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A MIXTURE THEORY FOR MICROSTRETCH THERMOVISCOELASTIC SOLIDS

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A nonlinear theory is developed for a heat-conducting viscoelastic composite which is modelled as a mixture consisting of a microstretch Kelvin–Voigt material and a microstretch elastic solid. The strain measures, the basic laws and the constitutive equations are established and presented in Lagrangian description. The initial boundary value problem associated to such model is also formulated. Then the linearized theory is considered and the constitutive equations are given for both anisotropic and isotropic bodies. Finally, a uniqueness result is established within the framework of the linear theory.

Keywords: Microstretch continua; Thermoviscoelastic mixtures; Uniqueness

INTRODUCTION

One of the mathematical models which describe the interactions between fluids, gases and solids is the theory of mixtures. Much effort has been expended to formulate the basic concepts of the theory of mixtures in the 1960s and the 1970s, when various types of mixtures have been introduced in Eulerian description (see, e.g., [1–9], the review articles [10–13] and the more recent books [14–16]).

The theory of mixtures uses the concept of superposition. This means that each place in the mixture is occupied simultaneously by different particles, one from each constituent. For theories based on Eulerian description, the typical particles of constituents are assumed to occupy the same position at current time t . In recent years a great attention has been given also to theories developed in Lagrangian description (see, e.g., [17–27]). For theories of mixtures developed in Lagrangian description the typical particles of constituents occupy the same position in the reference configuration. Ieșan [21, 23, 26] pointed out that the Lagrangian description and the Eulerian description lead to different theories and explained why it is important to use the Lagrangian description when dealing with theories of mixtures whose one component is a solid.

The Lagrangian description has been used firstly by Bedford and Stern [17, 18] in order to describe the mechanical behaviour of composites modelled as a mixture

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of elastic solids. This idea has been developed further in [19–22]. Moreover, some terms have been included in the theory in order to reflect the microstructure of the constituents (see [23–27]). In [25] a nonlinear theory for binary mixtures of micropolar thermoelastic solids is derived, while in the reference [26] a linear theory for a mixture consisting of a micropolar elastic solid and a micropolar Kelvin–Voigt material is developed. In the case of micropolar media, each particle can independently translate and rotate, so that it has six degrees of freedom.

The purpose of this paper is to generalize the theories presented in [25, 26] to the case of microstretch thermoviscoelastic solids. We suppose that the material points of the bodies can stretch and contract independently of their translations and rotations, so that there are required seven variables to describe their motion. The microstretch theory for a single continua was introduced by Eringen [28–30] and it was extensively studied in the literature. No results are in the case of mixtures.

We formulate a nonlinear theory for a viscoelastic composite which is modelled as a mixture consisting of two components: a microstretch elastic solid and a microstretch Kelvin–Voigt material. Thus, we generalize both theories presented in [25] and [26], respectively. We develop a kinematical study of the motion in which we choose the appropriate measures of deformation and determine their rates. Then, the balance laws for mass, microinertia, energy and production of entropy are formulated. We assume that the two constituents have a common temperature and every thermodynamical process that takes place in the mixture satisfies the Clausius–Duhem inequality. By using the constitutive axioms, we express the dependent constitutive variables in an invariant form, and then we use the Clausius–Duhem inequality to develop constitutive equations. Initial boundary value problems for the nonlinear theory are formulated. In the second part of the paper the theory is linearized and a uniqueness result is established. Moreover, we prove that in the absence of stretch the linear equations reduce to the equations obtained by Ieșan [26].

KINEMATICS

We consider a mixture of two interacting continua s_1 and s_2 . The mixture is viewed as a superposition of two continua each following its own motion and at any time each place in the mixture is occupied simultaneously by two different particles, one from each constituent.

We suppose that at time t_0 the body occupies the regular region B of Euclidean three-dimensional space and is bounded by piecewise smooth surface ∂B . We refer the motion of the body to a reference configuration, taken at time $t = 0$ and a fixed system of rectangular Cartesian axes. We employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers (1, 2, 3), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. Greek indices are understood to range over integers (1, 2) and summation convention is not used for these indices. We use a superposed dot to denote differentiation with respect to time variable.

The position of the two typical particles, say $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$, of s_1 and s_2 at time t are $\mathbf{x}$ and $\mathbf{y}$, respectively, where

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \quad \mathbf{y} = \mathbf{y}(\mathbf{Y}, t), \quad \mathbf{X}, \mathbf{Y} \in B, \quad t \in I$$

Here $\mathbf{X}$ and $\mathbf{Y}$ are reference positions of the two particles and $I = [t_0, t_1]$, where t_1 is an instant that may be also infinity.

In this paper we propose a binary mixture theory of microstretch solids, where the deformable particles occupy the same position in the reference configuration, that is $\mathbf{X} = \mathbf{Y}$. The translations of the body are described by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \quad \mathbf{y} = \mathbf{y}(\mathbf{X}, t), \quad \mathbf{X} \in B, \quad t \in I \quad (1)$$

According to the theory of microstretch continua [28–30], we suppose that each particle of the mixture can also independently rotate and stretch (expand and contract). The micromotion of the constituent s_α is described through the deformation of a vector, called deformable vector, attached to $\mathcal{P}^{(\alpha)}$. Thus, associating with the particle $\mathcal{P}^{(\alpha)}$ an arbitrary vector $\Xi^{(\alpha)}$ at $\mathbf{X}$, in the case of microstretch continua this vector undergoes the transformation $\Xi^{(\alpha)} \rightarrow \xi^{(\alpha)}$ expressed by the mapping $\xi_k^{(\alpha)} = \chi_{kK}^{(\alpha)} \Xi_K^{(\alpha)}$, where $\chi^{(\alpha)}$ satisfies the relations (see [30])

$$j_{(\alpha)} = \det \chi_{kK}^{(\alpha)} > 0, \quad \chi_{iK}^{(\alpha)} \chi_{iL}^{(\alpha)} = j_{(\alpha)}^2 \delta_{KL}, \quad \chi_{iK}^{(\alpha)} \chi_{jK}^{(\alpha)} = j_{(\alpha)}^2 \delta_{ij} \quad (2)$$

and completely describes the rotation and the stretch of particle. For micromorphic continua $\chi_{kK}^{(\alpha)}$ is supposed to satisfy just the relation (2)₁. In the above relations δ_{ij} and δ_{KL} are Kronecker deltas. The tensor $\chi_{iK}^{(\alpha)}(\mathbf{X}, t)$ is usually called micromotion (see [30]).

Setting

$$\bar{\chi}_{iK}^{(\alpha)} = \frac{1}{j_{(\alpha)}} \chi_{iK}^{(\alpha)} \quad (3)$$

with $\bar{\chi}_{iK}^{(\alpha)}$ subject to

$$\bar{\chi}_{iK}^{(\alpha)} \bar{\chi}_{iL}^{(\alpha)} = \delta_{KL}, \quad \bar{\chi}_{iK}^{(\alpha)} \bar{\chi}_{jK}^{(\alpha)} = \delta_{ij}, \quad \epsilon_{LMN} \bar{\chi}_{iL}^{(\alpha)} \bar{\chi}_{mM}^{(\alpha)} \bar{\chi}_{nN}^{(\alpha)} = \epsilon_{lmn} \quad (4)$$

where ϵ_{LMN} and ϵ_{lmn} are the alternating symbols, then the kinematical description of a microstretch binary mixture is complete once the following functions are known:

$$x_i = x_i(\mathbf{X}, t), \quad y_i = y_i(\mathbf{X}, t) \quad (5)$$

$$\bar{\chi}_{iK}^{(\alpha)} = \bar{\chi}_{iK}^{(\alpha)}(\mathbf{X}, t), \quad j_{(\alpha)} = j_{(\alpha)}(\mathbf{X}, t), \quad \mathbf{X} \in B, \quad t \in I, \quad \alpha = 1, 2$$

We suppose that the above functions are sufficiently smooth for the ensuing analysis to be valid.

Denoting by $\dot{f} = \frac{Df}{Dt}$ the partial derivative of $f = f(\mathbf{x}, t)$ with respect to t holding X_K fixed, we introduce the notations

$$v_{ij}^{(z)}(\mathbf{X}, t) \equiv \dot{\tilde{\chi}}_{iK}^{(z)} \tilde{\chi}_{jK}^{(z)} = -v_{ji}^{(z)}, \quad v^{(z)}(\mathbf{X}, t) \equiv \frac{1}{j_{(z)}} \frac{Dj_{(z)}}{Dt} \quad (6)$$

The corresponding angular velocity is given by

$$v_i^{(z)} = -\frac{1}{2} \epsilon_{ijk} v_{jk}^{(z)} \quad (7)$$

From (6) and (7), we deduce

$$v_i^{(z)} = -\frac{1}{2} \epsilon_{ijk} \dot{\tilde{\chi}}_{jL}^{(z)} \tilde{\chi}_{kL}^{(z)}, \quad \dot{\tilde{\chi}}_{iK}^{(z)} = \epsilon_{ijk} v_j^{(z)} \tilde{\chi}_{kK}^{(z)} \quad (8)$$

Clearly, a rigid motion of the mixture is described by the relations

$$\begin{aligned} x_i(\mathbf{X}, t) &= y_i(\mathbf{X}, t) = Q_{iK}(t)X_K + \tilde{c}_i(t) \\ \tilde{\chi}_{iK}^{(1)}(\mathbf{X}, t) &= \tilde{\chi}_{iK}^{(2)}(\mathbf{X}, t) = Q_{iK}(t), \quad j_{(1)}(\mathbf{X}, t) = j_{(2)}(\mathbf{X}, t) = 1 \end{aligned} \quad (9)$$

where $\mathbf{Q}$ is proper orthogonal. For rigid motions, from (8) and (9), we deduce

$$\dot{x}_i = \epsilon_{ijk} b_j x_k + c_i, \quad \dot{y}_i = \epsilon_{ijk} b_j y_k + c_i, \quad v_i^{(1)} = v_i^{(2)} = b_i, \quad v^{(1)} = v^{(2)} = 0 \quad (10)$$

where b_i is the axial vector of the skew-symmetric tensor $\dot{Q}_{iK} Q_{jK}$ and $c_i = \tilde{c}_i - \dot{Q}_{iK} Q_{jK} \tilde{c}_j$.

In the theories involving continuous media, many sets of strain measures may be used. The most appropriate set is suggested from an examination of the constitutive theory and the Clausius–Duhem inequality. Following [25] we consider the following set of strain measures

$$\begin{aligned} E_{KL} &= x_{i,K} \tilde{\chi}_{iL}^{(1)} - \delta_{KL}, & 2\Gamma_{KL}^{(1)} &= \epsilon_{LMN} \tilde{\chi}_{iN}^{(1)} \tilde{\chi}_{iM,K}^{(1)} \\ G_{KL} &= y_{i,K} \tilde{\chi}_{iL}^{(1)} - \delta_{KL}, & 2\Gamma_{KL}^{(2)} &= \epsilon_{LMN} \tilde{\chi}_{iN}^{(2)} \tilde{\chi}_{iM,K}^{(2)} \\ D_K &= \tilde{\chi}_{iK}^{(1)}(x_i - y_i), & \Delta_{KL} &= \tilde{\chi}_{iK}^{(1)} \tilde{\chi}_{iL}^{(2)} - \delta_{KL}, \\ \Gamma_K^{(1)} &= \frac{1}{j_{(1)}} [j_{(1)}]_{,K}, & \Gamma_K^{(2)} &= \frac{1}{j_{(2)}} [j_{(2)}]_{,K}, & E^{(1)} &= \frac{1}{2} [j_{(1)}^2 - 1], & E^{(2)} &= \frac{1}{2} [j_{(2)}^2 - 1] \end{aligned} \quad (11)$$

This set is form-invariant under rigid motions of the spatial frame of reference (i.e., it is objective) and it determines uniquely the motion and micromotion to within the same rigid motion, provided some compatibility conditions are satisfied. We shall see that this set of strain suit in a good enough manner the constitutive theory and thermodynamics.

The time rates of these measures, computed from (4), (6)–(8) are

$$\begin{aligned}
 \dot{E}_{KL} &= \bar{\chi}_{iL}^{(1)} (\dot{x}_{i,K} + \epsilon_{ijk} x_{j,K} v_k^{(1)}), & \dot{\Gamma}_{KL}^{(1)} &= \bar{\chi}_{iL}^{(1)} v_{i,K}^{(1)} \\
 \dot{G}_{KL} &= \bar{\chi}_{iL}^{(1)} (\dot{y}_{i,K} + \epsilon_{ijk} y_{j,K} v_k^{(1)}), & \dot{\Gamma}_{KL}^{(2)} &= \bar{\chi}_{iL}^{(2)} v_{i,K}^{(2)} \\
 \dot{D}_K &= \bar{\chi}_{iK}^{(1)} (\dot{x}_i - \dot{y}_i + \epsilon_{ijk} (x_j - y_j) v_k^{(1)}), & \dot{\Delta}_{KL} &= \epsilon_{ijm} \bar{\chi}_{mK}^{(1)} \bar{\chi}_{iL}^{(2)} (v_j^{(1)} - v_j^{(2)}) \\
 \dot{\Gamma}_K^{(1)} &= v_{i,K}^{(1)}, & \dot{\Gamma}_K^{(2)} &= v_{i,K}^{(2)}, & \dot{E}^{(1)} &= j_{(1)}^2 v^{(1)}, & \dot{E}^{(2)} &= j_{(2)}^2 v^{(2)}
 \end{aligned} \tag{12}$$

By using the relation (4) and the properties of the alternating symbol, from (12) we obtain

$$\begin{aligned}
 \dot{x}_{i,K} + \epsilon_{ijk} x_{j,K} v_k^{(1)} &= \bar{\chi}_{iL}^{(1)} \dot{E}_{KL}, & v_{i,K}^{(1)} &= \bar{\chi}_{iL}^{(1)} \dot{\Gamma}_{KL}^{(1)} \\
 \dot{y}_{i,K} + \epsilon_{ijk} y_{j,K} v_k^{(1)} &= \bar{\chi}_{iL}^{(1)} \dot{G}_{KL}, & v_{i,K}^{(2)} &= \bar{\chi}_{iL}^{(2)} \dot{\Gamma}_{KL}^{(2)} \\
 \dot{x}_i - \dot{y}_i + \epsilon_{ijk} (x_j - y_j) v_k^{(1)} &= \bar{\chi}_{iK}^{(1)} \dot{D}_K, & v_i^{(1)} - v_i^{(2)} &= \epsilon_{ijm} \bar{\chi}_{jK}^{(1)} \bar{\chi}_{mL}^{(2)} \dot{\Delta}_{KL} \\
 v_{i,K}^{(1)} &= \dot{\Gamma}_K^{(1)}, & v_{i,K}^{(2)} &= \dot{\Gamma}_K^{(2)}, & v^{(1)} &= \frac{1}{j_{(1)}^2} \dot{E}^{(1)}, & v^{(2)} &= \frac{1}{j_{(2)}^2} \dot{E}^{(2)}
 \end{aligned} \tag{13}$$

BASIC EQUATIONS

This section is devoted to discussing the balance laws governing the behavior of the microstretch mixture. We postulate the following balance laws: conservation of mass for each constituent, conservation of microinertia for each constituent, conservation of energy for the mixture as a whole and the law of entropy. We suppose that the mixture is chemical inert so that we do not consider the axiom of balance of mass (or microinertia) for the mixture. The balance of momentum and balance of moment of momentum for each constituent are derived by subjecting the global laws to Galilean invariance. As regards the entropy production inequality, we suppose that the constituents have a common temperature, so that the axiom involves only the statement concerning the mixture as a whole.

