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CONVOLUTION SEMIGROUPS ON THE HEISENBERG GROUP

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Convolution semigroups on Heisenberg group

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Abstract

The purpose of this paper is to see if using the Fourier transform one can establish a relation between convolution semigroups on a locally compact non-Abelian group and something to replace the notion of negative-definite function that appears in the Abelian case. We will see where leads us the Fourier transform on the measures of the semigroup. We will do that only on a particular case, that of the **Heisenberg group** H_n, with the "heat" semigroup corresponding to the laplacean Δ_H on H_n.

The paper is organized in three parts. In the first one some results concerning locally compact groups are presented. The Haar measure is of particular interest because of the role it plays in convolution. The case of the Heisenberg group will be presented here.

In the second part some elements of representation theory will be given together with the dual space of a group G which consists of all equivalence classes of irreducible representations of G. The special case of a locally compact Abelian (LCA) group is presented, followed by a method of determining irreducible representations of a locally compact non-Abelian group. The method, called "Mackey machine", will be applied to obtain irreducible representations for the Heisenberg group. We conclude this part with the Fourier transform - definition in the general case and the particular form in the abelian case.

The last part is reserved to convolution of functions and/or measures and convolution semigroups. We will study convolution semigroups in both LCA and LC non-Abelian case from the point of view presented in abstract .

My personal contributions to this paper can be found in subsection 3.2.2 and are related with the Heisenberg group: the forms for the Fourier transform on the Heisenberg group for the measures of the "heat kernel", proposition 3.2 that contains basic properties for the operators obtain doing a Fourier transformation and proposition 3.3 which states that the Fourier transform of a vaguely continuous convolution semigroup at a representation ρ_h is a C_0 compact semigroup of contractions.

1 Locally compact groups

1.1 Topological groups

1.1.1 Generalities

In this section basic elements of harmonic analysis are presented together with some results. All proofs will be omitted. For proofs and further results see [1].

Definition 1.1 *A topological group is a group G equipped with a topology with respect to which the group operation and the inverting operation are continuous.*

If G is a topological group, the group operation will be denoted as multiplication and the unit element by 1. If A and B are two subsets of G and $x \in G$ we define

$$Ax = \{yx|\ x \in A\} \tag{1}$$

$$xA = \{xy|\ y \in A\} \tag{2}$$

$$A^{-1} = \{y^{-1}|\ y \in A\} \tag{3}$$

$$AB = \{xy|\ x \in A\ \ y \in B\}. \tag{4}$$

The following proposition gives some basic properties of topological groups.

Proposition 1.1 *Let G be a topological group.*

1. *The topology of G is invariant under translations: if U is a open subset of G, $x \in G$ is an arbitrary point and A is a subset of G then Ux, xU, U^{-1}, AU and UA are all open in G.*
2. *For every neighborhood U of 1 there is a symmetric neighborhood V of 1 such that $VV \subset U$.*
3. *If H is a subgroup of G, then so is the closure, $\overline{H}$ of H.*
4. *Every open subgroup is also closed.*
5. *If A and B are compact subsets of G then so is AB.*

Let H be a subgroup of G, G/H the space of all left cosets of H and $q : G \to G/H$ the canonical quotient map. We can define a (natural) topology on G/H by saying that a subset U of G/H is open if and only if $q^{-1}(U)$ is open in G. Obviously q maps open subsets of G to open subsets of G/H.

Proposition 1.2 *Let H be a subgroup of the topological group G. Then:*

1. *If H is closed, G/H is Hausdorff.*
2. *If G is locally compact, so is G/H.*
3. *If H is normal, then G/H is a topological group.*

Corollary 1.1 *If G is T_1 then G is Hausdorff. If G is not T_1 then $\overline{\{1\}}$ is a closed normal subgroup of G and $G/\overline{\{1\}}$ is a Hausdorff topological group.*

In view of this corollary we may assume with no loss of generality that every topological group is Hausdorff (if not, we simply replace G by $G/\overline{\{1\}}$) so by a locally compact group we mean a topological group whose topology is locally compact *and Hausdorff.*

Proposition 1.3 *If G is a locally compact group then G has a subgroup H that is open (hence closed) and σ-compact.*

Using this proposition we can say that G is the disjoint union of cosets of H, each of which is closed and open in G and homeomorphic with H. So, from a topological point of view, G is a disjoint union of a σ-compact space; in particular, if G is connected then G is σ-compact.

If f is a function on a topological group G and $y \in G$, we define the left and right translations of f through y by

$$L_y f(x) = f(y^{-1}x), \qquad R_y f(x) = f(xy). \tag{5}$$

The reason for using y^{-1} instead of y is to make the maps $y \mapsto L_y$ and $y \mapsto R_y$ group homomorphisms:

$$L_{yz} = L_y L_z, \qquad R_{yz} = R_y R_z.$$

Definition 1.2 *We say that a function $f : G \to \mathbb{C}$ is left (or right) uniformly continuous if $\|L_y f - f\|_{sup} \to 0$ (resp. $\|R_y f - f\|_{sup} \to 0$) as $y \to 1$.*

Proposition 1.4 *If $f \in C_c(G)$ then f is left and right uniformly continuous.*

Example 1.1 The locally compact groups that arises most frequently in practice are the (finite-dimensional) Lie groups: the additive group $\mathbb{R}^n$, all closed subgroups of the group $GL(n, \mathbb{R})$ of invertible linear transformations of $\mathbb{R}^n$. The group $\mathbb{T}$ of complex numbers of modulus one,

$$\mathbb{T} = \{z \in \mathbb{C} |\ |z| = 1\}$$

will play a very important role. Because it is isomorphic with $\mathbb{R}/\mathbb{Z}$ we will make no difference between them.

Another example of topological group that we will discuss all through this paper is the *Heisenberg group*, H_n. This is actually $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the composition low:

$$(x, \xi, t)(x', \xi', t') = \big(x + x', \xi + \xi', t + t' + \frac{1}{2}(x'\xi - x\xi')\big) \tag{6}$$

where $x\xi'$ and $x'\xi$ are the usual products in $\mathbb{R}^n$. It verifies easily that H_n is a *non-Abelian* group, has the unit 1=(0,0,0) and the inverse of an element $X = (x, \xi, t) \in H_n$ is $(-x, -\xi, -t)$. It is also obvious that H_n endowed with the usual topology on $\mathbb{R}^{2n+1}$ is a locally compact group (it is actually σ-compact).

Two subgroups are most important here:

$$N_1 = \{X \in H_n |\ x = 0\} \ \ \textit{and}\ K_1 = \{X \in H_n |\ \ \xi = 0,\ t = 0\}. \tag{7}$$

We can consider as well

$$N_2 = \{X \in H_n |\ \xi = 0\} \ \ \textit{and}\ K_1 = \{X \in H_n |\ \ x = 0,\ t = 0\}. \tag{8}$$

But further on we will only work with N_1 and K_1. One can easily see that these subgroups are Abelian and moreover that N_1 is normal.

1.1.2 Semi-direct product of subgroups

Definition 1.3 *Let G be a topological group and N and H two closed subgroups of G. We say that G is the* semi-direct product *of N and H if:*

1. *N is normal*
2. *the map $(n, h) \to nh$ from $N \times H$ to G is a homeomorphism.*

In this case we write $G = N \times H$.

Thus, when $G = N \times H$ every element of G can be written uniquely as a product between an element from N and one from H. Of course, N and H have a single element in common, namely, the unit.

Example 1.2 For the Heisenberg group, H_n presented above, N_1 and K_1 are closed subgroups of H_n and N_1 is normal. If $X = (x, \xi, t)$ is an element of H_n and we wish to write

$$(x, \xi, t) = (0, \xi', t')(x', 0, 0)$$

we get

$$(x, \xi, t) = (x', \xi', t' + \frac{1}{2}x'\xi')$$

and this happens if and only if $x' = x$, $\xi' = \xi$ and $t' = t - \frac{1}{2}x\xi$. Thus, H_n is the semi-direct product of N_1 and K_1.

1.1.3 Homogeneous spaces

Consider now G a locally compact group and S a locally compact Hausdorff space.

Definition 1.4 *A (left)* action *of G on S is a continuous map $(x, s) \mapsto xs$ from $G \times S$ to S such that:*

1. *$s \mapsto xs$ is a homeomorphism of S for each $x \in G$, and*
2. *$x(ys) = (xy)s$ for all $x, y \in G$ and $s \in S$.*

A space S equipped with a left action of G is called a G-space. Further, a G-space is called *transitive* if for every $s, t \in S$ there exists $x \in G$ such that $xs = t$.

