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ON SOME NON-STANDARD PROBLEMS IN LINEAR THERMOELASTICITY

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In the present paper we consider a prismatic cylinder occupied by an anisotropic homogeneous compressible linear thermoelastic material that is subject to zero body force and heat supply and zero displacement-temperature variation on the lateral boundary. The motion is induced by a time-dependent displacement-temperature variation specified pointwise over the base. The motion is constrained such that the displacement-temperature variation and velocity at points in the cylinder and at a prescribed time are in given proportions to, but not identical with, their respective initial values. It is shown that certain integrals of the solution spatially evolve with respect to the axial variable. Conditions are derived that show the integrals exhibit alternative behavior and in particular for the semi-infinite cylinder that there is either at least exponential growth or at most exponential decay, provided the elasticity tensor is positive definite or satisfies a strong ellipticity condition.

Keywords: Growth and decay spatial estimates; Linear thermoelasticity; Non-standard problems

INTRODUCTION

It is well known that the backward-in-time problems of linear thermoelasticity are not well-posed or, equivalently, that the final time problems for the forward equations of linear thermoelasticity are not well-posed. In order to stabilize such kind of ill-posed problems many different techniques have been introduced such as those of “solving” ill-posed problems for equations of evolution. Some of these involve the altering of the governing equations in such a way as to make such problems well posed. Others involve constraining solutions to lie in some constraint set. Still others involve changing the initial and/or boundary conditions again in such a way as to make the problems well posed. This last technique consists of introducing a combination of the initial values with those at a later time. It is this latter method that provides the motivation for our study.

In fact, by changing the initial conditions we are led to so-called non-standard problems and in the recent years such problems are intensively studied in literature. To this end we have to cite the papers concerning the study of the heat equation

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(see, for example, Ames and Payne [1], Ames, Payne, and Schaefer [2, 3], Ames, Payne, and Song [4] and Song [5]) and those concerning the generalized heat equation (see, for example, Payne, Schaefer, and Song [6]) or those regarding fluid flows (see, e.g., Ames, Payne, and Song [7] and Quintanilla and Straughan [8]).

We have to note that Quintanilla and Straughan [9] derive energy bounds for a class of non-standard problems associated with linear thermoelasticity of second sound in which the initial data are substituted by a combination of data at initial time and at a later time.

Recently, Knops and Payne [10] consider a non-standard problem associated with the classical linear elasticity for a prismatic cylinder and establish decay and growth exponential estimates with respect to axial variable for certain time integrals of the cross-sectional energy, provided the elasticity tensor is positive definite. The initial displacement and velocity are not prescribed, nor are conditions specified on the upper end for a finite cylinder, or at asymptotically large axial distance for the semi-infinite cylinder. Possible applications are indicated in [10] for various practical problems ranging from geology to structural engineering. The example of a pile driven into a rigid foundation is given for preventing movement of the lateral boundary. The time-dependent displacement prescribed over the base end constrains the motion such that the displacement and velocity at some given time are proportional to their unknown initial values. It is desired to predict the deformation at each cross-section of the pile in terms of the base displacement.

The work in [10] suggests to consider the linear theory of thermoelasticity in order to study the elastic and thermal effects inside the cylinder. Thus, we consider a prismatic cylinder made of an anisotropic and homogeneous compressible linear thermoelastic material subjected to zero body supplies and zero lateral specific boundary conditions. The motion is induced by time-dependent displacement and temperature variation specified pointwise over the base and the displacement and velocity and temperature variation at points in the cylinder and at a prescribed time are in given proportions to, but not identical with, their respective initial values. The spatial evolution of the solution is studied by means of certain time-weighted integrals of the cross-sectional energetic terms, provided the specific heat is positive, the conductivity tensor is positive definite and the elasticity tensor is positive definite or satisfies a strong ellipticity condition. Consequently, when common zero lateral boundary conditions are considered then we assume the elasticity tensor is positive definite and, on this basis, we examine how certain integrals of the cross-sectional energy spatially evolve with respect to the axial distance to the end base. Conditions are derived that show these integrals exhibit Phragmén–Lindelöf alternative behavior, that is for a semi-infinite cylinder there is either least exponential growth or at most exponential decay.

When the cylinder is constrained to a fixed and thermal insulated lateral boundary we use the strong ellipticity of the elasticity tensor in order to construct an explicit time-weighted integral over the cylinder's cross-section able to furnish complete information on how the solution spatially evolves with respect to the axial distance to the excited base.

The technique employed in the present paper is different from that in [10] and it is based on the time-weighted method used in [11] in the study of the spatial behavior for forward thermoelastodynamic problems. However, we have to point out that in [10] a more general class of time-weight functions is discussed in short.

The time-weight function has been used in the study of the backward in time thermoelasticity [12] and in the study of other non-standard problems [5, 13].

FORMULATION OF PROBLEM

Consider a prismatic cylinder $B \subset \mathbb{R}^3$ whose bounded uniform cross-section $D \subset \mathbb{R}^2$ has piecewise continuously differentiable boundary ∂D . The origin of a rectangular Cartesian coordinate system is located in the cylinder's base and the positive x_3 -axis is directed along that of the cylinder. It is convenient to introduce the further abbreviation

$$B(z) = \{x \in B : z > x_3\}$$

and, moreover, we employ $D(x_3)$ to indicate that relevant quantities are to be evaluated over the cross-section whose distance from the origin is x_3 .