We consider an arbitrary material region P_α of constituent s_α at time t bounded by the surface ∂P_α , and we suppose that P_0 is the corresponding region at time t_0 bounded by the surface ∂P_0 . The equation of balance of mass for the constituent s_α is

$$\frac{d}{dt} \int_{P_\alpha} \rho_\alpha dv = \int_{P_\alpha} m_\alpha dv \tag{14}$$

where ρ_α is the mass density of the constituent s_α and m_α is the rate at which mass is supplied to s_α per unit volume from the other constituent. In this paper we assume that mass elements of each constituent are conserved so that $m_1 = 0$ and $m_2 = 0$. The relation (14) can be written in the form

$$\frac{d}{dt} \int_{P_0} J_{(z)} \rho_\alpha dV = 0 \tag{15}$$

where

$$J_{(1)} = \det\left(\frac{\partial x_i}{\partial X_A}\right), \quad J_{(2)} = \det\left(\frac{\partial y_i}{\partial X_A}\right) \quad (16)$$

With usual assumptions, from (15) we deduce

$$J_{(1)}\rho_1 = \rho_1^0, \quad J_{(2)}\rho_2 = \rho_2^0 \quad (17)$$

where ρ_α^0 is the mass density of the constituent s_α at time t_0 .

We ascribe to the particle $\mathcal{P}^{(z)}$ of constituent s_α an inertia tensor $I_{KL}(\mathbf{X})$ in the reference configuration, which is symmetric and positive definite. At time t , the inertia tensor of the particle in consideration is denoted by $i_{kl}^{(z)}(\mathbf{x}, t)$. Following the basic formulation concerning a single continua [30], we postulate the conservation of microinertia in the following form

$$\frac{d}{dt} \int_{P_\alpha} \frac{\rho_\alpha}{J_{(z)}} i_{kl}^{(z)} \bar{\chi}_{kK}^{(z)} \bar{\chi}_{lL}^{(z)} dv = 0 \quad (18)$$

If we proceed as above, from (17) and (18) we deduce

$$I_{KL}^{(z)} = \frac{1}{J_{(z)}^2} i_{kl}^{(z)} \bar{\chi}_{kK}^{(z)} \bar{\chi}_{lL}^{(z)} \quad (19)$$

Decomposing $i_{kl}^{(z)}$ and $I_{KL}^{(z)}$ as

$$\begin{aligned} i_{kl}^{(z)} &= \frac{1}{2} \mathcal{J}_0^{(z)} \delta_{kl} - \mathcal{J}_{kl}^{(z)}, & \mathcal{J}_0^{(z)} &= \mathcal{J}_{kk}^{(z)} \\ I_{KL}^{(z)} &= \frac{1}{2} J_0^{(z)} \delta_{KL} - J_{KL}^{(z)}, & J_0^{(z)} &= J_{KK}^{(z)} \end{aligned} \quad (20)$$

then, from (19) we have

$$J_0^{(z)} = \frac{\mathcal{J}_0^{(z)}}{J_{(z)}^2}, \quad J_{KL}^{(z)} = \frac{1}{J_{(z)}^2} \mathcal{J}_{kl}^{(z)} \bar{\chi}_{kK}^{(z)} \bar{\chi}_{lL}^{(z)} \quad (21)$$

In view of (4), (8) and symmetry of $\mathcal{F}_{ij}^{(z)}$ the above relations lead to

$$\frac{D\mathcal{J}_0^{(z)}}{Dt} - 2\mathcal{J}_0^{(z)} v^{(z)} = 0, \quad \frac{D\mathcal{J}_{kl}^{(z)}}{Dt} - 2v^{(z)} \mathcal{F}_{kl}^{(z)} + (\epsilon_{krm} \mathcal{F}_{lr}^{(z)} + \epsilon_{lrm} \mathcal{F}_{kr}^{(z)}) v_n^{(z)} = 0 \quad (22)$$

The kinetic energy per unit mass for the constituent s_α is [30]

$$\mathcal{K}^{(z)} = \frac{1}{2} v_i^{(z)} v_i^{(z)} + \frac{1}{4} \mathcal{J}_0^{(z)} v^{(z)} v^{(z)} + \frac{1}{2} \mathcal{F}_{kl}^{(z)} v_k^{(z)} v_l^{(z)} \quad (23)$$

where $\mathbf{v}^\alpha$ is the velocity vector field associated with constituent s_α .

Following [26, 28–30], we postulate an energy balance at time t in the form

$$\begin{aligned} \frac{d}{dt} \sum_{\alpha=1}^2 \int_{P_\alpha} \rho_\alpha (e + \mathcal{H}^{(\alpha)}) dv = \sum_{\alpha=1}^2 \left[\int_{P_\alpha} \rho_\alpha (F_i^{(\alpha)} v_i^{(\alpha)} + G_i^{(\alpha)} v_i^{(\alpha)} + L^{(\alpha)} v^{(\alpha)} + r) dv \right. \\ \left. + \int_{\partial P_\alpha} (t_i^{(\alpha)} v_i^{(\alpha)} + m_i^{(\alpha)} v_i^{(\alpha)} + \pi^{(\alpha)} v^{(\alpha)} + q^{(\alpha)}) da \right] \quad (24) \end{aligned}$$

where e is the internal energy of the mixture per unit mass; $\mathbf{F}^{(\alpha)}$ is the body force per unit mass acting on the constituent s_α ; $\mathbf{G}^{(\alpha)}$ is the body couple per unit mass; $L^{(\alpha)}$ is the generalized body load per unit mass; r is the external volume supply per unit mass and unit time; $\mathbf{t}^{(\alpha)}$ is the partial stress vector; $\mathbf{m}^{(\alpha)}$ is the partial couple stress vector; $\pi^{(\alpha)}$ is the partial microstress associated with s_α and $q^{(\alpha)}$ is the heat flux per unit area and unit time associated with the constituent s_α .

By using the relation (17), we may write (24) in the form

$$\begin{aligned} \frac{d}{dt} \sum_{\alpha=1}^2 \int_{P_0} \rho_\alpha^0 (e + \mathcal{H}^{(\alpha)}) dV = \sum_{\alpha=1}^2 \left[\int_{P_0} \rho_\alpha^0 (F_i^{(\alpha)} v_i^{(\alpha)} + G_i^{(\alpha)} v_i^{(\alpha)} + L^{(\alpha)} v^{(\alpha)} + r) dV \right. \\ \left. + \int_{\partial P_0} (T_i^{(\alpha)} v_i^{(\alpha)} + M_i^{(\alpha)} v_i^{(\alpha)} + \Pi^{(\alpha)} v^{(\alpha)} + Q^{(\alpha)}) dA \right] \quad (25) \end{aligned}$$

where $\mathbf{T}^{(\alpha)}$, $\mathbf{M}^{(\alpha)}$, $\Pi^{(\alpha)}$, $Q^{(\alpha)}$ are the partial stress, the partial couple stress, the partial microstress and the heat flux respectively, associated with the surface ∂P_α but measured per unit area of ∂P_0 .

Using (22) and (23), from (25) we deduce

$$\begin{aligned} \sum_{\alpha=1}^2 \int_{P_0} \rho_\alpha^0 \left(\dot{e} + v_i^{(\alpha)} \dot{v}_i^{(\alpha)} + \mathcal{F}_{kl}^{(\alpha)} v_k^{(\alpha)} \dot{v}_l^{(\alpha)} + \frac{1}{2} \mathcal{F}_0^{(\alpha)} v^{(\alpha)} \dot{v}^{(\alpha)} + \mathcal{F}_{kl}^{(\alpha)} v_k^{(\alpha)} v_l^{(\alpha)} v^{(\alpha)} + \frac{1}{2} \mathcal{F}_0^{(\alpha)} v^{(\alpha)} v^{(\alpha)} v^{(\alpha)} \right) dV \\ = \sum_{\alpha=1}^2 \left[\int_{P_0} \rho_\alpha^0 (F_i^{(\alpha)} v_i^{(\alpha)} + G_i^{(\alpha)} v_i^{(\alpha)} + L^{(\alpha)} v^{(\alpha)} + r) dV \right. \\ \left. + \int_{\partial P_0} (T_i^{(\alpha)} v_i^{(\alpha)} + M_i^{(\alpha)} v_i^{(\alpha)} + \Pi^{(\alpha)} v^{(\alpha)} + Q^{(\alpha)}) dA \right] \quad (26) \end{aligned}$$

Let us now require that the expression given by (26) is invariant under the Galilean group of transformations. Thus, considering a rigid motion of the frame of reference with constant translational and angular velocities, i.e., c_i and b_i constants (see (10)), at time t the frame being brought back to the original orientation, then we have

$$\begin{aligned} \tilde{v}_i^{(1)} = v_i^{(1)} + \epsilon_{ijk} b_j x_k + c_i, \quad \tilde{v}_i^{(2)} = v_i^{(2)} + \epsilon_{ijk} b_j y_k + c_i \\ \tilde{v}_i^{(\alpha)} = v_i^{(\alpha)} + b_i, \quad \tilde{v}^{(\alpha)} = v^{(\alpha)} \end{aligned} \quad (27)$$

where we used the symbol tilde for the quantities referred to the new frame of reference. Under these transformations ρ_α^0 , e , $\mathcal{F}_{kl}^{(\alpha)}$, $\mathcal{F}_0^{(\alpha)}$, $T_i^{(\alpha)}$, $M_i^{(\alpha)}$, $\Pi^{(\alpha)}$, $Q^{(\alpha)}$ and r are not affected since the motion is rigid. But $F_i^{(\alpha)}$, $G_i^{(\alpha)}$ and $L^{(\alpha)}$ must be

accommodated by corresponding accelerations and microaccelerations (spin inertia), i.e.,

$$\widetilde{F}_i^{(z)} - \dot{\widetilde{v}}_i^{(z)} = F_i^{(z)} - \dot{v}_i^{(z)}, \quad \widetilde{G}_i^{(z)} - \dot{\widetilde{\sigma}}_i^{(z)} = G_i^{(z)} - \dot{\sigma}_i^{(z)}, \quad \widetilde{L}^{(z)} - \dot{\widetilde{\sigma}}^{(z)} = L^{(z)} - \dot{\sigma}^{(z)} \quad (28)$$

where the microstretch rotatory inertia $\sigma_i^{(z)}$ and microstretch scalar inertia $\sigma^{(z)}$ are given by [30]

$$\begin{aligned} \sigma_i^{(z)} &= \mathcal{F}_{ij}^{(z)} \dot{v}_j^{(z)} + 2\mathcal{F}_{ij}^{(z)} v_j^{(z)} v^{(z)} + \epsilon_{ijm} \mathcal{F}_{mn}^{(z)} v_j^{(z)} v_n^{(z)} \\ \sigma^{(z)} &= \frac{1}{2} \mathcal{F}_0^{(z)} (\dot{v}^{(z)} + v^{(z)} v^{(z)}) - \mathcal{F}_{ij}^{(z)} v_i^{(z)} v_j^{(z)} \end{aligned} \quad (29)$$

Consider first transformations for which $b_i = 0$. It follows from (27)–(29) that (26) is also true when $\mathbf{v}^{(z)}$ is replaced by $\mathbf{v}^{(z)} + \mathbf{c}$, so that by subtraction we have

$$c_i \sum_{\alpha=1}^2 \left[\int_{P_0} \rho_\alpha^0 (\dot{v}_i^{(z)} - F_i^{(z)}) dV - \int_{\partial P_0} T_i^{(z)} dA \right] = 0$$

for all arbitrary constants c_i . Thus, we deduce that

$$\sum_{\alpha=1}^2 \left[\int_{P_0} \rho_\alpha^0 (\dot{v}_i^{(z)} - F_i^{(z)}) dV - \int_{\partial P_0} T_i^{(z)} dA \right] = 0 \quad (30)$$

for arbitrary region $P_0 \subset B$. From (30), by usual procedures we obtain

$$T_i^{(1)} + T_i^{(2)} = (T_{Ki}^{(1)} + T_{Ki}^{(2)}) N_K \quad (31)$$

where $T_{Ki}^{(z)}$ is the first Piola–Kirchhoff partial stress associated with the constituent s_α and N_K are the components of the unit outward normal vector to the surface ∂P_0 . It follows from (30) and (31) that

$$\sum_{\alpha=1}^2 [T_{Ki,K}^{(z)} + \rho_\alpha^0 (F_i^{(z)} - \dot{v}_i^{(z)})] = 0 \quad (32)$$