G-spaces are closely related with quotient spaces G/H where H is a subgroup of G. If S is a transitive G-space, let s_0 be a fixed element of S and the map

$$\phi : G \to S, \qquad \phi(x) = xs_0$$

and let

$$H = \{x \in G|\ xs_0 = s_0\}.$$

Then H is a closed subgroup of G and ϕ is a continuous surjection of G onto S that is constant on the left cosets of H. Thus, ϕ induces a continuous bijection $\Phi : G/H \to S$ such that $\Phi \circ q = \phi$, where $q : G \to G/H$ is the natural quotient map. So the only thing needed to identify S with G/H is that Φ^{-1} to be continuous. But this is not necessarily true. For example, if we consider $G = \mathbb{R}$ with discrete topology, acting by translations on $\mathbb{R}$ with the usual topology. Still, we have:

Proposition 1.5 *If S is a G-space and G is σ-compact then S can be identified with a quotient space G/H, where H is a closed subgroup of G (i.e. the above map, Φ, is a homeomorphism).*

We will use the term of homogeneous space to refer to a transitive G-space S that is isomorphic to a quotient space G/H and we shall generally identify S with G/H.

1.2 Haar measure on a locally compact group

This is a fundamental notion. It will allow us to define the convolution of functions and help us to give a structure of *-Banach algebra to $L^1(G, \lambda)$, where λ is the *left Haar measure on G.*

Let G be a locally compact group. Denote by $C_c(G)$ the space of real continuous functions on G with compact support and by

$$C_c^+(G) = \{f \in C_c(G)|\ f \geq 0 \textit{ and } f \textit{ is not identically } 0\}.$$

It is obvious that the linear span of $C_c^+(G)$ is $C_c(G)$.

Definition 1.5 *A left (resp. right) Haar measure on G is a nonzero Radon measure μ on G that satisfies*

$$\mu(xE) = \mu(E) \qquad (resp.\ \mu(Ex) = \mu(E)) \tag{9}$$

for every Borel set $E \subset G$ and every $x \in G$.

Proposition 1.6 *Consider μ a Radon measure on the locally compact group G, and let $\tilde{\mu}(E) = \mu(E^{-1})$.*

1. *μ is a left Haar measure if and only if $\tilde{\mu}$ is a right Haar measure.*
2. *μ is a left Haar measure if and only if*

$$\int L_y f d\mu = \int f d\mu \quad \forall f \in C_c^+(G) \text{ and } \forall\ y \in G \tag{10}$$

In view of this proposition it is the same thing if one chooses to study left or right Haar measure. We shall refrain here to the left one, being a more common choice. We have:

Proposition 1.7 *If λ is a left Haar measure on G, then:*

1. *$\lambda(U) > 0$ for any nonempty open set U.*
2. *$\int f d\lambda > 0$ for every $f \in C_c^+(G)$.*
3. *If μ is another Haar measure on G then there exists $c > 0$ such that $\mu = c\lambda$.*

The next theorem establishes the existence of Haar measure. A sketch of proof is presented (for full details, see [1], p. 36-40).

Theorem 1.1 *Every locally compact group G possesses a left Haar measure λ.*

Proof **F**or $f, \phi \in C_c^+(G)$ we define

$$(f : \phi) = inf\{\sum_{j=1}^{n} c_j|\ f \leq \sum_{j=1}^{n} c_j L_{x_j}\phi,\ c_j > 0,\ x_j \in G,\ j = 1..n,\ n \in \mathbb{N}^*\}. \tag{11}$$

The definition makes sense because the support of f can be covered by finitely many (say N) left translates of the set where $\phi > \frac{1}{2}\|\phi\|_{sup}$, so one can see immediately that

$$(f : \phi) \leq 2N\|f\|_{sup}/\|\phi\|_{sup}.$$

The quantity $(f : \phi)$ has the following properties:
1. $(f : \phi) = (L_y f : \phi)$ *for any* $y \in G$.

2. $(f_1 + f_2 : \phi) \leq (f_1 : \phi) + (f_2 : \phi)$.
3. $(cf : \phi) = c(f : \phi)$ *for any* $c > 0$.
4. $(f_1 : \phi) \leq (f_2 : \phi)$ *whenever* $f_1 \leq f_2$.
5. $(f : \phi) \geq \|f\|_{sup}/\|\phi\|_{sup}$.
6. $(f : \phi) \leq (f : \psi)(\psi : \phi)$ for any $\psi \in C_c^+(G)$.

Next step is to normalize by fixing an $f_0 \in C_c^+(G)$ and defining

$$I_\phi(f) = \frac{(f : \phi)}{f_0 : \phi} \quad f, \phi \in C_c^+(G). \tag{12}$$

By properties listed before, I_ϕ is left-invariant, homogeneous of degree 1, monotone and *sub-additive* for each ϕ and satisfies

$$(f_0 : f)^{-1} \leq I_\phi(f) \leq (f : f_0).$$

Moreover, if $f_1, f_2 \in C_c^+(G)$ and $\varepsilon > 0$ there is a neighborhood V of 1 in G such that

$$I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \varepsilon, \quad \forall \phi \in C_c^+ G, \ supp\phi \subset V.$$

For each $f \in C_c^+(G)$ consider X_f the interval $\left[(f_0 : f)^{-1}, (f : f_0)\right]$ and let X be the compact space of the Cartesian product of all X_f's. It is easy to see that each I_ϕ is in X. If V is a neighborhood of 1 in G let $K(V)$ be the closure in X of $\{I_\phi | \ supp\phi \subset V\}$. The sets $K(V)$ have the finite intersection property so by compactness, there exists a point $I \in X$ that lies in every $K(V)$. I has all the good properties and can be extended to a positive linear functional on $C_c(G)$. The desired measure λ is then the one associated with I by the Riesz representation theorem. Q.E.D.

Example 1.3 Here are a few examples of locally compact groups with their left Haar measures:

1. $dx/|x|$ is a left and right Haar measure on the multiplicative group $\mathbb{R} \setminus \{0\}$.
2. $dxdy/(x^2 + y^2)$ is a left and right Haar measure on the multiplicative group $\mathbb{C} \setminus \{0\}$ with coordinates $z = x + iy$.
3. On the additive group of $n \times n$ real matrices (x_{ij}) such that $x_{ij} = 0$ for $i > j$ and $x_{ii} = 1$ for $1 \leq i \leq n$ the Haar measure is $\Pi_{i<j} dx_{ij}$.
4. The **ax+b**-group is the group of all affine transformations $x \mapsto ax + b$ of $\mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}$. On this group $dadb/a^2$ is a left Haar measure and $dadb/a$ is a right Haar measure.

5. If $G_1, ..., G_n$ are locally compact groups with left Haar measures $\lambda_1, ..., \lambda_n$, the left Haar measure on $G = \prod_1^n G_j$ is, of course, the Radon measure on G associated to the linear functional
$$f \mapsto \int ... \int f(x_1, x_2, ..., x_n) d\lambda_1(x_1) ... d\lambda_n(x_n).$$

6. For the Heisenberg group (see 1.8), the Haar measure is nothing else but the Lebesgue measure on $\mathbb{R}^{2n+1}$. Indeed, if $X_0 = (x_0, \xi_0, t_0) \in H_n$ and $f \in C_c^+(G)$ we have:
$$\int_{H_n} L_{X_0} f(x, \xi, t) dx d\xi dt = \int_{\mathbb{R}^{2n+1}} f(x - x_0, \xi - \xi_0, t - t_0 + \frac{1}{2}(x\xi_0 - x_0\xi)) dx d\xi dt$$
Making the change of variable $x = u + x_0$, $\xi = v + \xi_0$, $t = t_0 - \frac{1}{2}(u\xi_0 - x_0 v)$, the determinant of it is 1 and thus Lebesgue measure is the Haar measure on H_n. In the same manner one can prove that Lebesgue measure is the right Haar measure on H_n.

1.2.1 The modular function

Let G be a locally compact group with a fixed Haar measure λ. If we define for $x \in G$, $\lambda_x(E) = \lambda(Ex)$, then λ_x is again a left Haar measure. By proposition (2.3, iii), there is a positive real number $\Delta(x)$ such that $\lambda_x = \Delta(x)\lambda$, and $\Delta(x)$ is independent of the original choice of λ. The function $\Delta : G \to (0, \infty)$ defined this way is called *the modular function* of G.