We are concerned with the system of equations of classical linear thermoelasticity. Thus, we assume that the cylinder is occupied by an anisotropic compressible thermoelastic material and is subject to a deformation in which the displacement-temperature variation field $(u, \theta)(x, t)$ is a smooth function satisfying the requirements of the classical dynamical theory of thermoelasticity [14]. The corresponding stress tensor $S(x, t)$ has Cartesian components given by

$$S_{rs} = C_{rsmn} u_{m,n} + M_{rs} \theta \quad (1)$$

where the constant elasticities C_{rsmn} and stress-temperature components M_{rs} possess the symmetries

$$C_{rsmn} = C_{mnrs} = C_{srnm}, \quad M_{rs} = M_{sr} \quad (2)$$

The heat flux vector $Q(x, t)$ has Cartesian components

$$Q_r = -K_{rs} \theta_{,s} \quad (3)$$

where K_{rs} are the constant components of the conductivity tensor and they satisfy

$$K_{rs} \theta_{,r} \theta_{,s} \geq 0 \quad (4)$$

The cylinder is set in motion subject to a pointwise prescribed base time-dependent displacement-temperature variation, zero body force and heat supply and zero displacement-temperature variation on the lateral surface. Furthermore, the prescribed displacement-temperature variation is such that a classical solution exists on the interval $[0, T]$. Consequently, the problem to be considered is specified by the following second-order differential system [14]

$$(C_{rskl} u_{k,l} + M_{rs} \theta)_{,r} = \rho \ddot{u}_s, \quad (x, t) \in B \times (0, T) \quad (5)$$

$$(K_{rs} \theta_{,s})_{,r} + \theta_0 M_{rs} \dot{u}_{r,s} = c \dot{\theta}, \quad (x, t) \in B \times (0, T) \quad (6)$$

subject to the lateral boundary conditions

$$\dot{u}_r(C_{rskl}u_{k,l} + M_{rs}\theta)n_s = 0, \quad \theta(K_{rs}\theta_{,s}n_r) = 0, \quad (x, t) \in (\partial D \times [0, L]) \times [0, T] \quad (7)$$

and the base boundary conditions

$$u_r(x, t) = f_r(x_1, x_2, t), \quad \theta(x, t) = g(x_1, x_2, t), \quad (x, t) \in D(0) \times [0, T] \quad (8)$$

and a combination of the initial values of the displacement, velocity, and temperature variation with their values at a later time T , in the form

$$u_r(x, T) = \lambda u_r(x, 0), \quad \theta(x, T) = v\theta(x, 0), \quad \dot{u}_r(x, T) = \mu \dot{u}_r(x, 0), \quad x \in B \quad (9)$$

where we have used a superposed dot for denoting differentiation with respect to time and a subscript comma indicates partial differentiation and n_r are the components of the unit outward normal on $(\partial D \times [0, L])$. Moreover, ρ is the constant positive mass density, c is the specific heat, θ_0 is the uniform reference temperature, $f_r(x_1, x_2, t)$ and $g(x_1, x_2, t)$ are prescribed differentiable functions compatible with the initial and lateral boundary conditions and λ , μ , and v are given parameters satisfying the conditions

$$|\lambda| > 1, \quad |\mu| > 1, \quad |v| > 1 \quad (10)$$

We are interested in the study of the spatial behavior of the solution (u_r, θ) of the above initial boundary value problem (5) to (9).

The relation (9) represents a nonstandard condition that leads to an ill-posed problem when $\lambda = 0$, $\mu = 0$, and $v = 0$. The spatial behavior for this last problem was studied by Chiriță and Ciarletta [12] by using the time-weighted method. Other studies concerning the ill-posed problems in linear thermoelasticity are by Ames and Payne [15], Ciarletta [16], Chiriță and Ciarletta [17] and Quintanilla [18]. Here, we are interested in determining a range of values for the parameters λ , μ , and v for which the problem is well-posed and to obtain spatial estimates describing how the displacement and temperature variation evolve with respect to the axial distance to the cylinder's base.

A MEASURE BASED ON POSITIVE DEFINITENESS CONDITIONS

Throughout this section we assume the following positive definiteness conditions upon the thermoelastic constitutive coefficients

$$C_{ijkl}\xi_{ij}\xi_{kl} > 0 \quad \text{for all non-zero symmetric } \xi_{ij} \quad (11)$$

$$c > 0, \quad K_{ij}\xi_i\xi_j > 0 \quad \text{for all non-zero } \xi_i \quad (12)$$

A full discussion on upper and lower bounds for the magnitude of the stresses in linear anisotropic elastic materials is given in a recent paper by Mehrabadi et al. [19]. It is shown there that

$$\mu_m e_{ij}e_{ij} \leq C_{ijkl}e_{ij}e_{kl} \leq \mu_M e_{ij}e_{ij} \quad (13)$$

where μ_m and μ_M are the lowest and largest eigenvalues of the elasticity tensor. The explicit values for the lowest and largest eigenvalues of the elasticity tensor are given by Mehrabadi et al. [19] for different elastic symmetries. According to Gurtin [20], μ_M and μ_m denote the maximum and minimum elastic moduli for C_{ijkl} . Since we are considering the coupled theory of thermoelasticity it follows that we assume that

$$m = (M_{ij}M_{ij})^{1/2} > 0 \quad (14)$$

Note that the relations (1), (13), and (14) give

$$\begin{aligned} S_{ij}S_{ij} &= C_{ijkl}e_{kl}S_{ij} + \theta M_{ij}S_{ij} \leq (C_{rsmn}e_{rs}e_{mn})^{1/2}(C_{ijkl}S_{ij}S_{kl})^{1/2} \\ &\quad + |\theta|(M_{rs}M_{rs})^{1/2}(S_{ij}S_{ij})^{1/2} \leq (S_{ij}S_{ij})^{1/2}[(\mu_M C_{rsmn}e_{rs}e_{mn})^{1/2} + m|\theta|] \end{aligned} \quad (15)$$

so that we deduce

$$(S_{ij}S_{ij})^{1/2} \leq (\mu_M C_{rsmn}e_{rs}e_{mn})^{1/2} + m|\theta| \quad (16)$$