If we write the term $T_i^{(1)} v_i^{(1)} + T_i^{(2)} v_i^{(2)}$ in the form

$$T_i^{(1)} v_i^{(1)} + T_i^{(2)} v_i^{(2)} = \frac{1}{2} (T_i^{(1)} + T_i^{(2)}) (v_i^{(1)} + v_i^{(2)}) + \frac{1}{2} (T_i^{(1)} - T_i^{(2)}) (v_i^{(1)} - v_i^{(2)}) \quad (33)$$

and use (31), (32) and

$$\begin{aligned} & \sum_{\alpha=1}^2 \left[\mathcal{F}_{kl}^{(z)} v_k^{(z)} \dot{v}_l^{(z)} + \frac{1}{2} \mathcal{F}_0^{(z)} v^{(z)} \dot{v}^{(z)} + \mathcal{F}_{kl}^{(z)} v_k^{(z)} v_l^{(z)} v^{(z)} + \frac{1}{2} \mathcal{F}_0^{(z)} v^{(z)} v^{(z)} v^{(z)} \right] \\ &= \sum_{\alpha=1}^2 \left[\sigma_i^{(z)} v_i^{(z)} + \sigma^{(z)} v^{(z)} \right] \end{aligned} \quad (34)$$

then (26) reduces to

$$\begin{aligned}
 & \int_{P_0} \left[\rho_0 \dot{e} + \frac{1}{2} (\rho_1^0 \dot{v}_i^{(1)} - \rho_2^0 \dot{v}_i^{(2)} - \rho_1^0 F_i^{(1)} + \rho_2^0 F_i^{(2)}) (v_i^{(1)} - v_i^{(2)}) \right. \\
 & \quad \left. - \frac{1}{2} (T_{Ki}^{(1)} + T_{Ki}^{(2)}) (v_{i,K}^{(1)} + v_{i,K}^{(2)}) \right] dV \\
 & \quad + \sum_{\alpha=1}^2 \int_{P_0} \rho_\alpha^0 [(\sigma_i^{(\alpha)} - G_i^{(\alpha)}) v_i^{(\alpha)} + (\sigma^{(\alpha)} - L^{(\alpha)}) v^{(\alpha)} - r] dV \\
 & = \int_{\partial P_0} \left[\frac{1}{2} (T_i^{(1)} - T_i^{(2)}) (v_i^{(1)} - v_i^{(2)}) + M_i^{(1)} v_i^{(1)} \right. \\
 & \quad \left. + M_i^{(2)} v_i^{(2)} + \Pi^{(1)} v^{(1)} + \Pi^{(2)} v^{(2)} + Q \right] dA
 \end{aligned} \tag{35}$$

where

$$\rho_0 = \rho_1^0 + \rho_2^0 \tag{36}$$

$$Q = Q^{(1)} + Q^{(2)} \tag{37}$$

Using an argument similar to that used in obtaining the relation (31), from (35) we deduce

$$\begin{aligned}
 & \frac{1}{2} [T_i^{(1)} - T_i^{(2)} - (T_{Ki}^{(1)} - T_{Ki}^{(2)}) N_K] (v_i^{(1)} - v_i^{(2)}) + (M_i^{(1)} - M_{Ki}^{(1)} N_K) v_i^{(1)} \\
 & \quad + (M_i^{(2)} - M_{Ki}^{(2)} N_K) v_i^{(2)} + (\Pi^{(1)} - \Pi_K^{(1)} N_K) v^{(1)} + (\Pi^{(2)} - \Pi_K^{(2)} N_K) v^{(2)} + Q - Q_K N_K = 0
 \end{aligned} \tag{38}$$

where $M_{Ki}^{(\alpha)}$ and $\Pi_K^{(\alpha)}$ are the partial couple stress tensor and the partial microstress vector, respectively, associated with the constituent s_α and Q_K is the heat flux vector. Using (38) in (35) and applying the resulting equation to an arbitrary region P_0 , we obtain

$$\begin{aligned}
 \rho_0 \dot{e} &= \sum_{\alpha=1}^2 (T_{Ki}^{(\alpha)} v_{i,K}^{(\alpha)} + M_{Ki}^{(\alpha)} v_{i,K}^{(\alpha)} + \Pi_K^{(\alpha)} v_{,K}^{(\alpha)} + R_i^{(\alpha)} v_i^{(\alpha)} + g^{(\alpha)} v^{(\alpha)}) \\
 & \quad + P_i (v_i^{(1)} - v_i^{(2)}) + Q_{K,K} + \rho_0 r
 \end{aligned} \tag{39}$$

where

$$\begin{aligned}
 P_i &= \frac{1}{2} [T_{Ki,K}^{(1)} + \rho_1^0 F_i^{(1)} - \rho_1^0 \dot{v}_i^{(1)} - T_{Ki,K}^{(2)} - \rho_2^0 F_i^{(2)} + \rho_2^0 \dot{v}_i^{(2)}] \\
 R_i^{(\alpha)} &= M_{Ki,K}^{(\alpha)} + \rho_\alpha^0 G_i^{(\alpha)} - \rho_\alpha^0 \sigma_i^{(\alpha)} \\
 g^{(\alpha)} &= \Pi_{K,K}^{(\alpha)} + \rho_\alpha^0 L^{(\alpha)} - \rho_\alpha^0 \sigma^{(\alpha)}
 \end{aligned} \tag{40}$$

Let us consider now transformations with $\mathbf{c} = \mathbf{0}$. Thus, assuming that the quantities ρ_0 , e , $T_{Ki}^{(\alpha)}$, $M_{Ki}^{(\alpha)}$, $\Pi_K^{(\alpha)}$, r and $Q_{K,K}$ remain invariant, from (27), (28) and

(40) it follows

$$\begin{aligned}\frac{\partial \tilde{v}_i^{(1)}}{\partial x_j} &= \frac{\partial v_i^{(1)}}{\partial x_j} + \epsilon_{jis} b_s, & \frac{\partial \tilde{v}_i^{(2)}}{\partial y_j} &= \frac{\partial v_i^{(2)}}{\partial y_j} + \epsilon_{jis} b_s, & \tilde{v}^{(z)} &= v^{(z)} \\ \tilde{v}_i^{(1)} &= v_i^{(1)} + b_i, & \tilde{v}_i^{(2)} &= v_i^{(2)} + b_i, & \tilde{v}_i^{(1)} - \tilde{v}_i^{(2)} &= v_i^{(1)} - v_i^{(2)} + \epsilon_{jis} b_s (x_j - y_j) \\ \tilde{P}_i &= P_i, & \tilde{R}_i^{(z)} &= R_i^{(z)}, & \tilde{g}^{(z)} &= g^{(z)}\end{aligned}\quad (41)$$

Writing (39) with respect to the new frame of reference and using (41), we obtain by subtraction

$$R_s^{(1)} + R_s^{(2)} + \epsilon_{sji} [T_{Ki}^{(1)} x_{j,K} + T_{Ki}^{(2)} y_{j,K} + P_i (x_j - y_j)] = 0 \quad (42)$$

By using the relation (42), we have

$$\begin{aligned}R_s^{(1)} v_s^{(1)} + R_s^{(2)} v_s^{(2)} &= (R_s^{(1)} + R_s^{(2)}) v_s^{(1)} - R_s^{(2)} (v_s^{(1)} - v_s^{(2)}) \\ &= \epsilon_{sij} [T_{Ki}^{(1)} x_{j,K} + T_{Ki}^{(2)} y_{j,K} + P_i (x_j - y_j)] v_s^{(1)} - R_s^{(2)} (v_s^{(1)} - v_s^{(2)})\end{aligned}\quad (43)$$

With the help of the relation (43) we can write the energy equation (39) in the form

$$\begin{aligned}\rho_0 \dot{e} &= T_{Ki}^{(1)} (v_{i,K}^{(1)} + \epsilon_{ijk} x_{j,K} v_k^{(1)}) + T_{Ki}^{(2)} (v_{i,K}^{(2)} + \epsilon_{ijk} y_{j,K} v_k^{(1)}) + M_{Ki}^{(1)} v_{i,K}^{(1)} + M_{Ki}^{(2)} v_{i,K}^{(2)} \\ &\quad + \Pi_K^{(1)} v_{i,K}^{(1)} + \Pi_K^{(2)} v_{i,K}^{(2)} + P_i [v_i^{(1)} - v_i^{(2)} + \epsilon_{ijk} (x_j - y_j) v_k^{(1)}] - R_i^{(2)} (v_i^{(1)} - v_i^{(2)}) \\ &\quad + g^{(1)} v^{(1)} + g^{(2)} v^{(2)} + Q_{K,K} + \rho_0 r\end{aligned}\quad (44)$$

Moreover it follows from (32), (40) and (42) that the equations of motion are

$$\begin{aligned}T_{Ki,K}^{(1)} - P_i + \rho_1^0 F_i^{(1)} &= \rho_1^0 \ddot{x}_i \\ T_{Ki,K}^{(2)} + P_i + \rho_2^0 F_i^{(2)} &= \rho_2^0 \ddot{y}_i\end{aligned}\quad (45)$$

$$\begin{aligned}M_{Ki,K}^{(1)} + \epsilon_{ijs} [T_{Ks}^{(1)} x_{j,K} + T_{Ks}^{(2)} y_{j,K} + P_s (x_j - y_j)] + R_i^{(2)} + \rho_1^0 G_i^{(1)} &= \rho_1^0 \sigma_i^{(1)} \\ M_{Ki,K}^{(2)} - R_i^{(2)} + \rho_2^0 G_i^{(2)} &= \rho_2^0 \sigma_i^{(2)}\end{aligned}\quad (46)$$

$$\begin{aligned}\Pi_{K,K}^{(1)} - g^{(1)} + \rho_1^0 L^{(1)} &= \rho_1^0 \sigma^{(1)} \\ \Pi_{K,K}^{(2)} - g^{(2)} + \rho_2^0 L^{(2)} &= \rho_2^0 \sigma^{(2)}\end{aligned}\quad (47)$$

We assume that the constituents have a common temperature and we adopt the following entropy production inequality (see [25, 26, 31])

$$\sum_{\alpha=1}^2 \left[\frac{d}{dt} \int_{P_\alpha} \rho_\alpha \eta \, dv - \int_{P_\alpha} \frac{\rho_\alpha r}{\theta} \, dv - \int_{\partial P_\alpha} \frac{q^{(\alpha)}}{\theta} \, da \right] \geq 0 \quad (48)$$

where η is the entropy per unit mass of the mixture and θ is the absolute temperature. The relations (17) and (48) yield

$$\int_{P_0} \rho_0 \dot{\eta} dV - \int_{P_0} \frac{\rho_0 r}{\theta} dV - \int_{\partial P_0} \frac{Q}{\theta} dA \geq 0 \quad (49)$$

Following Green and Steel [32], from (49) we deduce

$$Q = Q_K N_K \quad (50)$$

and the inequality (49) reduces to

$$\rho_0 \theta \dot{\eta} - \rho_0 r - Q_{K,K} + \frac{1}{\theta} Q_K \theta_{,K} \geq 0 \quad (51)$$

The relation (13) and the second Piola–Kirchhoff quantities, defined by

$$\begin{aligned} T_{Ki}^{(1)} &= T_{KL}^{(1)} \chi_{iL}^{(1)}, & T_{Ki}^{(2)} &= T_{KL}^{(2)} \chi_{iL}^{(1)}, & M_{Ki}^{(1)} &= M_{KL}^{(1)} \chi_{iL}^{(1)}, & M_{Ki}^{(2)} &= M_{KL}^{(2)} \chi_{iL}^{(2)} \\ P_i &= \mathcal{P}_K \chi_{iK}^{(1)}, & R_j^{(2)} &= -\epsilon_{ijm} \chi_{mK}^{(1)} \chi_{iL}^{(2)} \mathcal{R}_{KL} \end{aligned} \quad (52)$$

may be used to write the equation of energy (44) in the following material form

$$\begin{aligned} \rho_0 \dot{e} &= T_{KL}^{(1)} \dot{E}_{KL} + T_{KL}^{(2)} \dot{G}_{KL} + M_{KL}^{(1)} \dot{\Gamma}_{KL}^{(1)} + M_{KL}^{(2)} \dot{\Gamma}_{KL}^{(2)} + \Pi_K^{(1)} \dot{\Gamma}_K^{(1)} + \Pi_K^{(2)} \dot{\Gamma}_K^{(2)} \\ &\quad + \mathcal{P}_K \dot{D}_K + \mathcal{R}_{KL} \dot{\Delta}_{KL} + \frac{1}{j_{(1)}^2} g^{(1)} \dot{E}^{(1)} + \frac{1}{j_{(2)}^2} g^{(2)} \dot{E}^{(2)} + Q_{K,K} + \rho_0 r \end{aligned} \quad (53)$$

Let us introduce the Helmholtz free energy $\Upsilon = e - \eta\theta$. Then the energy equation (53) may be written in the form

$$\begin{aligned} \rho_0 (\dot{\Upsilon} + \dot{\eta}\theta + \theta\dot{\eta}) &= T_{KL}^{(1)} \dot{E}_{KL} + T_{KL}^{(2)} \dot{G}_{KL} + M_{KL}^{(1)} \dot{\Gamma}_{KL}^{(1)} + M_{KL}^{(2)} \dot{\Gamma}_{KL}^{(2)} + \Pi_K^{(1)} \dot{\Gamma}_K^{(1)} + \Pi_K^{(2)} \dot{\Gamma}_K^{(2)} \\ &\quad + \mathcal{P}_K \dot{D}_K + \mathcal{R}_{KL} \dot{\Delta}_{KL} + \frac{1}{j_{(1)}^2} g^{(1)} \dot{E}^{(1)} + \frac{1}{j_{(2)}^2} g^{(2)} \dot{E}^{(2)} + Q_{K,K} + \rho_0 r \end{aligned} \quad (54)$$

With the help of (54), the inequality (51) becomes

$$\begin{aligned} T_{KL}^{(1)} \dot{E}_{KL} + T_{KL}^{(2)} \dot{G}_{KL} + M_{KL}^{(1)} \dot{\Gamma}_{KL}^{(1)} + M_{KL}^{(2)} \dot{\Gamma}_{KL}^{(2)} + \Pi_K^{(1)} \dot{\Gamma}_K^{(1)} + \Pi_K^{(2)} \dot{\Gamma}_K^{(2)} \\ + \mathcal{P}_K \dot{D}_K + \mathcal{R}_{KL} \dot{\Delta}_{KL} + \frac{1}{j_{(1)}^2} g^{(1)} \dot{E}^{(1)} + \frac{1}{j_{(2)}^2} g^{(2)} \dot{E}^{(2)} - \rho_0 \dot{\Upsilon} - \rho_0 \dot{\eta}\theta + \frac{1}{\theta} Q_K \theta_{,K} \geq 0 \end{aligned} \quad (55)$$

The quantities $T_{KL}^{(z)}$, $M_{KL}^{(z)}$, $\Pi_K^{(z)}$, $\mathcal{P}_K$, $\mathcal{R}_{KL}$, $g^{(z)}$, Υ , η and Q_K must be prescribed by constitutive equations.

CONSTITUTIVE EQUATIONS

In this section, we state the constitutive equations that serve to classify the particular types of mixtures to be studied throughout the remainder of the paper. We generalize the thermoviscoelastic model of interacting Cosserat continua proposed by Ieșan [26] to the case of heat-conducting microstretch mixtures. In addition, we consider a nonlinear theory. Thus, we assume that the constituent s_1 is a microstretch Kelvin–Voigt material and that the constituent s_2 is a microstretch elastic solid. According to basic postulates, we consider the following independent constitutive variables

$$\mathcal{A} = \left(x_i, \dot{x}_i, x_{i,K}, \dot{x}_{i,K}, \bar{\chi}_{iK}^{(1)}, \dot{\bar{\chi}}_{iK}^{(1)}, \bar{\chi}_{iK,L}^{(1)}, \dot{\bar{\chi}}_{iK,L}^{(1)}, j_{(1)}, \frac{Dj_{(1)}}{Dt}, [j_{(1)}]_{,K}, \left[\frac{Dj_{(1)}}{Dt} \right]_{,K}, \right. \\ \left. y_i, \dot{y}_i, y_{i,K}, \bar{\chi}_{iK}^{(2)}, \dot{\bar{\chi}}_{iK}^{(2)}, \bar{\chi}_{iK,L}^{(2)}, j_{(2)}, [j_{(2)}]_{,K}, \theta, \theta_{,K}; X_K \right) \quad (56)$$

so that, the constitutive equations are

$$T_{KL}^{(z)} = \widehat{T}_{KL}^{(z)}(\mathcal{A}), \quad M_{KL}^{(z)} = \widehat{M}_{KL}^{(z)}(\mathcal{A}), \quad \Pi_K^{(z)} = \widehat{\Pi}_K^{(z)}(\mathcal{A}), \quad g^{(z)} = \widehat{g}^{(z)}(\mathcal{A}) \\ \mathcal{P}_K = \widehat{\mathcal{P}}_K(\mathcal{A}), \quad \mathcal{R}_{KL} = \widehat{\mathcal{R}}_{KL}(\mathcal{A}), \quad \Upsilon = \widehat{\Upsilon}(\mathcal{A}), \quad \eta = \widehat{\eta}(\mathcal{A}), \quad Q_K = \widehat{Q}_K(\mathcal{A}) \quad (57)$$

where the constitutive functionals are assumed to be sufficiently smooth. For homogeneous continua the response functionals do not depend on X_K explicitly.