Proposition 1.8 *Δ is a continuous homomorphism from G to the multiplicative group of positive real numbers. Moreover, for any $f \in L^1(\lambda)$ we have*
$$\int_G R_y f d\lambda = \Delta(y^{-1}) \int_G d\lambda.$$

Remark 1.1 A more convenient form for the formula in proposition 8 is the following:
$$d\lambda(xy_0) = \Delta(y_0) d\lambda(x). \tag{13}$$
G is called *unimodular* if $\Delta \equiv 1$, which happens exactly when the left Haar measure is also a right Haar measure. Unimodularity makes things easier in many cases. Of course, Abelian groups and discrete groups are unimodular but these are not the only ones. For instance , the Heisenberg group is unimodular, as we've already seen earlier. Here are other examples:

Proposition 1.9 *Let G be a locally compact group.*

1. *If K is any compact subgroup of G then $\Delta | K \equiv 1$.*

2. *If G is compact, then G is unimodular.*

3. *If $G/[G,G]$ is compact, then G is unimodular, where $[G,G]$ is the commutator subgroup of G, that is*
$$[G,G] = \{[x,y] = xyx^{-1}y^{-1} : \ x, y \in G\}.$$

Remark 1.2 The unimodular function can be used to relate the left and right Haar measure: if we denote by λ the left Haar measure, then the measure ρ given by $\rho(E) = \lambda(E^{-1})$ is a right Haar measure (by proposition (2.2)). Then, we have:

$$d\rho(x) = \Delta(x^{-1})d\lambda(x) \tag{14}$$

or, equivalent
$$d\lambda(x^{-1}) = \Delta(x^{-1})d\lambda(x), \quad d\rho(x^{-1}) = \Delta(x)d\rho(x). \tag{15}$$

2 Fourier transform on locally compact (non-Abelian) groups

2.1 Representations of a group

2.1.1 Basic notions

In order to define the Fourier transform for certain functions we need another notion, that of irreducible representation of a group G. If G is Abelian, we can define it more naturally, but essentially is a particular case of the first one. Thus, we begin with G being a locally compact group and not make the assumption for this group to be Abelian.

Definition 2.1 *Let G be a locally compact group. A unitary representation of G is a homomorphism $\pi : G \to \mathcal{U}(H_\pi)$ where H_π is a Hilbert space, $\mathcal{U}(H_\pi)$ is the group of unitary operators on H_π and π is continuous in the strong operator topology.*

In other words, $\pi : G \to \mathcal{U}(H_\pi)$ is a unitary representation of G onto H_π if:

1. $\pi(xy) = \pi(x)\pi(y)$
2. $\pi(x^{-1}) = (\pi(x))^{-1}$
3. $x \mapsto \pi(x)u$ is continuous for every $u \in H_\pi$.

H_π is said to be the *representation space* for π and the dimension of H_π is by definition, the *dimension* or *degree* of π. Since we will only use unitary representations, we will simply say representation to denote one.

A simple example is provided by the action of G on itself by left and right translations. Left translations determines the *left regular representation* of G on $L^2(G)$:

$$\pi_L : G \mapsto L^2(G), \quad [\pi_L(x)f](y) = L_x f(y) = f(x^{-1}y).$$

Right translations determines the *right regular translation*, π_R in a similar way.

Definition 2.2 *If $\pi_1 : G \to \mathcal{U}(H_{\pi_1})$ and $\pi_2 : G \to \mathcal{U}(H_{\pi_2})$ are two representations of G on H_{π_1} and H_{π_2} respectively, a bounded linear map $T : H_{\pi_1} \to H_{\pi_2}$ that satisfy $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$ is called an intertwining operator.*

The set of all intertwining operators is denoted by $\mathcal{C}(\pi_1, \pi_2)$. π_1 and π_2 are (unitary) equivalent if $\mathcal{C}(\pi_1, \pi_2)$ contains at least one unitary operator.

For example, the left and right regular representations are equivalent and the intertwining operator is $f \mapsto Uf$, where $Uf(x) = f(x^{-1})$.

We will write $\mathcal{C}(\pi)$ instead of $\mathcal{C}(\pi, \pi)$. This is the space of bounded operators on H_π that commutes with $\pi(x)$ for every $x \in G$; it is called the *centralizer* of π. Obviously $\mathcal{C}(\pi)$ is an algebra that is closed under weak limits and it is also closed under taking adjoint for if $T \in \mathcal{C}(\pi)$: $T^*\pi(x) = (\pi(x^{-1}T))^* = (T\pi(x^{-1}))^* = \pi(x)T^*$. Thus, $\mathcal{C}(\pi)$ is a weakly closed C^* algebra of operators on H_π, that is, a von Neumann algebra.

Definition 2.3 *Let G a locally compact group and π a representation of G on H_π.*

1. *A closed subspace $\mathcal{M}$ of H_π is called an invariant subspace for π if $\pi(x)\mathcal{M} \subset \mathcal{M}$ for every $x \in G$.*

2. *If $\mathcal{M}$ is invariant and non trivial, the restriction of π to $\mathcal{M}$, $\pi^{\mathcal{M}}(x) = \pi(x)|_{\mathcal{M}}$, defines a representation of G on $\mathcal{M}$, called a sub-representation of π. In this case the representation π is called reducible. Otherwise π is said to be irreducible.*

Proposition 2.1 ***(Schur's lemma)***

1. *A unitary representation π of G is irreducible if and only if $\mathcal{C}(\pi)$ contains only scalar multiples of the identity.*

2. *Suppose π_1 and π_2 are irreducible unitary representations of G. If π_1 and π_2 are equivalent then $\mathcal{C}(\pi_1, \pi_2)$ is one-dimensional; if not, $\mathcal{C}(\pi_1, \pi_2) = \{0\}$.*

2.1.2 Representations of LCA-groups

Schur's lemma allows us for an instance to see that if G is an LCA-group every irreducible representation of G is 1-dimensional. Indeed, if π is a representation of G the operators $\pi(x)$ all commute with one-another so they all belong to $\mathcal{C}(\pi)$. Moreover, if π is irreducible, then by Schur's lemma $\mathcal{C}(\pi)$ consists only of scalar multiples of the identity hence $\pi(x) = c_x I$ for every $x \in G$ so every 1-dimensional subspace of $H_\pi$is π-invariant.

Thus, necessarily, π is 1-dimensional for every such representation π we can take $H_\pi = \mathbb{C}$, and then $\pi(x)(z) = \xi(x)z$ $(z \in \mathbb{C})$ where ξ is a continuous homomorphism of G into the circle group $\mathbb{T}$. Such homomorphisms are called (unitary) *characters* of G, and we will identify a irreducible representation of a LCA-group with a character. The set of all characters of G is denoted by $\hat{G}$. It is obvious that $\hat{G}$ is an Abelian group under pointwise multiplication; its identity element is the constant function 1 and $\xi^{-1} = \overline{\xi}$.

It can be shown (*for full details see [1], pages 88-89*), that $\hat{G}$ is the set of extreme points of $\mathcal{P}_1(G) \subset L^\infty(G)$ of all functions of positive type on G, where $L^\infty(G)$ is with respect to the (left) Haar measure on G. On $\hat{G}$ we consider the topology of compact convergence, under which the group operations are obviously continuous. This topology coincides with the weak$*$ topology that $\hat{G}$ inherits as a subset of Ł$^\infty$. But $\hat{G} \cup \{0\}$ are **all** homomorphisms from L^1 to $\mathbb{C}$ which is a subset of the closed unit ball of L^∞ and is weak$*$ closed and hence is weak$*$ compact. Therefore, $\hat{G}$ is a locally compact Abelian group, called the *dual group* of G.

Example 2.1 *1.* $\hat{\mathbb{R}} \cong \mathbb{R}$, with the pairing $\xi(x) =< x, \xi >= e^{2\pi i \xi x}$

2. $\hat{\mathbb{T}} \cong \mathbb{Z}$, with the pairing $< \alpha, n >= \alpha^n$

3. If $G_1, G_2, ..., G_n$ are locally compact Abelian groups, then

$$(G_1 \times G_2 \times ... \times G_n)\hat{} \cong \hat{G}_1 \times \hat{G}_2 \times ... \times \hat{G}_n.$$

4. By c. $\hat{\mathbb{R}^n} \simeq \mathbb{R}^n$ and $\mathbb{T}^n \simeq \mathbb{Z}^n$.

More details about harmonic analysis on a locally compact abelian group can be found in [2].

2.2 Induced representations

Irreducible representations are the main ingredients for harmonic analysis on locally compact groups. In case of a LCA-group we've already seen precisely what are these representations. In the non-Abelian case obtaining the irreducible representations is a more difficult task to perform.

The inducing construction is a way of manufacturing a unitary (not necessarily irreducible) representation of a locally compact group G out of a unitary representation of a closed subgroup. If this subgroup is normal, via "Mackey machine" we can obtain *irreducible* representations in the non-compact non-Abelian case.

Let G be a locally compact group, H a closed subgroup, $q : G \to G|H$ the canonical quotient map and σ a unitary representation of H on H_σ. If $f \in C(G, H_\sigma)$ we will write $\sigma(\xi)f(x)$ instead of $\sigma(\xi)[f(x)]$, $\xi \in H$.