Thus, by means of the arithmetic–geometric mean inequality, from (16) we get

$$S_{ij}S_{ij} \leq (1 + \varepsilon_1)\mu_M C_{rsmn}e_{rs}e_{mn} + \left(1 + \frac{1}{\varepsilon_1}\right)m^2\theta^2, \quad \forall \varepsilon_1 > 0 \quad (17)$$

If k_M and k_m are the largest and lowest eigenvalues of the conductivity tensor K_{ij} , then we have

$$k_m \xi_i \xi_i \leq K_{ij} \xi_i \xi_j \leq k_M \xi_i \xi_i \quad (18)$$

Then the relations (3) and (18) give

$$\begin{aligned} Q_i Q_i &= -K_{rs}\theta_{,s}Q_r \leq (K_{rs}\theta_{,r}\theta_{,s})^{1/2}(K_{ij}Q_i Q_j)^{1/2} \\ &\leq (k_M K_{rs}\theta_{,r}\theta_{,s})^{1/2}(Q_i Q_i)^{1/2} \end{aligned} \quad (19)$$

so that we obtain the estimate

$$Q_i Q_i \leq k_M K_{rs}\theta_{,r}\theta_{,s} \quad (20)$$

The function considered is defined by

$$I(x_3) = \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left(S_{3i} \dot{u}_i - \frac{1}{\theta_0} Q_3 \theta \right) da \, d\tau, \quad x_3 \in [0, L] \quad (21)$$

in which σ is a positive parameter at our disposal whose values will be explicitly given later. Moreover, we used the alternative notation for emphasis, and, moreover, employ $D(x_3, t)$ to indicate that relevant quantities are to be evaluated at time t over the cross-section whose distance from the origin is x_3 .

By direct differentiation with respect to x_3 in (21) we get

$$\frac{dI}{dx_3}(x_3) = \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left(S_{3i,3} \dot{u}_i - \frac{1}{\theta_0} Q_{3,3} \theta + S_{3i} \dot{u}_{i,3} - \frac{1}{\theta_0} Q_{3,3} \theta_{,3} \right) da \, d\tau \quad (22)$$

so that, by means of the basic equations (5) and (6) and the divergence theorem, we obtain

$$\begin{aligned} \frac{dI}{dx_3}(x_3) = & \int_0^T \int_{\partial D(x_3, \tau)} e^{-\sigma\tau} \left(\frac{1}{\theta_0} Q_\alpha n_\alpha \theta - S_{\alpha i} n_\alpha \dot{u}_i \right) ds \, d\tau \\ & + \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left\{ \frac{\partial}{\partial \tau} \left[\frac{1}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) \right] \right. \\ & \left. + \frac{1}{\theta_0} K_{ij} \theta_{,i} \theta_{,j} \right\} da \, d\tau \end{aligned} \quad (23)$$

In view of the lateral boundary conditions (7) we have

$$\begin{aligned} \frac{dI}{dx_3}(x_3) = & \int_{D(x_3, T)} \frac{1}{2} e^{-\sigma T} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) da \\ & - \int_{D(x_3, 0)} \frac{1}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) da \\ & + \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) + \frac{1}{\theta_0} K_{ij} \theta_{,i} \theta_{,j} \right] da \, d\tau \end{aligned} \quad (24)$$

and by the constraint relations (9), we obtain

$$\begin{aligned} \frac{dI}{dx_3}(x_3) = & \frac{1}{2} (e^{-\sigma T} \mu^2 - 1) \int_{D(x_3, 0)} \rho \dot{u}_i \dot{u}_i \, da + \frac{1}{2} (e^{-\sigma T} v^2 - 1) \int_{D(x_3, 0)} \frac{c}{\theta_0} \theta^2 \, da \\ & + \frac{1}{2} (e^{-\sigma T} \lambda^2 - 1) \int_{D(x_3, 0)} C_{ijkl} u_{i,j} u_{k,l} \, da \\ & + \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) + \frac{1}{\theta_0} K_{ij} \theta_{,i} \theta_{,j} \right] da \, d\tau \end{aligned} \quad (25)$$

At this instant we invoke the assumption (10) and then it is possible to choose the parameter σ in such way to have

$$e^{-\sigma T} \mu^2 - 1 > 0, \quad e^{-\sigma T} v^2 - 1 > 0, \quad e^{-\sigma T} \lambda^2 - 1 > 0 \quad (26)$$

that is we assume that σ ranges in the set

$$0 < \sigma < \frac{2}{T} \min(\ln |\mu|, \ln |v|, \ln |\lambda|) \quad (27)$$

Further, we can set

$$0 < \kappa_\sigma = \frac{1}{2} \min(e^{-\sigma T} \mu^2 - 1, e^{-\sigma T} \nu^2 - 1, e^{-\sigma T} \lambda^2 - 1) \quad (28)$$

and therefore, we have

$$\begin{aligned} \frac{dI}{dx_3}(x_3) &\geq \kappa_\sigma \int_{D(x_3, 0)} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) da \\ &\quad + \int_0^T \int_{D(x_3, \tau)} e^{-\sigma \tau} \left[\frac{\sigma}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) + \frac{1}{\theta_0} K_{ij} \theta_{,i} \theta_{,j} \right] da d\tau \end{aligned} \quad (29)$$

Consequently, the hypotheses described in (11) and (12) implies that $I(x_3)$ is a non-decreasing function with respect to x_3 on $[0, L]$.