The constitutive functionals must be form invariant under rigid motions of the frame of reference. Proceeding as in [25], that is considering a translational motion of the frame of reference described by the vector $c_i(t) = y_i(X_K, t)$ followed by a rotation expressed by the proper orthogonal tensor $\mathbf{Q} = [\bar{\chi}^{(1)}]^T$, where $[\bar{\chi}^{(1)}]^T$ denotes the transpose of $\bar{\chi}^{(1)}$, it follows that the dependent constitutive variables must be expressible in the following invariant form:

$$T_{KL}^{(z)} = \widehat{T}_{KL}^{(z)}(\mathcal{S}), \quad M_{KL}^{(z)} = \widehat{M}_{KL}^{(z)}(\mathcal{S}), \quad \Pi_K^{(z)} = \widehat{\Pi}_K^{(z)}(\mathcal{S}), \quad g^{(z)} = \widehat{g}^{(z)}(\mathcal{S}) \\ \mathcal{P}_K = \widehat{\mathcal{P}}_K(\mathcal{S}), \quad \mathcal{R}_{KL} = \widehat{\mathcal{R}}_{KL}(\mathcal{S}), \quad \Upsilon = \widehat{\Upsilon}(\mathcal{S}), \quad \eta = \widehat{\eta}(\mathcal{S}), \quad Q_K = \widehat{Q}_K(\mathcal{S}) \quad (58)$$

where

$$\mathcal{S} = (E_{MN}, \dot{E}_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \dot{\Gamma}_{MN}^{(1)}, \Gamma_{MN}^{(2)}, \Gamma_M^{(1)}, \dot{\Gamma}_M^{(1)}, \Gamma_M^{(2)}, E^{(1)}, \dot{E}^{(1)}, E^{(2)}, \\ D_M, \dot{D}_M, \Delta_{MN}, \dot{\Delta}_{MN}, \theta, \theta_{,M}; X_M) \quad (59)$$

With the help of (58) and (59), inequality (55) becomes

$$\left(T_{KL}^{(1)} - \frac{\partial \mathcal{W}}{\partial E_{KL}} \right) \dot{E}_{KL} + \left(T_{KL}^{(2)} - \frac{\partial \mathcal{W}}{\partial G_{KL}} \right) \dot{G}_{KL} + \left(M_{KL}^{(1)} - \frac{\partial \mathcal{W}}{\partial \Gamma_{KL}^{(1)}} \right) \dot{\Gamma}_{KL}^{(1)} \\ + \left(M_{KL}^{(2)} - \frac{\partial \mathcal{W}}{\partial \Gamma_{KL}^{(2)}} \right) \dot{\Gamma}_{KL}^{(2)} + \left(\Pi_K^{(1)} - \frac{\partial \mathcal{W}}{\partial \Gamma_K^{(1)}} \right) \dot{\Gamma}_K^{(1)} + \left(\Pi_K^{(2)} - \frac{\partial \mathcal{W}}{\partial \Gamma_K^{(2)}} \right) \dot{\Gamma}_K^{(2)}$$

$$\begin{aligned}
& + \left(\mathcal{P}_K - \frac{\partial \mathcal{W}}{\partial D_K} \right) \dot{D}_K + \left(\mathcal{R}_{KL} - \frac{\partial \mathcal{W}}{\partial \Delta_{KL}} \right) \dot{\Delta}_{KL} + \left(\frac{1}{j_{(1)}^2} g^{(1)} - \frac{\partial \mathcal{W}}{\partial E^{(1)}} \right) \dot{E}^{(1)} \\
& + \left(\frac{1}{j_{(2)}^2} g^{(2)} - \frac{\partial \mathcal{W}}{\partial E^{(2)}} \right) \dot{E}^{(2)} - \left(\rho_0 \eta + \frac{\partial \mathcal{W}}{\partial \theta} \right) \dot{\theta} - \frac{\partial \mathcal{W}}{\partial \dot{E}_{KL}} \ddot{E}_{KL} - \frac{\partial \mathcal{W}}{\partial \dot{\Gamma}_{KL}^{(1)}} \ddot{\Gamma}_{KL}^{(1)} \\
& - \frac{\partial \mathcal{W}}{\partial \dot{E}^{(1)}} \ddot{E}^{(1)} - \frac{\partial \mathcal{W}}{\partial \dot{\Gamma}_K^{(1)}} \ddot{\Gamma}_K^{(1)} - \frac{\partial \mathcal{W}}{\partial \dot{D}_K} \ddot{D}_K - \frac{\partial \mathcal{W}}{\partial \dot{\Delta}_{KL}} \ddot{\Delta}_{KL} - \frac{\partial \mathcal{W}}{\partial \theta_{,K}} \dot{\theta}_{,K} + \frac{1}{\theta} Q_K \theta_{,K} \geq 0 \quad (60)
\end{aligned}$$

where $\mathcal{W} = \rho_0 \mathcal{T}$.

From (60) we deduce

$$\mathcal{W} = \widehat{\mathcal{W}}(E_{MN}, G_{MN}, \Gamma_{MN}^{(1)}, \Gamma_{MN}^{(2)}, \Gamma_M^{(1)}, \Gamma_M^{(2)}, E^{(1)}, E^{(2)}, D_M, \Delta_{MN}, \theta, X_M) \quad (61)$$

$$\begin{aligned}
T_{KL}^{(1)} &= \frac{\partial \mathcal{W}}{\partial E_{KL}} + T_{KL}^*, \quad T_{KL}^{(2)} = \frac{\partial \mathcal{W}}{\partial G_{KL}}, \quad M_{KL}^{(1)} = \frac{\partial \mathcal{W}}{\partial \Gamma_{KL}^{(1)}} + M_{KL}^*, \quad M_{KL}^{(2)} = \frac{\partial \mathcal{W}}{\partial \Gamma_{KL}^{(2)}} \\
\Pi_K^{(1)} &= \frac{\partial \mathcal{W}}{\partial \Gamma_K^{(1)}} + \Pi_K^*, \quad \Pi_K^{(2)} = \frac{\partial \mathcal{W}}{\partial \Gamma_K^{(2)}}, \quad \mathcal{P}_K = \frac{\partial \mathcal{W}}{\partial D_K} + \mathcal{P}_K^*, \quad \mathcal{R}_{KL} = \frac{\partial \mathcal{W}}{\partial \Delta_{KL}} + \mathcal{R}_{KL}^* \quad (62) \\
g^{(1)} &= j_{(1)}^2 \frac{\partial \mathcal{W}}{\partial E^{(1)}} + g^*, \quad g^{(2)} = j_{(2)}^2 \frac{\partial \mathcal{W}}{\partial E^{(2)}}, \quad \rho_0 \eta = -\frac{\partial \mathcal{W}}{\partial \theta}
\end{aligned}$$

where T_{KL}^* , M_{KL}^* , Π_K^* , $\mathcal{P}_K^*$, $\mathcal{R}_{KL}^*$, g^* and Q_K are functions of the independent constitutive variables of the set $\mathcal{S}$ and satisfy

$$T_{KL}^* \dot{E}_{KL} + M_{KL}^* \dot{\Gamma}_{KL}^{(1)} + \Pi_K^* \dot{\Gamma}_K^{(1)} + \mathcal{P}_K^* \dot{D}_K + \mathcal{R}_{KL}^* \dot{\Delta}_{KL} + \frac{1}{j_{(1)}^2} g^* \dot{E}^{(1)} + \frac{1}{\theta} Q_K \theta_{,K} \geq 0 \quad (63)$$

By the well-known method, the dissipation inequality (63) implies

$$\begin{aligned}
T_{KL}^*(\mathcal{S}_0) &= 0, \quad M_{KL}^*(\mathcal{S}_0) = 0, \quad \Pi_K^*(\mathcal{S}_0) = 0 \\
\mathcal{P}_K^*(\mathcal{S}_0) &= 0, \quad \mathcal{R}_{KL}^*(\mathcal{S}_0) = 0, \quad g^*(\mathcal{S}_0) = 0, \quad Q_K(\mathcal{S}_0) = 0
\end{aligned} \quad (64)$$

where

$$\mathcal{S}_0 = (E_{MN}, 0, G_{MN}, \Gamma_{MN}^{(1)}, 0, \Gamma_{MN}^{(2)}, \Gamma_M^{(1)}, 0, \Gamma_M^{(2)}, E^{(1)}, 0, E^{(2)}, D_M, 0, \Delta_{MN}, 0, \theta, 0, ; X_M)$$

From (52) and (61) we obtain

$$\begin{aligned}
T_{Ki}^{(1)} &= \bar{\chi}_{iL}^{(1)} \left(\frac{\partial \mathcal{W}}{\partial E_{KL}} + T_{KL}^* \right), \quad T_{Ki}^{(2)} = \bar{\chi}_{iL}^{(1)} \frac{\partial \mathcal{W}}{\partial G_{KL}}, \quad M_{Ki}^{(1)} = \bar{\chi}_{iL}^{(1)} \left(\frac{\partial \mathcal{W}}{\partial \Gamma_{KL}^{(1)}} + M_{KL}^* \right) \\
M_{Ki}^{(2)} &= \bar{\chi}_{iL}^{(2)} \frac{\partial \mathcal{W}}{\partial \Gamma_{KL}^{(2)}}, \quad \Pi_K^{(1)} = \frac{\partial \mathcal{W}}{\partial \Gamma_K^{(1)}} + \Pi_K^*, \quad \Pi_K^{(2)} = \frac{\partial \mathcal{W}}{\partial \Gamma_K^{(2)}} \\
P_i &= \bar{\chi}_{iK}^{(1)} \left(\frac{\partial \mathcal{W}}{\partial D_K} + \mathcal{P}_K^* \right), \quad R_i^{(2)} = \epsilon_{ilm} \bar{\chi}_{mK}^{(1)} \bar{\chi}_{iL}^{(2)} \left(\frac{\partial \mathcal{W}}{\partial \Delta_{KL}} + \mathcal{R}_{KL}^* \right) \\
g^{(1)} &= j_{(1)}^2 \frac{\partial \mathcal{W}}{\partial E^{(1)}} + g^*, \quad g^{(2)} = j_{(2)}^2 \frac{\partial \mathcal{W}}{\partial E^{(2)}}, \quad \rho_0 \eta = -\frac{\partial \mathcal{W}}{\partial \theta}
\end{aligned} \quad (65)$$

so that, the constitutive equations are given by (61), (65) and

$$\begin{aligned} T_{KL}^* &= T_{KL}^*(\mathcal{S}), & M_{KL}^* &= M_{KL}^*(\mathcal{S}), & \Pi_K^* &= \Pi_K^*(\mathcal{S}) \\ \mathcal{P}_K^* &= \mathcal{P}_K^*(\mathcal{S}), & \mathcal{R}_{KL}^* &= \mathcal{R}_{KL}^*(\mathcal{S}), & g^* &= g^*(\mathcal{S}), & Q_K &= Q_K(\mathcal{S}) \end{aligned} \quad (66)$$

where the functionals T_{KL}^* , M_{KL}^* , Π_K^* , $\mathcal{P}_K^*$, $\mathcal{R}_{KL}^*$, g^* and Q_K satisfy the inequality (63) and the restrictions (64).

By using the relations (61) and (62), the energy balance (56) reduces to

$$\begin{aligned} \rho_0 \theta \dot{\eta} &= T_{KL}^* \dot{E}_{KL} + M_{KL}^* \dot{I}_{KL}^{(1)} + \Pi_K^* \dot{I}_K^{(1)} + \mathcal{P}_K^* \dot{D}_K + \mathcal{R}_{KL}^* \dot{\Delta}_{KL} \\ &+ \frac{1}{j_{(1)}^2} g^* \dot{E}^{(1)} + Q_{K,K} + \rho_0 r \end{aligned} \quad (67)$$

In conclusion, the complete system of field equations of nonlinear theory consists of the equations of conservation of microinertia (21), equations of motion (45)–(47), energy equation (67), constitutive equations (61), (65), (66), and the geometric equations (11). To these equations we adjoin boundary and initial conditions. In the case of first boundary value problem the boundary conditions are

$$\begin{aligned} x_i &= \tilde{x}_i, & y_i &= \tilde{y}_i, & \tilde{\chi}_{iK}^{(1)} &= \tilde{\tilde{\chi}}_{iK}^{(1)}, & \tilde{\chi}_{iK}^{(2)} &= \tilde{\tilde{\chi}}_{iK}^{(2)} \\ j_{(1)} &= \tilde{j}_{(1)}, & j_{(2)} &= \tilde{j}_{(2)}, & \theta &= \tilde{\theta} & \text{on } \partial B \times I \end{aligned} \quad (68)$$

where $\tilde{x}_i$, $\tilde{y}_i$, $\tilde{\chi}_{iK}^{(1)}$, $\tilde{\chi}_{iK}^{(2)}$, $\tilde{j}_{(1)}$, $\tilde{j}_{(2)}$ and $\tilde{\theta}$ are prescribed functions. Clearly, $\tilde{\chi}_{iK}^{(1)}$ and $\tilde{\chi}_{iK}^{(2)}$ have to be proper orthogonal. In the second boundary value problem, the boundary conditions are

$$\begin{aligned} (T_{Ki}^{(1)} + T_{Ki}^{(2)}) N_K &= \tilde{T}_i, & (M_{Ki}^{(1)} + M_{Ki}^{(2)}) N_K &= \tilde{M}_i, & (\Pi_K^{(1)} + \Pi_K^{(2)}) N_K &= \tilde{\Pi} \\ x_i - y_i &= \tilde{d}_i, & \chi_{iK}^{(1)} \chi_{iL}^{(2)} &= \tilde{\Delta}_{KL}, & j_{(1)} - j_{(2)} &= \tilde{j}, & Q_K N_K &= \tilde{Q} & \text{on } \partial B \times I \end{aligned} \quad (69)$$

where $\tilde{T}_i$, $\tilde{M}_i$, $\tilde{\Pi}$, $\tilde{d}_i$, $\tilde{\Delta}_{KL}$, $\tilde{j}$ and $\tilde{Q}$ are given data and $\tilde{\Delta}_{KL}$ is proper orthogonal.