Consider the following space of vector-valued functions:

$$\mathcal{F}_0 = \{f \in C(G, H_\sigma)|\ q(supp\ f)\ is\ compact\ and\ f(x\xi) = \sigma(x^{-1})f(x)\}. \tag{16}$$

Proposition 2.2 *If $\alpha : G \to H_\sigma$ is continuous with compact support, then the function*

$$f_\alpha(x) = \int_H \sigma(\eta)\alpha(x\eta)d\eta$$

belongs to $\mathcal{F}_0$ and is uniformly continuous on G. Moreover, every element of $\mathcal{F}_0$ is of the form f_α for some $\alpha \in C_c(G, H_\sigma)$.

G acts on $\mathcal{F}_0$ by left translation, $f \to L_x f$, so we obtain a unitary representation of G if we can impose on $\mathcal{F}_0$ an inner product with respect to which these translations are isometries. This happens always when $G|H$ admits an invariant measure μ because if f and g are two elements of $\mathcal{F}_0$, $< f(x), g(x) >_\sigma$ depends only on the coset $q(x)$ of x since σ is unitary so it defines a function in $C_c(G|H)$ which can be integrated with respect to μ and setting

$$< f, g >= \int_{G|H} < f(x), g(x) >_\sigma d\mu(xH)$$

we obtain an inner product on $\mathcal{F}_0$ that is preserved under left translation since μ is invariant. Hence, if we denote by $\mathcal{F}$ the Hilbert space completion of $\mathcal{F}_0$, the translation operators L_x extends to unitary operators on $\mathcal{F}$ and so we obtain a representation of G on $\mathcal{F}$, called the *representation induced by* σ and denoted by $ind_H^G(\sigma)$.

For example, if σ is the trivial representation of H on $\mathbb{C}$, then $\mathcal{F}_0$ consists of functions on G that are constant on cosets of H. Hence, $\mathcal{F}_0$ can be naturally identified with $C_c(G|H)$. But then $\mathcal{F}$ is nothing else than $L^2(G|H)$ and $ind_H^G(\sigma)$ is just the natural representation of G on $L^2(G|H)$ by left translations. For further topics on this subject see [1],*pages 152-153, 182-187.*

2.2.1 Irreducible representations of the Heisenberg group

To obtain them we need a few more notions to give and observations to make.

Suppose G is a locally compact group and N is a commutative closed normal subgroup of G. Then G acts on N by conjugation: $(x, n) \mapsto x^{-1}nx$ and this induces another action, that of G on the dual space of N, $\hat{N}$: $(x, \nu) \mapsto x\nu$, where $< n, x\nu >=< x^{-1}nx, \nu >$.

For each $\nu \in \hat{N}$, we denote by G_ν the *stabilizer of* ν,

$$G_\nu = \{x \in G|\ x\nu = \nu\},$$

which is a closed subgroup of G, and by $\mathcal{O}_\nu$ we denote the *orbit of* ν:

$$\mathcal{O}_\nu = \{x\nu|\ x \in G\}.$$

The action of G on $\hat{N}$ is never transitive and the structure of the orbits can be very complicated. G is said to act regularly $\hat{N}$ if the following two conditions holds:

1. The orbit space is countably separated i.e. there exists a countable family E_j of G - invariant Borel sets in $\hat{N}$ such that each orbit is the intersection of all the E_j's that contain it;

2. For each $\nu \in \hat{N}$, the natural map $xG_\nu \mapsto x\nu$ from $G|G_\nu$ to $\mathcal{O}_\nu$ is a homeomorphism.

As a remark, when G is second countable these conditions are equivalent and they are both implied by:
3. There is a Borel set in $\hat{N}$ that intersect each orbit in exactly one point.

The little group H_ν associated with ν is the group $H_\nu = G_\nu \cap H$. Obviously, if N is Abelian then $N \subset G_\nu$, $G_\nu = N \times H_\nu$ and $H_\nu \simeq G_\nu|N$.

The character $\nu \in \hat{N}$ always extends to a homeomorphism $\tilde{\nu} : G_\nu \to \mathbb{T}$ given by $\tilde{\nu}(nh) = \nu(n) =< n, \nu >$:

$$\tilde{\nu}((n_1h_1)(n_2h_2)) =< n_1(h_1n_2h_1^{-1})h_1h_2, \nu >=< n_1, \nu >< h_1n_2h_1^{-1}, \nu >=$$

$$=< n_1, \nu >< n_2, \nu >= \tilde{\nu}(n_1)\tilde{\nu}(n_2) = \tilde{\nu}(n_1h_1)\tilde{\nu}(n_2h_2)$$

Finally if $\nu \in \hat{N}$ and ρ is a irreducible representation of H_ν then $\nu\rho$ defined by $(\nu\rho)(nh) =< n, \nu > \rho(h)$ is a irreducible representation of G_ν.

The next theorem connects all these things and shows how to produce irreducible representations out of those of a commutative normal subgroup.

Theorem 2.1 *Suppose $G = N \times H$, where N is Abelian and G acts regularly on $\hat{N}$. If $\nu \in \hat{N}$ and ρ is an irreducible representation of H_ν, then $ind_{G_\nu}^G(\nu\rho)$ is an irreducible representation of G, and every irreducible representation of G is equivalent to one of this form.*

Moreover, $ind_{G_\nu}^G(\nu\rho)$ and $ind_{G_{\nu'}}^G(\nu'\rho')$ are equivalent if and only if ν and ν' belong to the same orbit, say $\nu' = x\nu$, and $h \mapsto \rho(h)$ and $h \mapsto \rho'(x^{-1}hx)$ are equivalent representations of H_ν.

Example 2.2 ***The Heisenberg group***

To understand how this theorem works take the case of the Heisenberg group H_n. Recall that:

- The Heisenberg group is the set $H_n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the operation:

  $$(x, \xi, t) \cdot (x', \xi', t') = (x + x', \xi + \xi', t + t' + \frac{1}{2}(x'\xi - x\xi')),$$

  where $x, x', \xi,\ \xi' \in \mathbb{R}^n$ and $t,\ t' \in \mathbb{R}$.

- The neutral element is 1=(0,0,0) and the inverse of an element $X = (x, \xi, t) \in H_n$ is $X^{-1} = (x, \xi, t)^{-1} = (-x, -\xi, -t)$.

- H_n is a locally compact non-commutative group with the usual topology on $\mathbb{R}^{2n+1}$

- The Haar measure for H_n is the Lebesgue measure on $\mathbb{R}^{2n+1}$.

- H_n is the semidirect product of N_1 and K_1 where

  $$N_1 = \{X \in H_n |\ x = 0\} \ \ and\ K_1 = \{X \in H_n |\ \ \xi = 0,\ t = 0\}.$$

  Every element of H_n can be written $(x, \xi, t) = (0, \xi, t - \frac{1}{2}x\xi)(x, 0, 0)$.

We will only take the case where n=1 to find the irreducible representations, the ones for the general case being found in a similar way. For this we will place ourself in the conditions of the above theorem. N_1 is an Abelian closed normal subgroup of H_1 and K_1 is a closed Abelian subgroup of H_1.

Due to the fact that N_1 is isomorph with $\mathbb{R}^2$ it follows that $\hat{N_1} \simeq \hat{\mathbb{R}^2}$:

$$\gamma \in \hat{N} \Rightarrow \exists! \ \ (\xi_0, t_0) \in \mathbb{R}^2 \quad \gamma(\xi, t) = e^{2\pi i(\xi_0,t_0)(\xi,t)}$$

H_1 acts on N_1 by conjugation:

$$(X, N) \mapsto XNX^{-1} = (x, \xi, t)(0, \xi_1, t_1)(-x, -\xi, -t) = (0, \xi_1, t_1 - x\xi_1) \in N_1$$

where $X = (x, \xi, t) \in H_1$ and $N = (0, \xi, t) \in N_1$. This action induces another one, of H_1 on $\hat{N_1} : (X, \gamma) \mapsto X\gamma$, where $X\gamma$ is given by

$$X\gamma(N) = \gamma(XNX^{-1}) = \gamma(0, \xi_1, t_1 + x\xi_1) = e^{2\pi i(\xi_0,t_0)(\xi_1,t_1+x\xi_1)}.$$

The stabilizer of γ is the group

$$H_{1\gamma} = \{X \in H_1 |\ X\gamma = \gamma\} = \{X \in H_1 |\ X\gamma(N) = \gamma(N),\ \forall\ N \in N_1\}$$

With the notations above, $X \in H_{1\gamma}$ if and only if

$$e^{2\pi i(\xi_0,t_0)(\xi_1,t_1-x\xi_1)} = e^{2\pi i(\xi_0,t_0)(\xi_1,t_1)} \Leftrightarrow e^{2\pi i(\xi_0,t_0)(0,x\xi_1)} = 1 \Leftrightarrow t_0 x\xi_1 \in \mathbb{Z}$$

for every $\xi_1 \in \mathbb{R}$.