On the other hand, we use the Schwarz's inequality and the arithmetic-geometric mean inequality in conjunction with relations (17) and (20) in order to obtain from (21) that

$$\begin{aligned} |I(x_3)| &\leq \frac{1}{2} \int_0^T \int_{D(x_3, \tau)} e^{-\sigma \tau} \left(\frac{\varepsilon_2}{\rho} S_{ij} S_{ij} + \frac{1}{\varepsilon_2} \rho \dot{u}_i \dot{u}_i + \frac{\varepsilon_3}{c \theta_0} Q_i Q_i + \frac{1}{\varepsilon_3 \theta_0} c \theta^2 \right) da d\tau \\ &\leq \int_0^T \int_{D(x_3, \tau)} e^{-\sigma \tau} \left\{ \frac{1}{\sigma \varepsilon_2} \left(\frac{\sigma}{2} \rho \dot{u}_i \dot{u}_i \right) + \frac{\varepsilon_2 (1 + \varepsilon_1) \mu_M}{\sigma \rho} \left(\frac{\sigma}{2} C_{ijkl} u_{i,j} u_{k,l} \right) \right. \\ &\quad \left. + \left[\frac{\varepsilon_2 m^2 \theta_0}{\sigma \rho c} \left(1 + \frac{1}{\varepsilon_1} \right) + \frac{1}{\sigma \varepsilon_3} \right] \left(\frac{\sigma}{2} \frac{c}{\theta_0} \theta^2 \right) + \frac{\varepsilon_3 k_M}{2c} \left(\frac{1}{\theta_0} K_{ij} \theta_{,i} \theta_{,j} \right) \right\} da d\tau \end{aligned} \quad (30)$$

for all $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$. We now equate the coefficients of the various energetic terms in the last integral in (30), that is we set

$$\frac{1}{\sigma \varepsilon_2} = \frac{\varepsilon_2 (1 + \varepsilon_1) \mu_M}{\sigma \rho} = \frac{\varepsilon_2 m^2 \theta_0}{\sigma \rho c} \left(1 + \frac{1}{\varepsilon_1} \right) + \frac{1}{\sigma \varepsilon_3} = \frac{\varepsilon_3 k_M}{2c} \quad (31)$$

and hence we have

$$\varepsilon_1 = \varepsilon, \quad \varepsilon_2 = \frac{1}{\varkappa}, \quad \varepsilon_3 = \frac{2c\varkappa}{\sigma k_M} \quad (32)$$

where

$$\begin{aligned} \varkappa &= \sqrt{\frac{(1 + \varepsilon) \mu_M}{\rho}} \\ \varepsilon &= \frac{1}{2} \left(-1 + \frac{m^2 \theta_0}{c \mu_M} + \frac{\sigma \rho k_M}{2c \mu_M} + \sqrt{\left(1 - \frac{m^2 \theta_0}{c \mu_M} - \frac{\sigma \rho k_M}{2c \mu_M} \right)^2 + \frac{4m^2 \theta_0}{c \mu_M}} \right) \end{aligned} \quad (33)$$

With these choices the relation (30) gives

$$\frac{\sigma}{\varkappa} |I(x_3)| \leq \frac{dI}{dx_3}(x_3) \quad \text{for all } x_3 \in [0, L] \quad (34)$$

To discuss the implications of (34), we have to observe that the non-decreasing function $I(x_3)$ implies the only two possibilities: (i) $I(x_3) < 0$ for all $x_3 \in [0, L]$, or (ii) there is $x_3^* \in [0, L]$ so that $I(x_3^*) > 0$.

Let us first discuss the case (i), that is we assume that $I(x_3) < 0$ for all $x_3 \in [0, L]$. Then the differential inequality (34) becomes

$$\frac{dI}{dx_3}(x_3) + \frac{\sigma}{\varkappa} I(x_3) \geq 0 \quad \text{for all } x_3 \in [0, L] \quad (35)$$

which after integration furnishes the decay estimate

$$0 \leq -I(x_3) \leq -I(0)e^{-\frac{\sigma}{\varkappa}x_3} \quad \text{for all } x_3 \in [0, L] \quad (36)$$

An immediate consequence of (36) for the semi-infinite cylinder is that $I(x_3)$ vanishes as $x_3 \rightarrow \infty$, which in combination with (29) and (36) leads to the following decay estimate for the total energy

$$\begin{aligned} & \kappa_\sigma \int_{B(x_3, 0)} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) dv \\ & + \int_0^T \int_{B(x_3, \tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) + \frac{1}{\theta_0} K_{ij} \theta_{,i} \theta_{,j} \right] dv d\tau \\ & \leq -I(0)e^{-\frac{\sigma}{\varkappa}x_3} \quad \text{for all } x_3 \in [0, \infty) \end{aligned} \quad (37)$$

We next discuss the second case, that is we suppose that there is $x_3^* \in [0, L]$ so that $I(x_3^*) > 0$. Because $I(x_3)$ is non-decreasing with respect to x_3 , it follows that $I(x_3) \geq I(x_3^*) > 0$ for all $x_3 \in [x_3^*, L]$ and hence (34) gives

$$0 \leq \frac{dI}{dx_3}(x_3) - \frac{\sigma}{\varkappa} I(x_3) \quad \text{for all } x_3 \in [x_3^*, L] \quad (38)$$

which after integration leads to the growth estimate

$$I(x_3) \geq I(x_3^*)e^{\frac{\sigma}{\varkappa}(x_3 - x_3^*)} \quad \text{for all } x_3 \in [x_3^*, L] \quad (39)$$