The initial conditions are

$$\begin{aligned} x_i(\mathbf{X}, 0) &= x_i^0(\mathbf{X}), & y_i(\mathbf{X}, 0) &= y_i^0(\mathbf{X}), & \dot{x}_i(\mathbf{X}, 0) &= v_i^{(1)0}(\mathbf{X}) \\ \dot{y}_i(\mathbf{X}, 0) &= v_i^{(2)0}(\mathbf{X}), & \tilde{\chi}_{iK}^{(z)}(\mathbf{X}, 0) &= \tilde{\chi}_{iK}^{(z)0}(\mathbf{X}), & v_i^{(z)}(\mathbf{X}, 0) &= v_i^{(z)0}(\mathbf{X}) \\ j_{(z)}(\mathbf{X}) &= j_{(z)}^0(\mathbf{X}, 0), & v^{(z)}(\mathbf{X}, 0) &= v^{(z)0}(\mathbf{X}), & \eta(\mathbf{X}, 0) &= \eta^0(\mathbf{X}), & \mathbf{X} &\in B \end{aligned} \quad (70)$$

where x_i^0 , y_i^0 , $v_i^{(z)0}$, $\tilde{\chi}_{iK}^{(z)0}$, $j_{(z)}^0$, $v^{(z)0}$, η^0 are prescribed functions.

THE LINEAR THEORY

This section is devoted to the discussion of a linear theory. The linear constitutive equations are obtained for both anisotropic and isotropic thermomicrostretch mixture consisting of a Kelvin–Voigt material and an elastic solid.

We use the notations

$$X_i = \delta_{iK} X_K, \quad \bar{\chi}_{ij}^{(z)} = \delta_{jK} \bar{\chi}_{iK}^{(z)}, \quad \frac{\partial f}{\partial X_i} = f_{,i} \quad (71)$$

where δ_{iK} is the Kronecker delta. Let us introduce the displacement vectors $\mathbf{u}$ and $\mathbf{w}$ associated with s_1 and s_2 , respectively, and the temperature variation T from the constant absolute temperature T_0 of the body in the reference configuration. So, we have

$$x_i = X_i + u_i, \quad y_i = X_i + w_i, \quad T = \theta - T_0 \quad (72)$$

Being concerned with first-order approximations, following [30] and [33] (see (1.2.19) and (1.6.6)), we take

$$\bar{\chi}_{ij}^{(z)} \simeq \delta_{ij} - \epsilon_{ijs} \varphi_s^{(z)}, \quad j_{(z)} \simeq 1 + \phi^{(z)} \quad (73)$$

where $\varphi^{(z)}$ is the microrotation vector associated with the constituent s_z and $\phi^{(z)}$ is the microstretch function associated with the constituent s_z . In the linear theory, u_i , w_i , T , $\varphi_i^{(z)}$ and $\phi^{(z)}$ are small, that is

$$u_i = \varepsilon \tilde{u}_i, \quad w_i = \varepsilon \tilde{w}_i, \quad T = \varepsilon \tilde{T}, \quad \varphi_i^{(z)} = \varepsilon \tilde{\varphi}_i^{(z)}, \quad \phi^{(z)} = \varepsilon \tilde{\phi}^{(z)}, \quad \alpha = 1, 2 \quad (74)$$

where ε is a constant small enough for squares and higher powers to be neglected and $\tilde{u}_i$, $\tilde{w}_i$, $\tilde{T}$, $\tilde{\varphi}_i^{(z)}$ and $\tilde{\phi}^{(z)}$ are independent of ε . It follows from (6), (8), (73) and (74) that

$$v_i^{(z)} = \dot{\varphi}_i^{(z)}, \quad v^{(z)} = \dot{\phi}^{(z)} \quad (75)$$

The strain measures E_{KL} , G_{KL} , $\Gamma_{KL}^{(z)}$, $\Gamma_K^{(z)}$, $E^{(z)}$, D_K and Δ_{KL} defined by (11) reduce to

$$\begin{aligned} e_{ji} &= u_{i,j} + \epsilon_{sij} \varphi_s^{(1)}, & g_{ji} &= w_{i,j} + \epsilon_{sij} \varphi_s^{(1)}, & \gamma_{ij}^{(z)} &= \varphi_{j,i}^{(z)} \\ \gamma_i^{(z)} &= \phi_{,i}^{(z)}, & e^{(z)} &= \phi^{(z)}, & d_i &= u_i - w_i, & \Delta_{ij} &= \epsilon_{ijs} (\varphi_s^{(1)} - \varphi_s^{(2)}) \end{aligned} \quad (76)$$

and the microinertia coefficients $\mathcal{J}_0^{(z)}$, $\mathcal{J}_{ij}^{(z)}$ (see (21)) are given by

$$\mathcal{J}_0^{(z)} = J_0^{(z)} + 2\phi^{(z)} J_0^{(z)}, \quad \mathcal{J}_{ij}^{(z)} = J_{ij}^{(z)} - \epsilon_{iks} J_{kj}^{(z)} \varphi_s^{(z)} - \epsilon_{jks} J_{ik}^{(z)} \varphi_s^{(z)} - 2\phi^{(z)} J_{ij}^{(z)} \quad (77)$$

Let us introduce the notations

$$\begin{aligned} t_{ji} &= \delta_{jK} T_{Ki}^{(1)}, & s_{ji} &= \delta_{jK} T_{Ki}^{(2)}, & m_{ji}^{(z)} &= \delta_{jK} M_{Ki}^{(z)}, & \pi_i^{(z)} &= \delta_{iK} \Pi_K^{(z)} \\ p_i &= P_i, & r_i &= -R_i^{(2)}, & q_i &= \delta_{iK} Q_K, & t_{ij}^* &= \delta_{iK} \delta_{jL} T_{KL}^* \\ m_{ij}^* &= \delta_{iK} \delta_{jL} M_{KL}^*, & \pi_i^* &= \delta_{iK} \Pi_K^*, & p_i^* &= \delta_{iK} \mathcal{P}_K^*, & r_i^* &= \epsilon_{ikl} \delta_{kK} \delta_{lL} \mathcal{R}_{KL}^* \end{aligned} \quad (78)$$

and

$$\Delta_i = \varphi_i^{(1)} - \varphi_i^{(2)} \quad (79)$$

From (61), (76) and (79) it follows that $\mathcal{W}$ depends on the variables

$$\mathcal{W} = \widehat{\mathcal{W}}(e_{ij}, g_{ij}, \gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}, \gamma_i^{(1)}, \gamma_i^{(2)}, \phi^{(1)}, \phi^{(2)}, d_i, \Delta_i, T; X_i) \quad (80)$$

Collecting (65), (73), (74), (76), (78) and (79) we deduce

$$\begin{aligned} t_{ij} &= \frac{\partial \mathcal{W}}{\partial e_{ij}} + t_{ij}^*, & s_{ij} &= \frac{\partial \mathcal{W}}{\partial g_{ij}}, & m_{ij}^{(1)} &= \frac{\partial \mathcal{W}}{\partial \gamma_{ij}^{(1)}} + m_{ij}^*, & m_{ij}^{(2)} &= \frac{\partial \mathcal{W}}{\partial \gamma_{ij}^{(2)}} \\ \pi_i^{(1)} &= \frac{\partial \mathcal{W}}{\partial \gamma_i^{(1)}} + \pi_i^*, & \pi_i^{(2)} &= \frac{\partial \mathcal{W}}{\partial \gamma_i^{(2)}}, & p_i &= \frac{\partial \mathcal{W}}{\partial d_i} + p_i^*, & \rho_0 \eta &= -\frac{\partial \mathcal{W}}{\partial T} \\ r_i &= \epsilon_{ijk} \frac{\partial \mathcal{W}}{\partial \Delta_s} \frac{\partial \Delta_s}{\partial \Delta_{jk}} + r_i^* = \frac{\partial \mathcal{W}}{\partial \Delta_i} + r_i^*, & g^{(1)} &= \frac{\partial \mathcal{W}}{\partial \phi^{(1)}} + g^*, & g^{(2)} &= \frac{\partial \mathcal{W}}{\partial \phi^{(2)}} \end{aligned} \quad (81)$$

Assuming that the initial body is free from stress, couple stress and microstress, in the context of linear theory we have

$$\mathcal{W} = \mathcal{U} - \mathcal{F} \quad (82)$$

where

$$\begin{aligned} \mathcal{U} &= \frac{1}{2} A_{ijrs} e_{ij} e_{rs} + B_{ijrs} e_{ij} g_{rs} + \frac{1}{2} C_{ijrs} g_{ij} g_{rs} + \sum_{\alpha=1}^2 \left(F_{ijrs}^{(\alpha)} e_{ij} \gamma_{rs}^{(\alpha)} + H_{ijrs}^{(\alpha)} g_{ij} \gamma_{rs}^{(\alpha)} \right. \\ &\quad \left. + F_{ijk}^{(\alpha)} e_{ij} \gamma_k^{(\alpha)} + H_{ijk}^{(\alpha)} g_{ij} \gamma_k^{(\alpha)} + A_{ij}^{(\alpha)} e_{ij} \phi^{(\alpha)} + C_{ij}^{(\alpha)} g_{ij} \phi^{(\alpha)} \right) + a_{ijk} e_{ij} d_k + b_{ijk} e_{ij} \Delta_k \\ &\quad + c_{ijk} g_{ij} d_k + d_{ijk} g_{ij} \Delta_k + \sum_{\alpha, \beta=1}^2 \left(\frac{1}{2} D_{ijrs}^{(\alpha\beta)} \gamma_{ij}^{(\alpha)} \gamma_{rs}^{(\beta)} + \frac{1}{2} D_{ij}^{(\alpha\beta)} \gamma_i^{(\alpha)} \gamma_j^{(\beta)} \right. \\ &\quad \left. + D_{ijk}^{(\alpha\beta)} \gamma_{ij}^{(\alpha)} \gamma_k^{(\beta)} + f_{ij}^{(\alpha\beta)} \gamma_{ij}^{(\alpha)} \phi^{(\beta)} + f_i^{(\alpha\beta)} \gamma_i^{(\alpha)} \phi^{(\beta)} + \frac{1}{2} f^{(\alpha\beta)} \phi^{(\alpha)} \phi^{(\beta)} \right) \\ &\quad + \sum_{\alpha=1}^2 \left(\alpha_{ijk}^{(\alpha)} \gamma_{ij}^{(\alpha)} d_k + \beta_{ijk}^{(\alpha)} \gamma_{ij}^{(\alpha)} \Delta_k + \alpha_{ij}^{(\alpha)} \gamma_i^{(\alpha)} d_j + \beta_{ij}^{(\alpha)} \gamma_i^{(\alpha)} \Delta_j + \tau_i^{(\alpha)} \phi^{(\alpha)} d_i + \sigma_i^{(\alpha)} \phi^{(\alpha)} \Delta_i \right) \\ &\quad + \frac{1}{2} a_{ij} d_i d_j + \frac{1}{2} b_{ij} \Delta_i \Delta_j + c_{ij} d_i \Delta_j \end{aligned} \quad (83)$$

$$\begin{aligned} \mathcal{F} &= \frac{1}{2} a T^2 + \left(a_{ij}^{\circ} e_{ij} + c_{ij}^{\circ} g_{ij} + \alpha_{ij}^{\circ} \gamma_{ij}^{(1)} + \beta_{ij}^{\circ} \gamma_{ij}^{(2)} + \alpha_i^{\circ} \gamma_i^{(1)} + \beta_i^{\circ} \gamma_i^{(2)} \right. \\ &\quad \left. + \tau^{\circ} \phi^{(1)} + \sigma^{\circ} \phi^{(2)} + \tau_i^{\circ} d_i + \sigma_i^{\circ} \Delta_i \right) T \end{aligned} \quad (84)$$

The constitutive coefficients have the following symmetries

$$\begin{aligned} A_{ijrs} &= A_{rsij}, & C_{ijrs} &= C_{rsij}, & D_{ijrs}^{(\alpha\beta)} &= D_{rsij}^{(\beta\alpha)} \\ D_{ij}^{(\alpha\beta)} &= D_{ji}^{(\beta\alpha)}, & a_{ij} &= a_{ji}, & b_{ij} &= b_{ji}, & f^{(12)} &= f^{(21)} \end{aligned} \quad (85)$$

In the framework of linear theory, the relations (64), (66) and (78) assure that

$$\begin{aligned}
 t_{ij}^* &= A_{ijrs}^* \dot{e}_{rs} + B_{ijrs}^* \dot{\gamma}_{rs}^{(1)} + A_{ijk}^* \dot{\gamma}_k^{(1)} + A_{ij}^* \dot{\phi}^{(1)} + B_{ijk}^* \dot{d}_k + C_{ijk}^* \dot{\Delta}_k + a_{ijk}^* T_{,k} \\
 m_{ij}^* &= C_{ijrs}^* \dot{e}_{rs} + D_{ijrs}^* \dot{\gamma}_{rs}^{(1)} + D_{ijk}^* \dot{\gamma}_k^{(1)} + B_{ij}^* \dot{\phi}^{(1)} + E_{ijk}^* \dot{d}_k + F_{ijk}^* \dot{\Delta}_k + b_{ijk}^* T_{,k} \\
 \pi_i^* &= G_{irs}^* \dot{e}_{rs} + H_{irs}^* \dot{\gamma}_{rs}^{(1)} + C_{ij}^* \dot{\gamma}_j^{(1)} + A_i^* \dot{\phi}^{(1)} + D_{ij}^* \dot{d}_j + E_{ij}^* \dot{\Delta}_j + a_{ij}^* T_{,j} \\
 p_i^* &= J_{irs}^* \dot{e}_{rs} + K_{irs}^* \dot{\gamma}_{rs}^{(1)} + F_{ij}^* \dot{\gamma}_j^{(1)} + B_i^* \dot{\phi}^{(1)} + G_{ij}^* \dot{d}_j + H_{ij}^* \dot{\Delta}_j + b_{ij}^* T_{,j} \\
 r_i^* &= L_{irs}^* \dot{e}_{rs} + M_{irs}^* \dot{\gamma}_{rs}^{(1)} + J_{ij}^* \dot{\gamma}_j^{(1)} + C_i^* \dot{\phi}^{(1)} + K_{ij}^* \dot{d}_j + L_{ij}^* \dot{\Delta}_j + c_{ij}^* T_{,j} \\
 g^* &= M_{ij}^* \dot{e}_{ij} + N_{ij}^* \dot{\gamma}_{ij}^{(1)} + D_i^* \dot{\gamma}_i^{(1)} + A^* \dot{\phi}^{(1)} + E_i^* \dot{d}_i + F_i^* \dot{\Delta}_i + a_i^* T_{,i} \\
 q_i &= \alpha_{irs}^* \dot{e}_{rs} + \beta_{irs}^* \dot{\gamma}_{rs}^{(1)} + \alpha_{ij}^* \dot{\gamma}_j^{(1)} + \alpha_i^* \dot{\phi}^{(1)} + \beta_{ij}^* \dot{d}_j + \zeta_{ij}^* \dot{\Delta}_j + k_{ij} T_{,j}
 \end{aligned} \tag{86}$$