We separate two possible cases:

1. If t_0=0 then $t_0 x\xi_1 = 0 \in \mathbb{Z}\ \forall\ x \in \mathbb{R}$ so $H_{1\gamma} = H_1$.
2. If $t_0 \neq 0$ then $t_0 x\xi_1 \in \mathbb{Z}$ for all $\xi_1 \in \mathbb{R}$ if and only if x=0, that is, $H_{1\gamma} = N_1$. This means that the stabilizer of $\gamma \in \hat{N_1}$ is N_1.

   From here, to determine the orbit of a character γ, $O_\gamma = \{X\gamma |\ X \in H_1\}$ we get:

1. If $\gamma = (\xi_0, 0)$, (meaning $\gamma(\xi_1, t_1) = e^{2\pi i\xi_0\xi_1}$) then

   $$(X\gamma)(\xi_1, t_1) = \gamma(\xi_1, t_1 + x\xi_1) = e^{2\pi i\xi_0\xi_1} = \gamma(\xi_1, t_1)$$

   and so $O_\gamma = \{\gamma\}$.

2. If $\gamma = (\xi_0, t_0)$ with $t_0 \neq 0$ then

$$(X\gamma)(\xi_1, t_1) = \gamma(\xi_1, t_1 - x\xi_1) = e^{2\pi i(\xi_0,t_0)(\xi_1,t_1-x\xi_1)} = \gamma(\xi_1, t_1) \cdot \gamma(0, -x\xi_1).$$

We obtain that $O_\gamma \simeq \mathbb{R}$.

To show that H_1 acts regularly on $\hat{N}_1$ we need to prove that the application $XH_{1\gamma} \mapsto X\gamma$ is an isomorphism from $H_1|H_{1\gamma}$ to O_γ. For $t_0 = 0$ this is obvious since $H_{1\gamma} = H_1$ so $H_1|H_1 = \{0\}$, and $O_\gamma = \{\gamma\}$.

When $t_0 \neq 0$ we have $H_{1\gamma} = N_1$ and $O_\gamma \simeq \mathbb{R}$. As $H_1|N_1$ is obviously isomorph with $\mathbb{R}$ the conclusion follows.

Finally, the little group, $H_\gamma = H_{1\gamma} \cap K_1$ is K_1 when $H_{1\gamma} = H_1$ ($t_0 = 0$) and $\{0\}$ when $H_{1\gamma} = N_1$ ($t_0 \neq 0$).

To determine the irreducible representations of H_1 we will separate, as we did until now, the two cases: $t_0 = 0$ and $t_0 \neq 0$.

I.) If $H_\gamma = K_1$ ($t_0 = 0$) and ρ is a irreducible representation of K_1 then ρ is given by $\rho(x, 0, 0) = e^{2\pi i x_0 x}$, where x_0 is a real number uniquely determined for ρ. The representation $\gamma\rho$ is given by:

$$(\gamma\rho)(x, \xi, t) = (\gamma\rho)((0, \xi, t - \frac{1}{2}x\xi)(x, 0, 0)) = \gamma(\xi, t - \frac{1}{2}x\xi) \cdot \rho(x, 0, 0).$$

Hence, $(\gamma\rho)(x, \xi, t) = e^{2\pi i(\xi_0,0)(\xi,t-\frac{1}{2}x\xi)} \cdot e^{2\pi i x_0 x} = e^{2\pi i(\xi_0\xi + x_0 x)}$. Since $H_{1\gamma} = H_1$ we get:

$$ind_{H_{1\gamma}}^{H_1}(\gamma\rho) = ind_{H_1}^{H_1}(\gamma\rho) = \gamma\rho$$

II.) When $H_\gamma = \{0\}$ then $\widehat{H}_\gamma = \{\mathbf{1}\}$ and so

$$(\gamma\rho)(x, \xi, t) = \gamma(\xi, t - \frac{1}{2}x\xi) = e^{2\pi i\xi_0\xi + 2\pi i t_0 t - \pi i t_0 x\xi}.$$

We will build now $ind_{H_{1\gamma}}^{H_1}(\gamma\rho) = ind_{N_1}^{H_1}(\gamma\rho)$.

$$\mathcal{F}_0 = \{f \in \mathcal{C}(H_1, \mathbb{C})|\ q(suppf) - compact,\ f(X \cdot (0, \xi', t')) = \overline{\gamma(0, \xi', t')} f(X)\}$$

Since every $X \in H_1$ can be written $X = (x, \xi, t) = (x, 0, 0)(0, \xi, t + \frac{1}{2}x\xi)$ we get for $f \in \mathcal{F}_0$:

$$f(X) = \gamma(0, -\xi, -t + \frac{1}{2}x\xi) \cdot f(x, 0, 0)$$

so $q(supp\ f)$ is compact if and only if the set $\{x|(x, 0, 0) \in supp\ f\}$ is compact. This way we can identify f with a function in $C_c(\mathbb{R})$. Conversely, if $f \in C_c(\mathbb{R})$ we can define $\tilde{f}$ on H_1 given by

$$\tilde{f}(X) = \gamma(0, -\xi, -t - \frac{1}{2}x\xi) \cdot f(x), \quad \forall X \in H_1.$$

Obviously, $q(supp\ \tilde{f}) = q(supp\ f)$ is compact in $H_1|N_1$. Moreover, if $X = (x, \xi, t) \in H_1$ and $N = (0, m, n) \in N_1$ we have:

$$\tilde{f}(XN) = \tilde{f}(x, \xi + m, t + n - \frac{1}{2}mx) =$$

$$= \gamma(0, -\xi - m, -t - n - \frac{1}{2}x\xi) \cdot f(x) =$$

$$= \gamma(0, -m, -n)\gamma(0, -\xi, -t - \frac{1}{2}x\xi) \cdot f(x) = \gamma(N^{-1})\tilde{f}(X).$$

Thus, $\tilde{f}$ belongs to $\mathcal{F}_0$ and we can identify $\mathcal{F}_0$ with $C_c(\mathbb{R})$ since the correspondence $f \mapsto \tilde{f}$ is a isometry. But on $C_c(\mathbb{R})$ we have the usual inner product $< f, g >= \int f\overline{g}$ and the completion of $C_c(\mathbb{R})$ in the sense of this inner product is $L^2(\mathbb{R})$.

We only need to see what means $L_{X_0}f(y)$ for $X_0 \in H_1$ and $f \in L^2(\mathbb{R})$ since we would like to see this as acting on $\tilde{f}$. We get:

$$L_{X_0}f(y) = \tilde{f}(X_0^{-1}(y, \xi, t)) = \tilde{f}(-x_0 + y, -\xi_0 + \xi, -t_0 + t + \frac{1}{2}(x_0\xi - y\xi_0)) =$$

$$= \gamma(0, \xi_0 - \xi, t_0 - t + y\xi_0 - \frac{1}{2}x_0\xi_0 - \frac{1}{2}y\xi) \cdot f(y - x_0)$$

and all this happens for any ξ and t. We recall that γ is a character of N_1 given by $\gamma(0, \xi, t) = e^{2\pi i(a\xi + bt)}$ with a and b, $b \neq 0$ uniquely determined for γ. Separating the cases $a = 0$ and $a \neq 0$ we can choose convenient ξ and t such that

$$L_{X_0}f(y) = e^{2\pi ib(t_0 + \frac{1}{2}x_0\xi_0 - y\xi_0)}f(y - x_o).$$

Thus, $ind_{N_1}^{H_1}(\gamma\rho)$ depend by a single parameter, b, and we can say that irreducible representations of H_1 in this case are of the form (changing the notation a bit) $\rho_h : H_1 \to \mathcal{U}(L^2(\mathbb{R}))$ where

$$\rho_h(X)f(y) = e^{2\pi ih(t + \frac{1}{2}x\xi - y\xi_0)}f(y - x).$$

If $h_1 \neq 0 \neq h_2$ then the representations ρ_{h_1} and ρ_{h_2} are not equivalent. So, **all** irreducible representations of H_1 (until equivalence) are of the type $\pi_{b,\beta}$ or ρ_h.

Remark 2.1 ***(The dual space)*** For a locally compact group G the dual space consists of all classes of equivalence of it's irreducible representations and it is denoted by $\hat{G}$. In the Abelian case, we've seen that $\hat{G}$ can be organized as a locally compact Abelian group. In the non-Abelian case it is obvious that the structure of $\hat{G}$ as a group is no longer possible. Moreover, on the dual space can be given several topologies but these topologies have few chances to be Hausdorff. For a detailed exposition see [1], pages 201-210.