Thus, for a semi-infinite cylinder it follows from (39) that $I(x_3)$ becomes unbounded for asymptotically large values of x_3 , which by (29) implies the unboundedness as $L \rightarrow \infty$ of the energetic integral

$$\begin{aligned} & \kappa_\sigma \int_{x_3}^L d\xi \int_{D(\xi, 0)} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) da \\ & + \int_0^T d\tau \int_{x_3}^L d\xi \int_{D(\xi, \tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + C_{ijkl} u_{i,j} u_{k,l} \right) + \frac{1}{\theta_0} K_{ij} \theta_{,i} \theta_{,j} \right] da \end{aligned} \quad (40)$$

SOME MEASURES BASED ON THE STRONG ELLIPTICITY OF THE ELASTICITY TENSOR

Throughout this section we restrict our considerations to the lateral boundary condition

$$u_r(x, t) = 0, \quad \theta(x, t) = 0, \quad (x, t) \in (\partial D \times [0, L]) \times [0, T] \quad (41)$$

and relax the conditions of positive definiteness upon the elasticity tensor by assuming instead the strong ellipticity condition. Thus, in the present section we assume that

$$C_{ijkl}\xi_i\xi_k\xi_j\xi_l > 0 \quad \text{for all non-zero } \xi_r, \xi_s, \quad (42)$$

$$c > 0, \quad K_{ij}\xi_i\xi_j > 0 \quad \text{for all non-zero } \xi_i \quad (43)$$

The strong ellipticity condition was recently studied in [21] and necessary and sufficient conditions in terms of the constitutive coefficients only have been established for anisotropic materials with various elastic symmetries. In what follows we will use these conditions in order to obtain some measures describing the spatial behavior for the considered non-standard problems. We exemplify our procedure for the class of rhombic materials. With a standard notation for the elastic coefficients, for this class of elastic materials the strong ellipticity condition (42) takes the form

$$\begin{aligned} c_{11}\xi_1^2\xi_1^2 + c_{22}\xi_2^2\xi_2^2 + c_{33}\xi_3^2\xi_3^2 + c_{66}(\xi_1\xi_2 + \xi_2\xi_1)^2 + c_{55}(\xi_1\xi_3 + \xi_3\xi_1)^2 \\ + c_{44}(\xi_2\xi_3 + \xi_3\xi_2)^2 + 2c_{12}\xi_1\xi_1\xi_2\xi_2 + 2c_{13}\xi_1\xi_1\xi_3\xi_3 + 2c_{23}\xi_2\xi_2\xi_3\xi_3 > 0 \end{aligned} \quad (44)$$

for all non-zero vectors (ξ_1, ξ_2, ξ_3) , (ξ_1, ξ_2, ξ_3) . It was shown in [21] that the condition (44) is equivalent to the following restrictions

$$c_{11} > 0, \quad c_{22} > 0, \quad c_{33} > 0, \quad c_{44} > 0, \quad c_{55} > 0, \quad c_{66} > 0 \quad (45)$$

$$\begin{aligned} -2c_{66} + \kappa_3^i\sqrt{c_{11}c_{22}} < c_{12} < \kappa_3^s\sqrt{c_{11}c_{22}} \\ -2c_{44} + \kappa_1^i\sqrt{c_{22}c_{33}} < c_{23} < \kappa_1^s\sqrt{c_{22}c_{33}} \\ -2c_{55} + \kappa_2^i\sqrt{c_{33}c_{11}} < c_{13} < \kappa_2^s\sqrt{c_{33}c_{11}} \end{aligned} \quad (46)$$

where (κ_1^i, κ_1^s) , (κ_2^i, κ_2^s) , and (κ_3^i, κ_3^s) are solutions with respect to x , y , and z of the equation $x^2 + y^2 + z^2 - 2xyz - 1 = 0$ satisfying $|x| < 1$, $|y| < 1$, $|z| < 1$, with

$$x \in \left\{ \frac{c_{23}}{\sqrt{c_{22}c_{33}}}, \frac{c_{23} + 2c_{44}}{\sqrt{c_{22}c_{33}}} \right\}, \quad y \in \left\{ \frac{c_{13}}{\sqrt{c_{11}c_{33}}}, \frac{c_{13} + 2c_{55}}{\sqrt{c_{11}c_{33}}} \right\}, \quad z \in \left\{ \frac{c_{12}}{\sqrt{c_{11}c_{22}}}, \frac{c_{12} + 2c_{66}}{\sqrt{c_{11}c_{22}}} \right\}.$$

The differential system (5) and (6) becomes

$$\begin{aligned} \rho\ddot{u}_1 &= c_{11}u_{1,11} + c_{66}u_{1,22} + c_{55}u_{1,33} + (c_{12} + c_{66})u_{2,21} + (c_{13} + c_{55})u_{3,31} \\ &\quad + M_{11}\theta_{,1} + M_{12}\theta_{,2} \\ \rho\ddot{u}_2 &= (c_{12} + c_{66})u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} + c_{44}u_{2,33} + (c_{23} + c_{44})u_{3,32} \\ &\quad + M_{12}\theta_{,1} + M_{22}\theta_{,2} \end{aligned}$$

$$\begin{aligned}
\rho \ddot{u}_3 &= (c_{13} + c_{55})u_{1,13} + (c_{23} + c_{44})u_{2,23} + c_{55}u_{3,11} + c_{44}u_{3,22} + c_{33}u_{3,33} + M_{33}\theta_{,3} \\
c\dot{\theta} &= K_{11}\theta_{,11} + 2K_{12}\theta_{,12} + K_{22}\theta_{,22} + K_{33}\theta_{,33} \\
&\quad + \theta_0 M_{11}\dot{u}_{1,1} + \theta_0 M_{12}(\dot{u}_{1,2} + \dot{u}_{2,1}) + \theta_0 M_{22}\dot{u}_{2,2} + \theta_0 M_{33}\dot{u}_{3,3}
\end{aligned} \tag{47}$$