From (81)–(84) it follows that the constitutive equations are

$$\begin{aligned}
 t_{ij} &= A_{ijrs} e_{rs} + B_{ijrs} g_{rs} + \sum_{\alpha=1}^2 (F_{ijrs}^{(\alpha)} \gamma_{rs}^{(\alpha)} + F_{ijk}^{(\alpha)} \gamma_k^{(\alpha)} + A_{ij}^{(\alpha)} \phi^{(\alpha)}) \\
 &\quad + a_{ijk} d_k + b_{ijk} \Delta_k - a_{ij}^{\otimes} T + t_{ij}^* \\
 s_{ij} &= B_{rsij} e_{rs} + C_{ijrs} g_{rs} + \sum_{\alpha=1}^2 (H_{ijrs}^{(\alpha)} \gamma_{rs}^{(\alpha)} + H_{ijk}^{(\alpha)} \gamma_k^{(\alpha)} + C_{ij}^{(\alpha)} \phi^{(\alpha)}) + c_{ijk} d_k + d_{ijk} \Delta_k - c_{ij}^{\otimes} T \\
 m_{ij}^{(1)} &= F_{rsij}^{(1)} e_{rs} + H_{rsij}^{(1)} g_{rs} + \sum_{\alpha=1}^2 (D_{ijrs}^{(1\alpha)} \gamma_{rs}^{(\alpha)} + D_{ijk}^{(1\alpha)} \gamma_k^{(\alpha)} + f_{ij}^{(1\alpha)} \phi^{(\alpha)}) \\
 &\quad + \alpha_{ijk}^{(1)} d_k + \beta_{ijk}^{(1)} \Delta_k - \alpha_{ij}^{\otimes} T + m_{ij}^* \\
 m_{ij}^{(2)} &= F_{rsij}^{(2)} e_{rs} + H_{rsij}^{(2)} g_{rs} + \sum_{\alpha=1}^2 (D_{ijrs}^{(2\alpha)} \gamma_{rs}^{(\alpha)} + D_{ijk}^{(2\alpha)} \gamma_k^{(\alpha)} + f_{ij}^{(2\alpha)} \phi^{(\alpha)}) + \alpha_{ijk}^{(2)} d_k + \beta_{ijk}^{(2)} \Delta_k - \beta_{ij}^{\otimes} T \\
 \pi_i^{(1)} &= F_{rsi}^{(1)} e_{rs} + H_{rsi}^{(1)} g_{rs} + \sum_{\alpha=1}^2 (D_{rsi}^{(\alpha 1)} \gamma_{rs}^{(\alpha)} + D_{ik}^{(1\alpha)} \gamma_k^{(\alpha)} + f_i^{(1\alpha)} \phi^{(\alpha)}) \\
 &\quad + \alpha_{ik}^{(1)} d_k + \beta_{ik}^{(1)} \Delta_k - \alpha_i^{\otimes} T + \pi_i^* \\
 \pi_i^{(2)} &= F_{rsi}^{(2)} e_{rs} + H_{rsi}^{(2)} g_{rs} + \sum_{\alpha=1}^2 (D_{rsi}^{(\alpha 2)} \gamma_{rs}^{(\alpha)} + D_{ik}^{(2\alpha)} \gamma_k^{(\alpha)} + f_i^{(2\alpha)} \phi^{(\alpha)}) \\
 &\quad + \alpha_{ik}^{(2)} d_k + \beta_{ik}^{(2)} \Delta_k - \beta_i^{\otimes} T \\
 p_i &= a_{rsi} e_{rs} + c_{rsi} g_{rs} + \sum_{\alpha=1}^2 (\alpha_{rsi}^{(\alpha)} \gamma_{rs}^{(\alpha)} + \alpha_{ki}^{(\alpha)} \gamma_k^{(\alpha)} + \tau_i^{(\alpha)} \phi^{(\alpha)}) + a_{ik} d_k + c_{ik} \Delta_k - \tau_i^{\otimes} T + p_i^* \\
 r_i &= b_{rsi} e_{rs} + d_{rsi} g_{rs} + \sum_{\alpha=1}^2 (\beta_{rsi}^{(\alpha)} \gamma_{rs}^{(\alpha)} + \beta_{ki}^{(\alpha)} \gamma_k^{(\alpha)} + \sigma_i^{(\alpha)} \phi^{(\alpha)}) + c_{ki} d_k + b_{ik} \Delta_k - \sigma_i^{\otimes} T + r_i^* \\
 g^{(1)} &= A_{ij}^{(1)} e_{ij} + C_{ij}^{(1)} g_{ij} + \sum_{\alpha=1}^2 (f_{ij}^{(\alpha 1)} \gamma_{ij}^{(\alpha)} + f_i^{(\alpha 1)} \gamma_i^{(\alpha)} + f^{(1\alpha)} \phi^{(\alpha)}) + \tau_i^{(1)} d_i + \sigma_i^{(1)} \Delta_i - \tau^{\otimes} T + g^*
 \end{aligned}$$

$$\begin{aligned}
g^{(2)} &= A_{ij}^{(2)} e_{ij} + C_{ij}^{(2)} g_{ij} + \sum_{\alpha=1}^2 (f_{ij}^{(\alpha 2)} \gamma_{ij}^{(\alpha)} + f_i^{(\alpha 2)} \gamma_i^{(\alpha)} + f^{(2\alpha)} \phi^{(\alpha)}) + \tau_i^{(2)} d_i + \sigma_i^{(2)} \Delta_i - \sigma^{\circ} T \\
\rho_0 \eta &= a_{ij}^{\circ} e_{ij} + c_{ij}^{\circ} g_{ij} + \alpha_{ij}^{\circ} \gamma_{ij}^{(1)} + \beta_{ij}^{\circ} \gamma_{ij}^{(2)} + \alpha_i^{\circ} \gamma_i^{(1)} + \beta_i^{\circ} \gamma_i^{(2)} + \tau^{\circ} \phi^{(1)} \\
&\quad + \sigma^{\circ} \phi^{(2)} + \tau_i^{\circ} d_i + \sigma_i^{\circ} \Delta_i + aT
\end{aligned} \tag{87}$$

and (86). The dissipation inequality (63) becomes

$$\mathcal{G} = t_{ij}^* \dot{e}_{ij} + m_{ij}^* \dot{\gamma}_{ij}^{(1)} + \pi_i^* \dot{\gamma}_i^{(1)} + p_i^* \dot{d}_i + r_i^* \dot{\Delta}_i + g^* \dot{\phi}^{(1)} + \frac{1}{T_0} q_i T_{,i} \geq 0 \tag{88}$$

where t_{ij}^* , m_{ij}^* , π_i^* , p_i^* , r_i^* , g^* and q_i are given by (86). This inequality holds for any variables $\dot{e}_{ij}$, $\dot{\gamma}_{ij}^{(1)}$, $\dot{\gamma}_i^{(1)}$, $\dot{d}_i$, $\dot{\Delta}_i$, $\dot{\phi}^{(1)}$ and $T_{,i}$.

For isotropic solids, odd order constitutive tensors vanish and even order tensors can be constituted by the products of δ_{ij} . We also note that e_{ij} , g_{ij} , d_i , $\gamma_i^{(\alpha)}$, $\phi^{(\alpha)}$, T , t_{ij} , s_{ij} , $\pi_i^{(\alpha)}$, p_i , $g^{(\alpha)}$ and η are polar tensors, while $\gamma_{ij}^{(\alpha)}$, Δ_i , $m_{ij}^{(\alpha)}$ and r_i are axial tensors. By examining (86) and (87), we find that the only surviving isotropic material constants are

$$\begin{aligned}
A_{ijrs} &= \lambda_1 \delta_{ij} \delta_{rs} + (\mu_1 + \kappa_1) \delta_{ir} \delta_{js} + \mu_1 \delta_{is} \delta_{jr}, \\
C_{ijrs} &= \lambda_2 \delta_{ij} \delta_{rs} + (\mu_2 + \kappa_2) \delta_{ir} \delta_{js} + \mu_2 \delta_{is} \delta_{jr} \\
B_{ijrs} &= \nu \delta_{ij} \delta_{rs} + \xi \delta_{ir} \delta_{js} + \zeta \delta_{is} \delta_{jr}, \quad D_{ijrs}^{(\alpha\alpha)} = \alpha_{\alpha} \delta_{ij} \delta_{rs} + \gamma_{\alpha} \delta_{ir} \delta_{js} + \beta_{\alpha} \delta_{is} \delta_{jr} \\
D_{ijrs}^{(12)} &= D_{rsij}^{(21)} = \alpha_3 \delta_{ij} \delta_{rs} + \gamma_3 \delta_{ir} \delta_{js} + \beta_3 \delta_{is} \delta_{jr}, \quad D_{ijk}^{(11)} = D_1 \epsilon_{ijk}, \quad D_{ijk}^{(22)} = D_2 \epsilon_{ijk} \\
D_{ijk}^{(12)} &= D_3 \epsilon_{ijk}, \quad D_{ijk}^{(21)} = D_4 \epsilon_{ijk}, \quad D_{ij}^{(11)} = F_1 \delta_{ij}, \quad D_{ij}^{(22)} = F_2 \delta_{ij}, \quad D_{ij}^{(12)} = D_{ji}^{(21)} = F_3 \delta_{ij} \\
A_{ij}^{(\alpha)} &= A^{(\alpha)} \delta_{ij}, \quad C_{ij}^{(\alpha)} = C^{(\alpha)} \delta_{ij}, \quad b_{ijk} = b^0 \epsilon_{ijk}, \quad d_{ijk} = d^0 \epsilon_{ijk}, \quad \alpha_{ijk}^{(1)} = \alpha^0 \epsilon_{ijk} \\
\alpha_{ijk}^{(2)} &= \beta^0 \epsilon_{ijk}, \quad \alpha_{ij}^{(1)} = A^0 \delta_{ij}, \quad \alpha_{ij}^{(2)} = B^0 \delta_{ij}, \quad a_{ij} = a^0 \delta_{ij}, \quad b_{ij} = c^0 \delta_{ij} \\
f^{(11)} &= f_1, \quad f^{(22)} = f_2, \quad f^{(12)} = f^{(21)} = f_3, \\
a_{ij}^{\circ} &= a^{\circ} \delta_{ij}, \quad c_{ij}^{\circ} = c^{\circ} \delta_{ij}, \quad \tau^{\circ}, \quad \sigma^{\circ}
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
A_{ijrs}^* &= \lambda_1^* \delta_{ij} \delta_{rs} + (\mu_1^* + \kappa_1^*) \delta_{ir} \delta_{js} + \mu_1^* \delta_{is} \delta_{jr}, \quad D_{ijrs}^* = \alpha_1^* \delta_{ij} \delta_{rs} + \gamma_1^* \delta_{ir} \delta_{js} + \beta_1^* \delta_{is} \delta_{jr} \\
A_{ij}^* &= B^* \delta_{ij}, \quad C_{ijk}^* = c^* \epsilon_{ijk}, \quad D_{ijk}^* = d^* \epsilon_{ijk}, \quad E_{ijk}^* = e^* \epsilon_{ijk}, \quad H_{ijk}^* = h^* \epsilon_{ijk} \\
C_{ij}^* &= C^* \delta_{ij}, \quad D_{ij}^* = D^* \delta_{ij}, \quad a_{ij}^* = \zeta^* \delta_{ij}, \quad K_{ijk}^* = k^* \epsilon_{ijk}, \quad F_{ij}^* = F^* \delta_{ij} \\
G_{ij}^* &= G^* \delta_{ij}, \quad b_{ij}^* = \zeta^* \delta_{ij}, \quad L_{ijk}^* = l^* \epsilon_{ijk}, \quad L_{ij}^* = L^* \delta_{ij}, \quad M_{ij}^* = M^* \delta_{ij} \\
\beta_{ijk}^* &= \gamma^* \epsilon_{ijk}, \quad \alpha_{ij}^* = \alpha^* \delta_{ij}, \quad \beta_{ij}^* = \beta^* \delta_{ij}, \quad k_{ij} = k \delta_{ij}, \quad A^*
\end{aligned} \tag{90}$$

The constitutive equations reduce to

$$\begin{aligned}
 t_{ij} &= \lambda_1 e_{rr} \delta_{ij} + (\mu_1 + \kappa_1) e_{ij} + \mu_1 e_{ji} + \nu g_{rr} \delta_{ij} + \zeta g_{ij} + \zeta g_{ji} \\
 &\quad + A^{(1)} \phi^{(1)} \delta_{ij} + A^{(2)} \phi^{(2)} \delta_{ij} + b^0 \epsilon_{ijk} \Delta_k - a^{\circ} T \delta_{ij} + t_{ij}^* \\
 s_{ij} &= \nu e_{rr} \delta_{ij} + \zeta e_{ij} + \zeta e_{ji} + \lambda_2 g_{rr} \delta_{ij} + (\mu_2 + \kappa_2) g_{ij} + \mu_2 g_{ji} \\
 &\quad + C^{(1)} \phi^{(1)} \delta_{ij} + C^{(2)} \phi^{(2)} \delta_{ij} + d^0 \epsilon_{ijk} \Delta_k - c^{\circ} T \delta_{ij} \\
 m_{ij}^{(1)} &= \alpha_1 \gamma_{rr}^{(1)} \delta_{ij} + \gamma_1 \gamma_{ij}^{(1)} + \beta_1 \gamma_{ji}^{(1)} + \alpha_3 \gamma_{rr}^{(2)} \delta_{ij} + \gamma_3 \gamma_{ij}^{(2)} + \beta_3 \gamma_{ji}^{(2)} \\
 &\quad + D_1 \epsilon_{ijk} \gamma_k^{(1)} + D_3 \epsilon_{ijk} \gamma_k^{(2)} + \alpha^0 \epsilon_{ijk} d_k + m_{ij}^* \\
 m_{ij}^{(2)} &= \alpha_3 \gamma_{rr}^{(1)} \delta_{ij} + \gamma_3 \gamma_{ij}^{(1)} + \beta_3 \gamma_{ji}^{(1)} + \alpha_2 \gamma_{rr}^{(2)} \delta_{ij} + \gamma_2 \gamma_{ij}^{(2)} + \beta_2 \gamma_{ji}^{(2)} \\
 &\quad + D_4 \epsilon_{ijk} \gamma_k^{(1)} + D_2 \epsilon_{ijk} \gamma_k^{(2)} + \beta^0 \epsilon_{ijk} d_k \\
 \pi_i^{(1)} &= D_1 \epsilon_{rsi} \gamma_{rs}^{(1)} + D_4 \epsilon_{rsi} \gamma_{rs}^{(2)} + F_1 \gamma_i^{(1)} + F_3 \gamma_i^{(2)} + A^0 d_i + \pi_i^* \\
 \pi_i^{(2)} &= D_3 \epsilon_{rsi} \gamma_{rs}^{(1)} + D_2 \epsilon_{rsi} \gamma_{rs}^{(2)} + F_3 \gamma_i^{(1)} + F_2 \gamma_i^{(2)} + B^0 d_i \\
 p_i &= \alpha^0 \epsilon_{rsi} \gamma_{rs}^{(1)} + \beta^0 \epsilon_{rsi} \gamma_{rs}^{(2)} + A^0 \gamma_i^{(1)} + B^0 \gamma_i^{(2)} + a^0 d_i + p_i^* \\
 r_i &= b^0 \epsilon_{rsi} e_{rs} + d^0 \epsilon_{rsi} g_{rs} + c^0 \Delta_i + r_i^* \\
 g^{(1)} &= A^{(1)} e_{rr} + C^{(1)} g_{rr} + f_1 \phi^{(1)} + f_3 \phi^{(2)} - \tau^{\circ} T + g^* \\
 g^{(2)} &= A^{(2)} e_{rr} + C^{(2)} g_{rr} + f_3 \phi^{(1)} + f_2 \phi^{(2)} - \sigma^{\circ} T \\
 \rho_0 \eta &= a^{\circ} e_{rr} + c^{\circ} g_{rr} + \tau^{\circ} \phi^{(1)} + \sigma^{\circ} \phi^{(2)} + aT \\
 q_i &= \gamma^* \epsilon_{ijk} \dot{\gamma}_{jk}^{(1)} + \alpha^* \dot{\gamma}_i^{(1)} + \beta^* \dot{d}_i + kT_{,i}
 \end{aligned} \tag{91}$$