2.3 Fourier transform

We will use Fourier transform of measures and functions to establish a connection between convolution semigroups on LCA-groups and negative-definite functions. Fourier transform is regarded as a map on the dual space of a group:

Definition 2.4 *Let G be a locally compact group. The Fourier transform of a function $f \in L^1(G)$ is the map $\pi \to \hat{f}(\pi)$, for each irreducible representation π of G on the Hilbert H_π, where*

$$\hat{f}(\pi) : H_\pi \to H_\pi, \quad \hat{f}(\pi)u = \int_G f(x)[\pi(x^{-1})]udx \tag{17}$$

with dx being the left Haar measure on G(for each irreducible representation π the value of $\hat{f}$ at π is an operator on H_π).

Remark 2.2 *1.* As we have seen earlier, all irreducible representations of a LCA-group are 1-dimensional so we can always take as representation space $\mathbb{C}$, the Hilbert space of all complex numbers. Thus, a simpler form for the Fourier transform is to be expected:

$$\hat{f} : \hat{G} \to \mathbb{C}, \quad \hat{f}(\gamma) = \int_G f(x)\overline{\gamma}(x)dx \tag{18}$$

for each character $\gamma \in \hat{G}$.

2. If G is the additive group of real numbers then the Fourier transform takes the more familiar form:

$$\hat{f}(y) = \int_{\mathbb{R}} e^{-2\pi ixy} f(x)dx. \tag{19}$$

In the case of an LCA-group we can extend the Fourier transformation from $L^1(G)$ to $M_b(G)$, also called the *Fourier-Stiltjes transformation*:

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma}(x)d\mu(x) \quad \textit{for } \mu \in M_b(G) \textit{ and } \gamma \in \hat{G}.$$

In this case we have the following properties:

Proposition 2.3 *1. For $\mu \in M_b(G)$ the Fourier transform $\hat{\mu}$ is a uniformly continuous and bounded function on $\hat{G}$ satisfying $\|\hat{\mu}\|_\infty \leq \|\mu\|$.*

2. For $f \in L^1(G)$ the Fourier transform $\hat{f}$ tends to 0 at infinity.

3. The set $\{\hat{f}|f \in L^1(G)\}$ is a dense self-adjoint subalgebra of $C_0(\hat{G})$.

4. The Fourier transform is injective as a map from $M_b(G)$.

5. *(Plancherel) The Fourier transform* $\hat{f}$ *of* $f \in L^1(G) \cap L^2(G)$ *is square integrable with respect to any Haar measure on* $\hat{G}$*. There is a uniquely determined Haar measure* ω *on* $\hat{G}$ *such that*

$$\int_G |f(x)|^2 dx = \int_{\hat{G}} |\hat{f}(\gamma)|^2 d\omega(\gamma)$$

If G is no longer Abelian things are a little more complicated but a inversion formula still exists. For a detailed exposition see [1], pages232-239. Some examples and applications in the non-Abelian case and Abelian case as well will be given in the next section.

3 Convolution semigroups

3.1 Convolutions

Let G be a locally compact group equipped with a (fixed) Haar measure λ. All L^p spaces will be regarded with respect to λ and we will also write dx for $d\lambda$. Denote by $M(G)$ the space of all complex Radon measures on G.

3.1.1 The M(G) algebra

Definition 3.1 *Let $\mu, \nu \in M(G)$. The map*

$$\phi \to I(\phi), \qquad I(\phi) = \int \int \phi(xy) d\mu(x) d\nu(y)$$

is clearly a linear functional on $C_0(G)$ and $|I(\phi)| \leq \|\phi\|_{sup} \|\mu\| \|\nu\|$. Hence, this functional defines a measure which is denoted by $\mu * \nu \in M(G)$ and is called the convolution *of μ and ν:*

$$\int \phi d(\mu * \nu) = \int \int \phi(xy) d\mu(x) d\nu(y).$$

Remark 3.1 *A few things are to worth emphasize here:*

1. Convolution is associative: if $\mu, \nu, \sigma \in M(G)$ and $\phi \in C_c(G)$ then

$$\int \phi d(\mu * (\nu * \sigma)) = \int \int \phi(xy) d\mu(x) d(\nu * \sigma(y)) =$$

$$= \int \int \int \phi(xyz) d\mu(x) d\nu(y) d\sigma(z)$$

$$= \int \int \phi(yz) d(\mu * \nu)(y) d\sigma(z) = \int \phi d((\mu * \nu) * \sigma).$$

2. Convolution is commutative if and only if so is G. This is due to the fact that the point mass at $x \in G$, $\delta_x \in M(G)$ and $\delta_x * \delta_y = \delta_{xy}$.

3. The estimate $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$ makes $M(G)$ a Banach algebra. $M(G)$ has a unit, namely the point mass at 1, δ_1:

$$\int \phi d(\delta_1 * \mu) = \int \int \phi(xy) d\delta_1(x) d\mu(y) = \int \phi(y) d\mu(y)$$

so $\delta_1 * \mu = \mu$ and in the same way we obtain $\mu * \delta_1 = \mu$. $M(G)$ also has a canonical involution $\mu \to \mu^*$ given by

$$\mu^*(E) = \overline{\mu(E^{-1})} \text{ or } \int \phi d\mu^* = \int \phi(x^{-1}) d\overline{\mu}(x).$$

We have:

$$\int \phi d(\mu * \nu)^* = \int \phi(x^{-1}) d\overline{(\mu * \nu)} = \int \int \phi(y^{-1}x^{-1}) d\overline{\mu}(x) d\overline{\nu}(y)$$

$$= \int \int \phi(yx) d\mu^*(x) d\nu^*(y) = \int \phi d(\nu^* * \mu^*)$$

and so $(\mu * \nu)^* = \nu^* * \mu^*$.

3.1.2 The $L^1(G)$ algebra

Identifying a function $f \in L^1(G)$ with the measure $f(x)dx \in M(G)$ we can restrict the convolution to $L^1(G)$. If $f, g \in L^1$, the *convolution* of f and g is the function given by

$$f * g(x) = \int f(y) g(y^{-1}x) dy.$$

The integral defining $f * g$ is absolutely convergent for almost every x and

$$\int \int |f(y) g(y^{-1}x)| dx dy = \int \int |f(y) g(x)| dx dy = \|f\|_1 \|g\|_1$$

so we also have that $\|f * g\| \leq \|f\|_1 \|g\|_1$.

The involution on $M(G)$, restricted to $L^1(G)$, is defined by the relation

$$f^*(x) dx = \overline{f(x^{-1})} d(x^{-1}) = \overline{f(x^{-1})} \Delta(x^{-1}) dx$$

and so

$$f^*(x) = \Delta(x^{-1}) \overline{f(x^{-1})}.$$

Thus, $L^1(G)$ becomes a Banach $*$-algebra without unit, called the L^1 *group algebra of G*.

Remark 3.2 *1.* One needs often to apply the Fourier transform to the convolution of two functions or two measures. When G is Abelian, we get:

$$\widehat{f * g}(\gamma) = \int_G f * g(x) \overline{\gamma}(x) dx = \int_G \int_G f(y) g(y^{-1}x) \overline{\gamma}(x) dy dx =$$

$$= \int_G \left(f(y) \int_G g(y^{-1}x) \overline{\gamma}(x) dx \right) dy = \int_G \left(f(y) \int_G g(x) \overline{\gamma}(yx) dx \right) dy$$

$$= \int_G f(y) \overline{\gamma}(y) dy \int_G g(x) \overline{\gamma}(x) dx = \hat{f}(\gamma) \cdot \hat{g}(\gamma)$$

for every $f, g \in L^1(G)$. In the same manner we obtain $\widehat{\mu * \nu} = \hat{\mu} \cdot \hat{\nu}$ for every $\mu, \nu \in M_b(G)$.

2. When G is non-Abelian and π is a irreducible representation of G we have for each $u \in H_\pi$:

$$\widehat{f * g}(\pi)u = \int_G \int_G f(y)g(y^{-1}x)[\pi(x^{-1})]udxdy =$$
$$= \int_G \left(f(y) \int_G g(x)[\pi(x^{-1}y^{-1})]dx \right) dy$$
$$= \int \left(f(y) \int_G g(x)[\pi^*(x) \circ \pi^*(y)]udx \right) dy$$
$$= \int_G f(y)[\hat{g}(\pi)](\pi^*(y)u)dy = \hat{g}(\pi) \circ \hat{f}(\pi)u$$

and thus $\widehat{f * g} = \hat{g}\hat{f}$ (we used the facts that π is a homomorphism, $\pi(x)$ are unitary operators and $\hat{g}(\pi)$ is a continuous operator to subtract it from the integral).

3.2 Convolution semigroups

We will discuss first the case of an locally compact Abelian group G. Afterwards we will place the same problems in the case of an locally compact non-Abelian group and see what we obtain and what problems needs to be solved.