Thus, in what follows we consider the problem specified by the differential system (47) with the lateral boundary conditions (41) and the conditions (8) and (9).

The function to be considered now is

$$\begin{aligned}
J(x_3) &= \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left\{ [c_{55}u_{1,3} + (c_{55} - \varkappa_2)u_{3,1}]\dot{u}_1 + [c_{44}u_{2,3} + (c_{44} - \varkappa_1)u_{3,2}]\dot{u}_2 \right. \\
&\quad \left. + [(c_{13} + \varkappa_2)u_{1,1} + (c_{23} + \varkappa_1)u_{2,2} + c_{33}u_{3,3}]\dot{u}_3 + M_{33}\dot{u}_3\theta + \frac{1}{\theta_0}K_{33}\theta\theta_{,3} \right\} da \, d\tau
\end{aligned} \tag{48}$$

where the parameters σ , $\varkappa_1$, and $\varkappa_2$ are at our disposal and their values will be individuated later in a convenient way.

By direct differentiation with respect to x_3 we get

$$\begin{aligned}
\frac{dJ}{dx_3}(x_3) &= \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \{ [c_{55}u_{1,33} + (c_{55} - \varkappa_2)u_{3,31}]\dot{u}_1 + [c_{44}u_{2,33} + (c_{44} - \varkappa_1)u_{3,32}]\dot{u}_2 \\
&\quad + [(c_{13} + \varkappa_2)u_{1,13} + (c_{23} + \varkappa_1)u_{2,23} + c_{33}u_{3,33}]\dot{u}_3 + \frac{1}{\theta_0}K_{33}\theta\theta_{,33} \\
&\quad + [c_{55}u_{1,3} + (c_{55} - \varkappa_2)u_{3,1}]\dot{u}_{1,3} + [c_{44}u_{2,3} + (c_{44} - \varkappa_1)u_{3,2}]\dot{u}_{2,3} \\
&\quad + [(c_{13} + \varkappa_2)u_{1,1} + (c_{23} + \varkappa_1)u_{2,2} + c_{33}u_{3,3}]\dot{u}_{3,3} + \frac{1}{\theta_0}K_{33}\theta_{,3}^2 \\
&\quad + M_{33}\dot{u}_{3,3}\theta + M_{33}\dot{u}_3\theta_{,3} \} da \, d\tau
\end{aligned} \tag{49}$$

so that, by means of the equations given in (47) and the lateral boundary condition (41), we obtain

$$\begin{aligned}
\frac{dJ}{dx_3}(x_3) &= \int_0^T \int_{D(x_3, \tau)} e^{-\sigma\tau} \left\{ \frac{\partial}{\partial\tau} \left[\frac{1}{2}(\rho\dot{u}_i\dot{u}_i + \frac{c}{\theta_0}\theta^2 + c_{11}u_{1,1}^2 + c_{22}u_{2,2}^2 + c_{33}u_{3,3}^2 \right. \right. \\
&\quad + c_{66}(u_{1,2}^2 + u_{2,1}^2) + c_{55}(u_{1,3}^2 + u_{3,1}^2) + c_{44}(u_{2,3}^2 + u_{3,2}^2) + 2(c_{12} + \varkappa_3)u_{1,1}u_{2,2} \\
&\quad + 2(c_{13} + \varkappa_2)u_{1,1}u_{3,3} + 2(c_{23} + \varkappa_1)u_{2,2}u_{3,3} + 2(c_{66} - \varkappa_3)u_{1,2}u_{2,1} \\
&\quad \left. \left. + 2(c_{44} - \varkappa_1)u_{2,3}u_{3,2} + 2(c_{55} - \varkappa_2)u_{1,3}u_{3,1}) \right] \right. \\
&\quad \left. + \frac{1}{\theta_0}K_{\alpha\beta}\theta_{,\alpha}\theta_{,\beta} + \frac{1}{\theta_0}K_{33}\theta_{,3}\theta_{,3} \right\} da \, d\tau
\end{aligned} \tag{50}$$

At this instant we choose the parameters $\varkappa_1 \in [0, 2c_{44}]$, $\varkappa_2 \in [0, 2c_{55}]$, and $\varkappa_3 \in [0, 2c_{66}]$ so that the point $P(x, y, z)$, with coordinates $x = \frac{c_{23} + \varkappa_1}{\sqrt{c_{22}c_{33}}}$, $y = \frac{c_{13} + \varkappa_2}{\sqrt{c_{11}c_{33}}}$,