where

$$\begin{aligned}
 t_{ij}^* &= \lambda_1^* \dot{e}_{rr} \delta_{ij} + (\mu_1^* + \kappa_1^*) \dot{e}_{ij} + \mu_1^* \dot{e}_{ji} + B^* \dot{\phi}^{(1)} \delta_{ij} + c^* \epsilon_{ijk} \dot{\Delta}_k \\
 m_{ij}^* &= \alpha_1^* \dot{\gamma}_{rr}^{(1)} \delta_{ij} + \gamma_1^* \dot{\gamma}_{ij}^{(1)} + \beta_1^* \dot{\gamma}_{ji}^{(1)} + d^* \epsilon_{ijk} \dot{\gamma}_k^{(1)} + e^* \epsilon_{ijk} \dot{d}_k \\
 \pi_i^* &= h^* \epsilon_{ijk} \dot{\gamma}_{jk}^{(1)} + C^* \dot{\gamma}_i^{(1)} + D^* \dot{d}_i + \zeta^* T_{,i} \\
 p_i^* &= k^* \epsilon_{ijk} \dot{\gamma}_{jk}^{(1)} + F^* \dot{\gamma}_i^{(1)} + G^* \dot{d}_i + \zeta^* T_{,i} \\
 r_i^* &= l^* \epsilon_{ijk} \dot{e}_{jk} + L^* \dot{\Delta}_i \\
 g^* &= M^* \dot{e}_{ii} + A^* \dot{\phi}^{(1)}
 \end{aligned} \tag{92}$$

With the help of the relations (29), (74), (77)–(79), the equations (45)–(47) and the energy equation (67) may be written in the form

$$\begin{aligned}
 t_{ji,j} - p_i + \rho_1^0 F_i^{(1)} &= \rho_1^0 \ddot{u}_i \\
 s_{ji,j} + p_i + \rho_2^0 F_i^{(2)} &= \rho_2^0 \ddot{w}_i
 \end{aligned} \tag{93}$$

$$\begin{aligned}
 m_{ji,j}^{(1)} + \epsilon_{ijk} [t_{jk} + s_{jk}] - r_i + \rho_1^0 G_i^{(1)} &= \rho_1^0 J_{ij}^{(1)} \ddot{\phi}_j^{(1)} \\
 m_{ji,j}^{(2)} + r_i + \rho_2^0 G_i^{(2)} &= \rho_2^0 J_{ij}^{(2)} \ddot{\phi}_j^{(2)}
 \end{aligned} \tag{94}$$

$$\begin{aligned}\pi_{i,i}^{(1)} - g^{(1)} + \rho_1^0 L^{(1)} &= \frac{1}{2} \rho_1^0 J_0^{(1)} \ddot{\phi}^{(1)} \\ \pi_{i,i}^{(2)} - g^{(2)} + \rho_2^0 L^{(2)} &= \frac{1}{2} \rho_2^0 J_0^{(2)} \ddot{\phi}^{(2)}\end{aligned}\quad (95)$$

and

$$\rho_0 T_0 \dot{\eta} = q_{i,i} + \rho_0 r \quad (96)$$

Thus, the basic equations in the linear theory are the equations of motions (93)–(95), the energy equation (96), the constitutive equations (86) and (87) (for anisotropic materials) or (91) and (92) (for isotropic bodies) and the geometric equations (76) and (79). In the case of first boundary value problem the boundary conditions are

$$u_i = \tilde{u}_i, \quad w_i = \tilde{w}_i, \quad \varphi_i^{(x)} = \tilde{\varphi}_i^{(x)}, \quad \phi_i^{(x)} = \tilde{\phi}_i^{(x)}, \quad T = \tilde{T} \quad \text{on } \partial B \times I \quad (97)$$

where $\tilde{u}_i$, $\tilde{w}_i$, $\tilde{\varphi}_i^{(x)}$, $\tilde{\phi}_i^{(x)}$ and $\tilde{T}$ are prescribed functions. For the second boundary value problem, the boundary conditions are

$$\begin{aligned}(t_{ji} + s_{ji})n_j &= \tilde{t}_i, \quad (m_{ji}^{(1)} + m_{ji}^{(2)})n_j = \tilde{m}_i, \quad (\pi_i^{(1)} + \pi_i^{(2)})n_i = \tilde{\pi} \\ d_i &= \tilde{d}_i, \quad \Delta_i = \tilde{\Delta}_i, \quad \phi^{(1)} - \phi^{(2)} = \tilde{\phi}, \quad q_i n_i = \tilde{q} \quad \text{on } \partial B \times I\end{aligned}\quad (98)$$

where $\tilde{t}_i$, $\tilde{m}_i$, $\tilde{\pi}$, $\tilde{d}_i$, $\tilde{\Delta}_i$, $\tilde{\phi}$ and $\tilde{q}$ are given data. The initial conditions are

$$\begin{aligned}u_i(\mathbf{X}, 0) &= \hat{a}_i(\mathbf{X}), \quad w_i(\mathbf{X}, 0) = \hat{b}_i(\mathbf{X}), \quad \dot{u}_i(\mathbf{X}, 0) = \hat{c}_i(\mathbf{X}), \quad \dot{w}_i(\mathbf{X}, 0) = \hat{f}_i(\mathbf{X}) \\ \varphi_i^{(x)}(\mathbf{X}, 0) &= \hat{\psi}_i^{(x)}(\mathbf{X}), \quad \dot{\varphi}_i^{(x)}(\mathbf{X}, 0) = \hat{\omega}_i^{(x)}(\mathbf{X}), \quad \phi^{(x)}(\mathbf{X}, 0) = \hat{\psi}^{(x)}(\mathbf{X}) \\ \dot{\phi}^{(x)}(\mathbf{X}, 0) &= \hat{\omega}^{(x)}(\mathbf{X}), \quad T(\mathbf{X}, 0) = \hat{T}(\mathbf{X}), \quad \mathbf{X} \in \bar{B}\end{aligned}\quad (99)$$

where the functions $\hat{a}_i$, $\hat{b}_i$, $\hat{c}_i$, $\hat{f}_i$, $\hat{\psi}_i^{(x)}$, $\hat{\omega}_i^{(x)}$, $\hat{\psi}^{(x)}$, $\hat{\omega}^{(x)}$ and $\hat{T}$ are prescribed.

UNIQUENESS THEOREM

In this section we establish a uniqueness result in the linear theory. To this aim we introduce the notation

$$\mathcal{H} = \frac{1}{2} \int_B \left[\rho_1^0 \dot{u}_i \dot{u}_i + \rho_2^0 \dot{w}_i \dot{w}_i + \sum_{\alpha=1}^2 \left(\rho_\alpha^0 J_{ij}^{(\alpha)} \dot{\varphi}_i^{(\alpha)} \dot{\varphi}_j^{(\alpha)} + \frac{1}{2} \rho_\alpha^0 J_0^{(\alpha)} \dot{\phi}^{(\alpha)} \dot{\phi}^{(\alpha)} \right) + a T^2 + 2U \right] dV \quad (100)$$

and then choose the initial instant t_0 to be zero. We may prove the following theorem:

Theorem 1. *Let us assume that*

- i) ρ_α^0 , $J_0^{(\alpha)}$ and a are strictly positive and $J_{ij}^{(\alpha)}$ is symmetric and positive definite;
- ii) the constitutive coefficients satisfy symmetry relations (85) and the inequality (88);
- iii) U as defined by (83) is positive definite.

Then the initial boundary value problem defined by the equations of motion (93)–(95), the energy equation (96), the constitutive equations (86) and (87), the geometric equations (76) and (79), the boundary conditions (97) (or (98)) and the initial conditions (99) has at most one solution.

Proof. In view of the constitutive equations (87), the symmetry relations (85), the relations (83) and (88), we deduce

$$\begin{aligned} & t_{ij}\dot{e}_{ij} + s_{ij}\dot{g}_{ij} + m_{ij}^{(1)}\dot{\gamma}_{ij}^{(1)} + m_{ij}^{(2)}\dot{\gamma}_{ij}^{(2)} + \pi_i^{(1)}\dot{\gamma}_i^{(1)} + \pi_i^{(2)}\dot{\gamma}_i^{(2)} + p_i\dot{d}_i \\ & + r_i\dot{\Delta}_i + g^{(1)}\dot{\phi}^{(1)} + g^{(2)}\dot{\phi}^{(2)} + \rho_0\dot{\eta}T = \frac{\partial}{\partial t}\left(\frac{1}{2}aT^2 + U\right) + \mathcal{G} - \frac{1}{T_0}q_iT_{,i} \end{aligned} \quad (101)$$

On the other hand, in view of the equations (93)–(95) and the geometric equations (76) and (79), we have

$$\begin{aligned} & t_{ij}\dot{e}_{ij} + s_{ij}\dot{g}_{ij} + m_{ij}^{(1)}\dot{\gamma}_{ij}^{(1)} + m_{ij}^{(2)}\dot{\gamma}_{ij}^{(2)} + \pi_i^{(1)}\dot{\gamma}_i^{(1)} + \pi_i^{(2)}\dot{\gamma}_i^{(2)} + p_i\dot{d}_i \\ & + r_i\dot{\Delta}_i + g^{(1)}\dot{\phi}^{(1)} + g^{(2)}\dot{\phi}^{(2)} + \rho_0\dot{\eta}T \\ & = [t_{ji}\dot{u}_i + s_{ji}\dot{w}_i + m_{ji}^{(1)}\dot{\phi}_i^{(1)} + m_{ji}^{(2)}\dot{\phi}_i^{(2)} + \pi_j^{(1)}\dot{\phi}^{(1)} + \pi_j^{(2)}\dot{\phi}^{(2)} + \frac{1}{T_0}q_jT]_{,j} + \rho_1^0F_i^{(1)}\dot{u}_i \\ & + \rho_2^0F_i^{(2)}\dot{w}_i + \rho_1^0G_i^{(1)}\dot{\phi}_i^{(1)} + \rho_2^0G_i^{(2)}\dot{\phi}_i^{(2)} + \rho_1^0L^{(1)}\dot{\phi}^{(1)} + \rho_1^0L^{(2)}\dot{\phi}^{(2)} + \frac{1}{T_0}\rho_0rT \\ & - \frac{1}{2}\frac{\partial}{\partial t}\left[\rho_1^0\dot{u}_i\dot{u}_i + \rho_2^0\dot{w}_i\dot{w}_i + \sum_{\alpha=1}^2\left(\rho_x^0J_{ij}^{(\alpha)}\dot{\phi}_i^{(\alpha)}\dot{\phi}_j^{(\alpha)} + \frac{1}{2}\rho_x^0J_0^{(\alpha)}\dot{\phi}^{(\alpha)}\dot{\phi}^{(\alpha)}\right)\right] - \frac{1}{T_0}q_iT_{,i} \end{aligned} \quad (102)$$

Then (101) and (102) imply

$$\begin{aligned} & \frac{1}{2}\frac{\partial}{\partial t}\left[\rho_1^0\dot{u}_i\dot{u}_i + \rho_2^0\dot{w}_i\dot{w}_i + \sum_{\alpha=1}^2\left(\rho_x^0J_{ij}^{(\alpha)}\dot{\phi}_i^{(\alpha)}\dot{\phi}_j^{(\alpha)} + \frac{1}{2}\rho_x^0J_0^{(\alpha)}\dot{\phi}^{(\alpha)}\dot{\phi}^{(\alpha)}\right) + aT^2 + 2U\right] + \mathcal{G} \\ & = \rho_1^0F_i^{(1)}\dot{u}_i + \rho_2^0F_i^{(2)}\dot{w}_i + \rho_1^0G_i^{(1)}\dot{\phi}_i^{(1)} + \rho_2^0G_i^{(2)}\dot{\phi}_i^{(2)} + \rho_1^0L^{(1)}\dot{\phi}^{(1)} + \rho_1^0L^{(2)}\dot{\phi}^{(2)} \\ & + \frac{1}{T_0}\rho_0rT + \frac{1}{2}\left[(t_{ji} + s_{ji})(\dot{u}_i + \dot{w}_i) + (t_{ji} - s_{ji})\dot{d}_i + (m_{ji}^{(1)} + m_{ji}^{(2)})(\dot{\phi}_i^{(1)} + \dot{\phi}_i^{(2)})\right. \\ & \quad + (m_{ji}^{(1)} - m_{ji}^{(2)})\dot{\Delta}_i + (\pi_j^{(1)} + \pi_j^{(2)})(\dot{\phi}^{(1)} + \dot{\phi}^{(2)}) \\ & \quad \left. + (\pi_j^{(1)} - \pi_j^{(2)})(\dot{\phi}^{(1)} - \dot{\phi}^{(2)}) + \frac{1}{T_0}q_jT\right]_{,j} \end{aligned} \quad (103)$$

