials with voids. *Arch. Ration. Mech. Anal.* **72**, 175–201.
- Payne, L. E. & Song, J. C. 1997 Continuous dependence on the initial-time geometry in generalized heat conduction. *Math. Models Meth. Appl. Sci.* **7**, 125–138.
- Payne, L. E. & Straughan, B. 1996 Stability in the initial-time geometry problem for the Brinkman and Darcy equations of flow in porous media. *J. Math. Pure Appl.* **75**, 225–271.
- Payne, L. E. & Straughan, B. 1998 Structural stability for the Darcy equations of flow in porous media. *Proc. R. Soc. A* **454**, 1691–1698. (doi:10.1098/rspa.1998.0227)
- Payne, L. E. & Weinberger, H. F. 1958 New bounds for solutions of second order elliptic partial differential equations. *Pac. J. Math.* **8**, 551–573.
- Quintanilla, R. 2003 Convergence and structural stability in thermoelasticity. *Appl. Math. Comput.* **135**, 287–300. (doi:10.1016/s0096-3003(01)00331-9)