$z = \frac{c_{12} + \varkappa_3}{\sqrt{c_{11}c_{22}}}$, lies inside the region limited by the surface $S(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1 = 0$, $|x| < 1$, $|y| < 1$, $|z| < 1$. With these choices we have

$$c_{44}(u_{2,3}^2 + u_{3,2}^2) + 2(c_{44} - \varkappa_1)u_{2,3}u_{3,2} \geq \chi_1(u_{2,3}^2 + u_{3,2}^2) \quad (51)$$

$$c_{55}(u_{1,3}^2 + u_{3,1}^2) + 2(c_{55} - \varkappa_2)u_{1,3}u_{3,1} \geq \chi_2(u_{1,3}^2 + u_{3,1}^2) \quad (52)$$

$$c_{66}(u_{1,2}^2 + u_{2,1}^2) + 2(c_{66} - \varkappa_3)u_{1,2}u_{2,1} \geq \chi_3(u_{1,2}^2 + u_{2,1}^2) \quad (53)$$

$$c_{11}u_{1,1}^2 + c_{22}u_{2,2}^2 + c_{33}u_{3,3}^2 + 2(c_{12} + \varkappa_3)u_{1,1}u_{2,2} + 2(c_{13} + \varkappa_2)u_{1,1}u_{3,3} + 2(c_{23} + \varkappa_1)u_{2,2}u_{3,3} \geq \chi_4(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2) \quad (54)$$

$$K_{\alpha\beta}\theta_{,\alpha}\theta_{,\beta} \geq k_0\theta_{,\rho}\theta_{,\rho} \quad (55)$$

where

$$\chi_1 = \min(2c_{44} - \varkappa_1, \varkappa_1), \quad \chi_2 = \min(2c_{55} - \varkappa_2, \varkappa_2), \quad \chi_3 = \min(2c_{66} - \varkappa_3, \varkappa_3) \quad (56)$$

χ_4 is the lowest positive eigenvalue of the linear transformation given by the 3×3 -matrix

$$\begin{pmatrix} c_{11} & c_{12} + \varkappa_3 & c_{13} + \varkappa_2 \\ c_{12} + \varkappa_3 & c_{22} & c_{23} + \varkappa_1 \\ c_{13} + \varkappa_2 & c_{23} + \varkappa_1 & c_{33} \end{pmatrix} \quad (57)$$

and k_0 is the lowest positive eigenvalue of $K_{\alpha\beta}$. On this basis we have

$$\Phi(u_{r,s}) \geq \chi_4(u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2) + \chi_3(u_{1,2}^2 + u_{2,1}^2) + \chi_2(u_{1,3}^2 + u_{3,1}^2) + \chi_1(u_{2,3}^2 + u_{3,2}^2) \quad (58)$$

where

$$\begin{aligned} \Phi(u_{r,s}) = & c_{11}u_{1,1}^2 + c_{22}u_{2,2}^2 + c_{33}u_{3,3}^2 + c_{66}(u_{1,2}^2 + u_{2,1}^2) + c_{55}(u_{1,3}^2 + u_{3,1}^2) \\ & + c_{44}(u_{2,3}^2 + u_{3,2}^2) + 2(c_{12} + \varkappa_3)u_{1,1}u_{2,2} + 2(c_{13} + \varkappa_2)u_{1,1}u_{3,3} \\ & + 2(c_{23} + \varkappa_1)u_{2,2}u_{3,3} + 2(c_{66} - \varkappa_3)u_{1,2}u_{2,1} + 2(c_{44} - \varkappa_1)u_{2,3}u_{3,2} \\ & + 2(c_{55} - \varkappa_2)u_{1,3}u_{3,1} \end{aligned} \quad (59)$$

In view of the conditions (9), (10) and by choosing σ to satisfy (27) and by taking into account the notation (28), from (50) we deduce

$$\begin{aligned} \frac{dJ}{dx_3}(x_3) \geq & \kappa_\sigma \int_{D(x_3,0)} \frac{1}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + \Phi(u_{i,j}) \right) da \\ & + \int_0^T \int_{D(x_3,\tau)} e^{-\sigma\tau} \left[\frac{\sigma}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + \Phi(u_{i,j}) \right) \right. \\ & \left. + \frac{1}{\theta_0} K_{\alpha\beta} \theta_{,\alpha} \theta_{,\beta} + \frac{1}{\theta_0} K_{33} \theta_{,3}^2 \right] da d\tau \end{aligned} \quad (60)$$

Now we write the relation (48) in the form

$$\begin{aligned}
 J(x_3) = & \int_0^T \int_{D(x_3, \tau)} \left\{ \sqrt{\frac{c_{55}}{\rho}} \left(\frac{1}{\sqrt{c_{55}}} [c_{55} u_{1,3} + (c_{55} - \varkappa_2) u_{3,1}] \right) (\sqrt{\rho} \dot{u}_1) \right. \\
 & + \sqrt{\frac{c_{44}}{\rho}} \left(\frac{1}{\sqrt{c_{44}}} [c_{44} u_{2,3} + (c_{44} - \varkappa_1) u_{3,2}] \right) (\sqrt{\rho} \dot{u}_2) \\
 & + \sqrt{\frac{c_{33}}{\rho}} \left[\frac{c_{13} + \varkappa_2}{\sqrt{c_{11} c_{33}}} (\sqrt{c_{11}} u_{1,1}) (\sqrt{\rho} \dot{u}_3) \right. \\
 & + \left. \left. \frac{c_{23} + \varkappa_1}{\sqrt{c_{22} c_{33}}} (\sqrt{c_{22}} u_{2,2}) (\sqrt{\rho} \dot{u}_3) + (\sqrt{c_{33}} u_{3,3}) (\sqrt{\rho} \dot{u}_3) \right] \right\} da d\tau \\
 & + \int_0^T \int_{D(x_3, \tau)} \left[\frac{M_{33} \sqrt{\theta_0}}{\sqrt{c \rho}} (\sqrt{\rho} \dot{u}_3) \left(\sqrt{\frac{c}{\theta_0}} \theta \right) + \sqrt{\frac{K_{33}}{c}} \left(\sqrt{\frac{c}{\theta_0}} \theta \right) \left(\sqrt{\frac{K_{33}}{\theta_0}} \theta_{,3} \right) \right] da d\tau
 \end{aligned} \quad (61)$$