By integration of the above relation over B and by using the divergence theorem and (100), we obtain

$$\begin{aligned} \dot{\mathcal{H}} + \int_B \mathcal{G} dV = & \int_B \left(\rho_1^0 F_i^{(1)} \dot{u}_i + \rho_2^0 F_i^{(2)} \dot{w}_i + \rho_1^0 G_i^{(1)} \dot{\varphi}_i^{(1)} + \rho_2^0 G_i^{(2)} \dot{\varphi}_i^{(2)} \right. \\ & \left. + \rho_1^0 L^{(1)} \dot{\phi}^{(1)} + \rho_1^0 L^{(2)} \dot{\phi}^{(2)} + \frac{1}{T_0} \rho_0 r T \right) dV \\ & + \frac{1}{2} \int_{\partial B} \left[(t_{ji} + s_{ji})(\dot{u}_i + \dot{w}_i) + (t_{ji} - s_{ji})\dot{d}_i + (m_{ji}^{(1)} + m_{ji}^{(2)})(\dot{\varphi}_i^{(1)} + \dot{\varphi}_i^{(2)}) \right. \\ & \quad \left. + (m_{ji}^{(1)} - m_{ji}^{(2)})\dot{\Delta}_i + (\pi_j^{(1)} + \pi_j^{(2)})(\dot{\phi}^{(1)} + \dot{\phi}^{(2)}) \right. \\ & \quad \left. + (\pi_j^{(1)} - \pi_j^{(2)})(\dot{\phi}^{(1)} - \dot{\phi}^{(2)}) + \frac{1}{T_0} q_j T \right] n_j dA \end{aligned} \quad (104)$$

Let us suppose that there are two solutions $\{^1u_i, ^1w_i, ^1\varphi_i^{(1)}, ^1\varphi_i^{(2)}, ^1\phi^{(1)}, ^1\phi^{(2)}, ^1T\}$ and $\{^2u_i, ^2w_i, ^2\varphi_i^{(1)}, ^2\varphi_i^{(2)}, ^2\phi^{(1)}, ^2\phi^{(2)}, ^2T\}$. Because of the linearity of the considered initial boundary value problem, their difference $\mathcal{P} = \{\bar{u}_i, \bar{w}_i, \bar{\varphi}_i^{(1)}, \bar{\varphi}_i^{(2)}, \bar{\phi}^{(1)}, \bar{\phi}^{(2)}, \bar{T}\}$ corresponds to null data. It follows from (97), (98) and (104) that

$$\dot{\bar{\mathcal{H}}} + \int_B \bar{\mathcal{G}} dV = 0 \quad (105)$$

where $\bar{\mathcal{H}}$ and $\bar{\mathcal{G}}$ are the functions associated with the process $\mathcal{P}$. Since $\bar{\mathcal{G}}$ is a nonnegative function from (105) we deduce that $\bar{\mathcal{H}}(t) \leq \bar{\mathcal{H}}(0)$, $t \in [0, t_1]$. From the initial conditions, we find $\bar{\mathcal{H}}(0) = 0$ and therefore $\bar{\mathcal{H}}(t) = 0$, $t \in [0, t_1]$. The hypotheses of the theorem assure that $\dot{\bar{\mathbf{u}}} = \mathbf{0}$, $\dot{\bar{\mathbf{w}}} = \mathbf{0}$, $\dot{\bar{\varphi}}^{(1)} = \mathbf{0}$, $\dot{\bar{\varphi}}^{(2)} = \mathbf{0}$, $\dot{\bar{\phi}}^{(1)} = 0$, $\dot{\bar{\phi}}^{(2)} = 0$, $T = 0$ on $B \times [0, t_1]$. Since $\bar{\mathbf{u}}, \bar{\mathbf{w}}, \bar{\varphi}^{(1)}, \bar{\varphi}^{(2)}, \bar{\phi}^{(1)}, \bar{\phi}^{(2)}$ vanish initially, we conclude that $\bar{\mathbf{u}} = \mathbf{0}$, $\bar{\mathbf{w}} = \mathbf{0}$, $\bar{\varphi}^{(1)} = \mathbf{0}$, $\bar{\varphi}^{(2)} = \mathbf{0}$, $\bar{\phi}^{(1)} = 0$, $\bar{\phi}^{(2)} = 0$, $T = 0$ on $B \times [0, t_1]$ and thus, the proof is complete.

MICROPOLAR MIXTURES

In the absence of stretch, from (76), (79), (86), (87), (93), (94) and (96) we deduce for anisotropic bodies the following equations:

- the equations of motion

$$t_{ji,j} - p_i + \rho_1^0 F_i^{(1)} = \rho_1^0 \ddot{u}_i \quad (106)$$

$$s_{ji,j} + p_i + \rho_2^0 F_i^{(2)} = \rho_2^0 \ddot{w}_i$$

$$\begin{aligned} m_{ji,j}^{(1)} + \epsilon_{ijk}[t_{jk} + s_{jk}] - r_i + \rho_1^0 G_i^{(1)} &= \rho_1^0 J_{ij}^{(1)} \ddot{\varphi}_j^{(1)} \\ m_{ji,j}^{(2)} + r_i + \rho_2^0 G_i^{(2)} &= \rho_2^0 J_{ij}^{(2)} \ddot{\varphi}_j^{(2)} \end{aligned} \quad (107)$$

- the energy equation

$$\rho_0 T_0 \dot{\eta} = q_{i,i} + \rho_0 r \quad (108)$$

- the constitutive equations

$$\begin{aligned}
t_{ij} &= A_{ijrs}e_{rs} + B_{ijrs}g_{rs} + F_{ijrs}^{(1)}\gamma_{rs}^{(1)} + F_{ijrs}^{(2)}\gamma_{rs}^{(2)} + a_{ijk}d_k + b_{ijk}\Delta_k - a_{ij}^{\circ}T + t_{ij}^* \\
s_{ij} &= B_{rsij}e_{rs} + C_{ijrs}g_{rs} + H_{ijrs}^{(1)}\gamma_{rs}^{(1)} + H_{ijrs}^{(2)}\gamma_{rs}^{(2)} + c_{ijk}d_k + d_{ijk}\Delta_k - c_{ij}^{\circ}T \\
m_{ij}^{(1)} &= F_{rsij}^{(1)}e_{rs} + H_{rsij}^{(1)}g_{rs} + D_{ijrs}^{(11)}\gamma_{rs}^{(1)} + D_{ijrs}^{(12)}\gamma_{rs}^{(2)} + \alpha_{ijk}^{(1)}d_k + \beta_{ijk}^{(1)}\Delta_k - \alpha_{ij}^{\circ}T + m_{ij}^* \\
m_{ij}^{(2)} &= F_{rsij}^{(2)}e_{rs} + H_{rsij}^{(2)}g_{rs} + D_{ijrs}^{(21)}\gamma_{rs}^{(1)} + D_{ijrs}^{(22)}\gamma_{rs}^{(2)} + \alpha_{ijk}^{(2)}d_k + \beta_{ijk}^{(2)}\Delta_k - \beta_{ij}^{\circ}T \\
p_i &= a_{rsi}e_{rs} + c_{rsi}g_{rs} + \alpha_{rsi}^{(1)}\gamma_{rs}^{(1)} + \alpha_{rsi}^{(2)}\gamma_{rs}^{(2)} + a_{ik}d_k + c_{ik}\Delta_k - \tau_i^{\circ}T + p_i^* \\
r_i &= b_{rsi}e_{rs} + d_{rsi}g_{rs} + \beta_{rsi}^{(1)}\gamma_{rs}^{(1)} + \beta_{rsi}^{(2)}\gamma_{rs}^{(2)} + c_{ki}d_k + b_{ik}\Delta_k - \sigma_i^{\circ}T + r_i^* \\
\rho_0\eta &= a_{ij}^{\circ}e_{ij} + c_{ij}^{\circ}g_{ij} + \alpha_{ij}^{\circ}\gamma_{ij}^{(1)} + \beta_{ij}^{\circ}\gamma_{ij}^{(2)} + \tau_i^{\circ}d_i + \sigma_i^{\circ}\Delta_i + aT
\end{aligned} \quad (109)$$

where

$$\begin{aligned}
t_{ij}^* &= A_{ijrs}^*\dot{e}_{rs} + B_{ijrs}^*\dot{\gamma}_{rs}^{(1)} + B_{ijk}^*\dot{d}_k + C_{ijk}^*\dot{\Delta}_k + a_{ijk}^*T_{,k} \\
m_{ij}^* &= C_{ijrs}^*\dot{e}_{rs} + D_{ijrs}^*\dot{\gamma}_{rs}^{(1)} + E_{ijk}^*\dot{d}_k + F_{ijk}^*\dot{\Delta}_k + b_{ijk}^*T_{,k} \\
p_i^* &= J_{irs}^*\dot{e}_{rs} + K_{irs}^*\dot{\gamma}_{rs}^{(1)} + G_{ij}^*\dot{d}_j + H_{ij}^*\dot{\Delta}_j + b_{ij}^*T_{,j} \\
r_i^* &= L_{irs}^*\dot{e}_{rs} + M_{irs}^*\dot{\gamma}_{rs}^{(1)} + K_{ij}^*\dot{d}_j + L_{ij}^*\dot{\Delta}_j + c_{ij}^*T_{,j} \\
q_i &= \alpha_{irs}^*\dot{e}_{rs} + \beta_{irs}^*\dot{\gamma}_{rs}^{(1)} + \beta_{ij}^*\dot{d}_j + \zeta_{ij}^*\dot{\Delta}_j + k_{ij}T_{,j}
\end{aligned} \quad (110)$$

- the geometric equations

$$\begin{aligned}
e_{ji} &= u_{i,j} + \epsilon_{sij}\varphi_s^{(1)}, \quad g_{ji} = w_{i,j} + \epsilon_{sij}\varphi_s^{(1)}, \quad \gamma_{ij}^{(z)} = \varphi_{j,i}^{(z)} \\
d_i &= u_i - w_i, \quad \Delta_i = \varphi_i^{(1)} - \varphi_i^{(2)}
\end{aligned} \quad (111)$$

The remaining constitutive coefficients have to satisfy the symmetry relations (85). These equations are identical with those obtained by Ieşan [26] (see the equations (57)–(59), (52), (51), (38), (35) and (23)). They appear to be different only because in [26] there are used the constitutive variables

$$\varepsilon_{ij} = u_{j,i} + \frac{1}{2}\varepsilon_{jik}(\varphi_k^{(1)} + \varphi_k^{(2)}), \quad \kappa_{ij} = w_{j,i} + \frac{1}{2}\varepsilon_{jik}(\varphi_k^{(1)} + \varphi_k^{(2)}) \quad (112)$$

instead of e_{ij} and g_{ij} and the equations (107) are written in a symmetric form. It is just a problem of calculus to see that the formulations in question are equivalent. For example, if we introduce the notation

$$h_i = r_i - \frac{1}{2}\varepsilon_{ijk}[t_{jk} + s_{jk}] \quad (113)$$

then the equations (107) may be written in the form

$$\begin{aligned} m_{ji,j}^{(1)} + \frac{1}{2}\epsilon_{ijk}[t_{jk} + s_{jk}] - h_i + \rho_1^0 G_i^{(1)} &= \rho_1^0 J_{ij}^{(1)} \ddot{\varphi}_j^{(1)} \\ m_{ji,j}^{(2)} + \frac{1}{2}\epsilon_{ijk}[t_{jk} + s_{jk}] + h_i + \rho_2^0 G_i^{(2)} &= \rho_2^0 J_{ij}^{(2)} \ddot{\varphi}_j^{(2)} \end{aligned} \quad (114)$$

namely, we obtain the same equations as those described by the relations (58) of [26]. Moreover, from (111) and (112) we deduce that the two sets of constitutive variables are related by

$$e_{ij} = \varepsilon_{ij} - \frac{1}{2}\varepsilon_{ijk}\Delta_k, \quad g_{ij} = \kappa_{ij} - \frac{1}{2}\varepsilon_{ijk}\Delta_k \quad (115)$$

Thus, from (109), (110) and (115) we can write h_i in the form

$$\begin{aligned} h_i &= \bar{b}_{rsi}\varepsilon_{rs} + \bar{d}_{rsi}\kappa_{rs} + \bar{\beta}_{rsi}^{(1)}\gamma_{rs}^{(1)} + \bar{\beta}_{rsi}^{(2)}\gamma_{rs}^{(2)} + \bar{c}_{ki}d_k + \bar{b}_{ik}\Delta_k - \bar{\sigma}_i^{\otimes}T \\ &+ \bar{L}_{irs}^*\dot{\varepsilon}_{rs} + \bar{M}_{irs}^*\dot{\gamma}_{rs}^{(1)} + \bar{K}_{ij}^*\dot{d}_j + \bar{L}_{ij}^*\dot{\Delta}_j + \bar{c}_{ij}^*T_{,j} \end{aligned} \quad (116)$$

where

$$\begin{aligned} \bar{b}_{rsi} &= b_{rsi} - \frac{1}{2}\varepsilon_{ijk}(A_{jkrs} + B_{rsjk}), \quad \bar{d}_{rsi} = d_{rsi} - \frac{1}{2}\varepsilon_{ijk}(B_{jkrs} + C_{jkrs}) \\ \bar{\beta}_{rsi}^{(\alpha)} &= \beta_{rsi}^{(\alpha)} - \frac{1}{2}\varepsilon_{ijk}(F_{jkrs}^{(\alpha)} + H_{jkrs}^{(\alpha)}), \quad \bar{c}_{mi} = c_{mi} - \frac{1}{2}\varepsilon_{ijk}(a_{jkm} + c_{jkm}) \\ \bar{b}_{im} &= b_{im} - \frac{1}{2}\varepsilon_{rsm}(b_{rsi} + d_{rsi}) - \frac{1}{2}\varepsilon_{ijk}(b_{jkm} + d_{jkm}) \\ &+ \frac{1}{4}\varepsilon_{ijk}\varepsilon_{rsm}(A_{jkrs} + B_{rsjk} + B_{jkrs} + C_{jkrs}) \\ \bar{\sigma}_i^{\otimes} &= \sigma_i^{\otimes} - \frac{1}{2}\varepsilon_{ijk}(a_{jk}^{\otimes} + c_{jk}^{\otimes}), \quad \bar{L}_{irs}^* = L_{irs}^* - \frac{1}{2}\varepsilon_{ijk}A_{jkrs}^*, \quad \bar{M}_{irs}^* = M_{irs}^* - \frac{1}{2}\varepsilon_{ijk}B_{jkrs}^* \\ \bar{K}_{im}^* &= K_{im}^* - \frac{1}{2}\varepsilon_{ijk}B_{jkm}^*, \quad \bar{c}_{im}^* = c_{im}^* - \frac{1}{2}\varepsilon_{ijk}a_{jkm}^* \\ \bar{L}_{im}^* &= L_{im}^* - \frac{1}{2}\varepsilon_{rsm}L_{irs}^* - \frac{1}{2}\varepsilon_{ijk}C_{jkm}^* + \frac{1}{4}\varepsilon_{ijk}\varepsilon_{rsm}A_{jkrs}^* \end{aligned} \quad (117)$$

that is, h_i has the form given by the constitutive equations (51) and (52) obtained in [26]. In the same way we can make the remaining identifications.

The measures of strain used here and in [25] are suggested from an examination of the Clausius–Duhem inequality and the nonlinear constitutive equations. That is, we have been able to write the entropy production inequality in the material form (54) and to see that, under the axioms of constitutive theory, the response functionals may be written in terms of the strain measures defined by the relations (11).

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