Definition 3.2 *A family $(\mu_t)_{t>0}$ consisting of positive and bounded measures on a group G with the following properties:*

1. $\mu_t(G) \leq 1 \ \forall \ t > 0$
2. $\mu_t * \mu_s = \mu_{t+s}, \ \forall t, s > 0$
3. $\mu_t \to \varepsilon_0$ *pt* $t \to 0$ *vaguely (in the* $\sigma(M(G), \mathcal{C}_c(G))$ *topology)*

is called a (vaguely) continuous convolution semigroup.

3.2.1 The Abelian case

There is a beautiful connection between vaguely continuous convolution semigroups on a LCA-group G and continuous negative-definite functions on the dual group $\hat{G}$.

Definition 3.3 *A function $\phi : G \to \mathbb{C}$ is* positive definite *if*

$$\forall n \in \mathbb{N}, \ x_1 ... x_n \in G \ c_1 ... c_n \in \mathbb{C} \quad \sum_{i,j=1}^{n} \phi(x_i x_j^{-1}) c_i \overline{c_j} \geq 0.$$

Definition 3.4 *A function $\psi : \hat{G} \to \mathbb{C}$ is negative definite if*

$$\forall n \in \mathbb{N},\ \gamma_1,\ \gamma_2, ..., \gamma_n \in \hat{G}\ c_1,\ c_2, ..., c_n \in \mathbb{C} \quad \sum_{i,j=1}^{n} (\psi(\gamma_i) + \overline{\psi(\gamma_j)} - \psi(\gamma_i \overline{\gamma}_j)) \geq 0$$

We have several simpler characterizations for the negative-definite functions. We only give here the one given by Schönberg:

Theorem 3.1 *A function $\psi : \hat{G} \to \mathbb{C}$ is negative-definite if and only if the following two conditions hold:*

1. $\psi(0) \geq 0$
2. *the function $\gamma \mapsto exp(-t\psi(\gamma))$ is positive-definite for all $t \geq 0$.*

The next theorem allows us to study properties of convolution semigroups by means of negative-definite functions:

Theorem 3.2 *There is a one-to-one correspondence between continuous negative-definite functions and vaguely continuous convolution semigroups on a locally compact Abelian group G. More specific, if $(\mu_t)_{t>0}$ is a vaguely continuous convolution semigroup on G there exists a uniquely determined continuous negative-definite function ψ on Γ such that:*

$$\hat{\mu}_t(\gamma) = exp(-t\psi(\gamma))\ \forall \gamma \in \Gamma\ t > 0$$

Conversely, if ψ is a continuous negative-definite function, the above formula defines a vaguely continuous convolution semigroup on G.

Example 3.1 The Brownian semigroup on $\mathbb{R}^n$ is given by:

$$\mu_t = (4\pi t)^{-n/2} e^{-\frac{\|x\|^2}{4t}} dx$$

where dx is the Haar (Lebesgue) measure on $\mathbb{R}^n$. The associated continuous negative-definite function is

$$\psi : \mathbb{R}^n \to \mathbb{C}, \quad \psi(y) = \|y\|^2$$

since

$$\hat{\mu}(y) = e^{-t\|y\|^2}, \quad \forall t > 0,\ y \in \mathbb{R}^n.$$

3.2.2 The non-Abelian case

Let G be a locally compact non-Abelian, non-compact group. We may define as well as in the Abelian case a *Fourier-Stiltjes* transform for a bounded measure $\mu \in M_b(G)$ as an operator-valued function on the dual space:

Definition 3.5 *If π is a irreducible representation of G on H_π the Fourier transform of μ at π is the operator*

$$\hat{\mu}(\pi) : H_\pi \to H_\pi, \qquad [\hat{\mu}(\pi)]u = \int_G \pi(x^{-1})ud\mu(x) = \int_G \pi^*(x)ud\mu(x) \quad \forall u \in H_\pi.$$

We obtain immediately:

Proposition 3.1 *Let G be a locally compact group and π a representation of G on H_π. The application*

$$T_\pi : M_b(G) \to \mathcal{L}(H_\pi),\ T_\pi(\mu) = \hat{\mu}(\pi)$$

is a continuous linear operator of norm 1.

Proof It's clear that for each $\mu \in M_b(G)$ and for each irreducible representation π the operator $\hat{\mu}(\pi)$ is bounded and

$$\|T_\pi(\mu)u\| \leq \int_G \|u\| d|\mu| = \|u\| \cdot \|\mu\|.$$

This shows that $\|T_\pi(\mu)\| \leq \|\mu\|$ and thus $\|T_\pi\| \leq 1$. But the measure $\varepsilon_0 \in M_b(G)$, is of norm 1 and

$$T_\pi(\varepsilon_0) = \hat{\varepsilon}_0(\pi) = I.$$

Thus, $\|T_\pi\| \geq 1$ and hence it equals 1. Q.E.D.

Remark 3.3 If $\mu, \nu \in M_b(G)$ then

$$[\widehat{\mu * \nu}(\pi)]u = \int_G \pi^*(x)ud(\mu * \nu)(x) = \int_G \int_G \pi^*(xy)ud\mu(x)d\nu(y) =$$

$$= \int_G \pi(y^{-1}) \left(\int_G \pi(x^{-1})ud\mu(x) \right) d\nu(y) =$$

$$= \hat{\nu}(\pi)[\hat{\mu}(\pi)u] = [\hat{\nu}(\pi) \circ \hat{\mu}(\pi)]u$$

and thus $\widehat{\mu * \nu}(\pi) = \hat{\nu}(\pi)\hat{\mu}(\pi)$.

This means, in particular that if $(\mu_t)_{t>0}$ is a convolution semigroup then all the operators $\hat{\mu}_t(\pi)$ that we obtain for a irreducible representation π on the Hilbert space H_π must commute with one-another, since

$$\hat{\mu}_t(\pi)\hat{\mu}_s(\pi) = \widehat{\mu_t * \mu_s}(\pi) = \hat{\mu}_{t+s}(\pi) = \hat{\mu}_{s+t}(\pi) = \hat{\mu}_s(\pi)\hat{\mu}_t(\pi).$$

The problems I want to solve here are as follows: what "qualities" have the operators $\hat{\mu}(\pi)$ and how can these properties give back the initial measure. This way, if $(\mu_t)_{t>0}$ is a convolution semigroup on G we would be able to reconstruct it from a certain semigroup of operators.

3.2.3 "The Heat kernel" on Heisenberg group

Before studying the properties of these operators in the general case I took the particular case of the Heisenberg group, H_n, and the convolution semigroup associated with the Laplacean Δ_H on H_n as it appears in [3] and [4].

The operator Δ_H has the form:

$$\Delta_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \xi^2} + \xi \frac{\partial^2}{\partial x \partial t} - x \frac{\partial^2}{\partial \xi \partial t} + \frac{1}{4}(x^2 + \xi^2)\frac{\partial^2}{\partial t^2} \; for n = 1$$

and, of course,

$$\Delta_H = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial \xi_i^2} + \xi_i \frac{\partial^2}{\partial x_i \partial t} - x_i \frac{\partial^2}{\partial \xi_i \partial t} + \frac{1}{4}(x_i^2 + \xi_i^2)\frac{\partial^2}{\partial t^2} \right).$$

The form of Laplacean comes in a natural way. Take the case $n = 1$ for example. The partial derivative of a specific function f at the variable x should have the form:

$$f_x(x, \xi, t) = \lim_{h \to 0} \frac{f\left((x, \xi, t)(h, 0, 0)\right) - f(x, \xi, t)}{h}$$

We obtain:

$$f_x(x, \xi, t) = \lim_{h \to 0} \frac{f(x + h, \xi, t + \frac{1}{2}h\xi) - f(x, \xi, t)}{h} = \frac{\partial f}{\partial x}(x, \xi, t) + \frac{1}{2}\xi \frac{\partial f}{\partial t}(x, \xi, t).$$

In the same way we get

$$f_\xi(x, \xi, t) = \frac{\partial f}{\partial \xi}(x, \xi, t) - \frac{1}{2}x \frac{\partial f}{\partial t}(x, \xi, t).$$

With respect to the classic formula of the Laplacean: $\Delta_H f = f_{xx} + f_{\xi\xi}$ we obtain the formula stated above.

By [3] and [4] the "heat" operator $exp(s\Delta_H)$ is a convolution operator on H_n and it's convolution kernel admits integral representation which have the form:

$$K_s(x,\xi,t) = \frac{1}{(4\pi s)^{n+1}} \int_{-\infty}^{+\infty} exp(-f(x,\xi,t,\tau)/2s)V(\tau)d\tau$$

where

$$f(x,\xi,t,\tau) = -it\tau + (\|x\|^2 + \|\xi\|^2)\frac{\tau}{4}ctgh\frac{\tau}{2}$$

and

$$V(\tau) = \left(\frac{\tau/2}{sh\frac{\tau}{2}}\right)^n.$$

We will take the case $n = 1$, calculus being the analogous for $n > 1$.