and use the inequalities $|c_{55} - \varkappa_2| < c_{55}$, $|c_{44} - \varkappa_1| < c_{44}$, $|c_{13} + \varkappa_2| < \sqrt{c_{11} c_{33}}$, $|c_{23} + \varkappa_1| < \sqrt{c_{22} c_{33}}$ and the arithmetic-geometric and Schwarz inequalities to obtain the estimate

$$|J(x_3)| \leq \frac{1}{2} \int_0^T \int_{D(x_3, \tau)} \left[\mathcal{M}_1 \rho \dot{u}_i \dot{u}_i + \mathcal{M}_2 \frac{c}{\theta_0} \theta^2 + \mathcal{M}_3 \Phi(u_{r,s}) + \mathcal{M}_4 \frac{1}{\theta_0} K_{33} \theta_{,3}^2 \right] da d\tau \quad (62)$$

where

$$\begin{aligned}
 \mathcal{M}_1 &= \frac{1}{\sqrt{\rho}} \max \left(\sqrt{c_{55}}, \sqrt{c_{44}}, 3\sqrt{c_{33}} + |M_{33}| \sqrt{\frac{\theta_0}{c}} \right), \quad \mathcal{M}_2 = |M_{33}| \sqrt{\frac{\theta_0}{\rho c}} + \sqrt{\frac{K_{33}}{c}} \\
 \mathcal{M}_3 &= \max \left(\frac{1}{\chi_4} \sqrt{\frac{c_{33}}{\rho}} \max(c_{11}, c_{22}, c_{33}), \frac{2c_{55}}{\chi_2} \sqrt{\frac{c_{55}}{\rho}}, \frac{2c_{44}}{\chi_1} \sqrt{\frac{c_{44}}{\rho}} \right), \quad \mathcal{M}_4 = \sqrt{\frac{K_{33}}{c}}
 \end{aligned} \quad (63)$$

Further, we set

$$v = \max \left(\frac{1}{\sigma} \max(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3), \mathcal{M}_4 \right) \quad (64)$$

so that the relations (60) and (62) give

$$|J(x_3)| \leq -v \frac{\partial J}{\partial x_3}(x_3), \quad (x_3) \in (0, L) \quad (65)$$

The implications of the differential inequality (65) can be studied by the same procedure as in the above section. So we obtain the following alternative for a semi-infinite thermoelastic cylinder: Either the energetic measure

$$\begin{aligned}
 \mathcal{E}(x_3) &= \kappa_\sigma \int_{B(x_3, 0)} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + \Phi(u_{i,j}) \right) dv \\
 &+ \int_0^T \int_{B(x_3, \tau)} e^{-\sigma \tau} \left[\frac{\sigma}{2} \left(\rho \dot{u}_i \dot{u}_i + \frac{c}{\theta_0} \theta^2 + \Phi(u_{i,j}) \right) + \frac{1}{\theta_0} K_{ij} \theta_{,i} \theta_{,j} \right] dv d\tau
 \end{aligned} \quad (66)$$

is finite and then we have

$$\mathcal{E}(x_3) \leq J(0)e^{-\frac{x_3}{v}} \quad \text{for all } x_3 \in [0, \infty) \quad (67)$$

or $\mathcal{E}(x_3)$ is unbounded and then $J(x_3)$ grows to infinity more rapidly than an appropriate increasing exponential.

CONCLUDING REMARKS

It may be observed that our results are established under positive conditions upon the thermoelastic coefficients for the class of lateral boundary conditions expressed in (7). Restricting this class to fixed and thermal insulated lateral boundary surface we are able to relax the positive definiteness condition by assuming instead the strong ellipticity condition. This last condition expresses a larger class of thermoelastic materials, including in particular the materials with negative Poisson's ratio (see, for example, Lakes [22]).

On the other hand, it is not immediately obvious how the terms $I(0)$ and $J(0)$ appearing in the decay estimates of the previous sections, depend upon data to the problem and in the absence of some explicit expressions diminishes the practical applicability of the decay estimates in concern. Such bounds should be obtained by adapting the procedure described in [10].

It can be remarked that the present results are established without prescribing explicitly of the initial displacement, velocity and temperature variation and without explicit boundary conditions specified over the ends and without any condition concerning the asymptotically behaviour at large distance for the semi-infinite cylinder.

Our analysis in this paper was developed under the assumption that the constraint parameters λ , μ and v satisfy inequalities (10), that is we have $|\lambda| > 1$, $|\mu| > 1$, $|v| > 1$. For the other values of the constraint parameters λ , μ , and v there seems to be not possible to develop an analysis giving results similar with those in the previous sections. For example, when $|\lambda| < 1$, $|\mu| < 1$, $|v| < 1$ the problem will be ill-posed (see an entire discussion on this argument in [9]). In the case $|\lambda| < 1$, $|\mu| > 1$, $|v| < 1$ (and the others similar) there seems to be non-uniqueness as result from the studies by Quintanilla and Straughan [9], Knops and Payne [10] and Payne and Schaefer [23]. An argument similar with that in [10] can be considered in the discussion of the case when $|\lambda| = 1$, $|\mu| = 1$, or $|v| = 1$.

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