We have already proved that there are two types of irreducible representations for this group:

1. 1-dimensional :

$$\pi_{b,\beta} : H_1 \longrightarrow \mathbb{T},\ \pi_{b,\beta}(x,\xi,t) = e^{2\pi i(bx+\beta\xi)}$$

   with $b, \beta \in \mathbb{R}$

2. ∞-dimensional:

$$\rho_h : G \longrightarrow \mathcal{U}(L^2(R)),\ [\rho_h(x,\xi,t)f](y) = e^{2\pi i h(t+\frac{1}{2}x\xi - y\xi)}f(y-x)$$

   where h is non-zero real number.

Thus, the Fourier transform will have 2 forms, corresponding to each type of representation. For the representations $\pi_{b,\beta}$ we have:

$$\hat{f}(\pi_{b,\beta}) = \int_{H_1} \pi_{b,\beta}(-x,-\xi,-t)f(x,\xi,t)dxd\xi dt =$$

$$= \int_{H_1} e^{-2\pi i(bx+\beta\xi)}f(x,\xi,t)dxd\xi dt$$

meaning that we actually have two successive Fourier transforms with respect to the first and second variable, respectively.

For the second form, the values of the Fourier transform is an operator on $L^2(\mathbb{R})$:

$$\hat{f}(\rho_h) = \int_{H_1} f(x,\xi,t)\rho_h(-x,-\xi,-t)dxd\xi dt$$

where

$$\hat{f}(\rho_h) : L^2(\mathbb{R}) \to \text{Ł}^2(\mathbb{R}),$$

$$[\hat{f}(\rho_h)\phi](y) = \int_{H_1} f(x,\xi,t)[\rho_h(-x,-\xi,-t)\phi](y)dxd\xi dt =$$

$$= \int_{H_1} f(x,\xi,t)e^{2\pi ih(-t+\frac{1}{2}x\xi+y\xi)}\phi(y+x)dxd\xi dt$$

Calculating the Fourier transform of the functions K_s above we get:

$$\hat{K}_s(\pi_{b,\beta}) = \frac{1}{(4\pi s)^2}\int_{H_1}\int_{\mathbb{R}} e^{\frac{it\tau}{2s}} \cdot e^{-(x^2+\xi^2)\frac{\tau}{8s}ctgh\frac{\tau}{2}} \cdot \frac{\tau}{2sh\frac{\tau}{2}} \cdot e^{-2\pi i(bx+\beta\xi)}d\tau dxd\xi dt$$

$$= \frac{1}{(4\pi s)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{it\tau}{2s}}\frac{\tau}{2sh\frac{\tau}{2}} \cdot \int_{\mathbb{R}} e^{-2\pi ibx}e^{-x^2\frac{\tau}{8s}ctgh\frac{\tau}{2}}dx \int_{\mathbb{R}} e^{-2\pi i\beta\xi}e^{-\xi^2\frac{\tau}{8s}ctgh\frac{\tau}{2}}d\xi d\tau dt$$

$$= \frac{1}{(4\pi s)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{it\tau}{2s}}\frac{\tau}{2sh\frac{\tau}{2}} \cdot \frac{8\pi s}{\tau ctgh\frac{\tau}{2}} \cdot e^{-\frac{8\pi^2 s(b^2+\beta^2)}{\tau ctgh\frac{\tau}{2}}}d\tau dt$$

$$= \frac{1}{4\pi s}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{it\tau}{2s}} \cdot \frac{1}{ch\frac{\tau}{2}} \cdot e^{-\frac{8\pi^2 s(b^2+\beta^2)}{\tau ctgh\frac{\tau}{2}}}d\tau dt = \frac{1}{4\pi s}\int_{\mathbb{R}} \mathcal{F}f(-\frac{t}{2s})dt = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}f(t)dt$$

$$= \frac{1}{2\pi}\mathcal{F}\mathcal{F}f(0) = f(0) = e^{-4\pi^2 s(b^2+\beta^2)}$$

where

$$f(\tau) = \frac{1}{ch\frac{\tau}{2}} \cdot e^{-\frac{8\pi^2 s(b^2+\beta^2)}{\tau ctgh\frac{\tau}{2}}}$$

which is interesting because this would suggest that the "negative-definite function" we are looking for is

$$\pi_{b,\beta} \mapsto 4\pi^2(b^2+\beta^2).$$

For the ρ_h representations we use the fact that we obviously can write

$$K_s(x,\xi,t) = \mathcal{F}g(\frac{-t}{2s})$$

where

$$g(\tau) = \frac{1}{(4\pi s)^2}V(\tau)e^{-(x^2+\xi^2)\frac{\tau}{8s}ctgh\frac{\tau}{2}}.$$

We have then:

$$[\hat{K}_s(\rho_h)\phi](y) = \int_{H_1} K_s(x,\xi,t)[\rho_h(-x,-\xi,-t)\phi](y)dxd\xi dt$$

$$= \int_{H_1} \mathcal{F}g(\frac{-t}{2s}) \cdot e^{-2\pi iht+\pi ihx\xi+2\pi ihy\xi}\phi(y+x)dxd\xi dt$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(y+x) e^{i\xi(\pi hx+2\pi hy)} \int_{\mathbb{R}} e^{-2\pi iht} \mathcal{F}g(\frac{-t}{2s}) dt dx d\xi$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(y+x) e^{i\xi(\pi hx+2\pi hy)} \cdot 4\pi s g(4\pi hs) d\xi dx$$

$$= \frac{1}{4\pi s}\int_{\mathbb{R}} \phi(y+x)\cdot e^{-x^2\frac{\pi h}{2}ctgh(2\pi hs)} \cdot \int_{\mathbb{R}} e^{i\xi\pi h(x+2y)} \frac{2\pi hs}{sh(2\pi hs)} e^{-\xi^2\frac{\pi h}{2}ctgh(2\pi hs)} d\xi dx$$

$$= \frac{h}{2sh(2\pi hs)} \cdot \sqrt{\frac{2}{h\cdot ctgh 2\pi hs}} \int_{\mathbb{R}} \phi(y+x)\cdot e^{-x^2\frac{\pi h}{2}ctgh(2\pi hs)\cdot e^{-\frac{\pi^2 h^2 (x+2y)^2}{2\pi h\cdot ctgh(2\pi hs)}}} dx$$

Denoting $a = ctgh(2\pi hs)$ and changing the variable $x+y \mapsto x$ we get:

$$[\hat{K}_s(\rho_h)\phi](y) = \sqrt{\frac{h}{sh(4\pi hs)}} \cdot \int_{\mathbb{R}} \phi(x) e^{-\frac{\pi h}{2}a(x-y)^2} \cdot e^{-\frac{\pi h}{2}\frac{1}{a}(x+y)^2} dx$$

This result does not indicate a substitute for the notion of "negative-definite function", as we obtained earlier.

For the general case the Fourier transform has similar forms:

$$\hat{K}_s(\pi_{b,\beta}) = e^{-4\pi^2(\|b\|^2+\|\beta\|^2)}$$

and

$$[\hat{K}_s(\rho_h)\phi](y) = \left(\frac{h}{sh(4\pi hs)}\right)^{\frac{n}{2}} \cdot \int_{\mathbb{R}^n} \phi(x) e^{-\frac{\pi h}{2}(a\|x-y\|^2+\frac{1}{a}\|x+y\|^2)} dx$$

If for $\hat{K}_s(\pi_{b,\beta})$ things are clear, being similar with the Fourier transform of the Brownian semigroup in the commutative case, we will look for properties of the operators $\hat{K}_s(\rho_h)$.

Proposition 3.2 *For each $h \neq 0$ the operators $\hat{K}_s(\rho_h)$ have the following properties:*

1. $\hat{K}_{s_1}(\rho_h)\hat{K}_{s_2}(\rho_h) = \hat{K}_{s_1+s_2}(\rho_h)$*;*
2. $\hat{K}_s(\rho_h)$ *is self-adjoint* $\forall s > 0$*;*
3. $\hat{K}_s(\rho_h)$ *is positive and* $< \hat{K}_s(\rho_h)\phi, \phi >= \|\hat{K}_{\frac{s}{2}}(\rho_h)\phi\|^2$.

Proof **1**. Although a more direct proof can be given, via remark 3.3, I will give here a complete calculus. Let $\phi \in L^2(\mathbb{R})$. Denoting by $a_1 = ctgh(2\pi hs_1)$ and $a_2 = ctgh(2\pi hs_2)$ we have:

$$\hat{K}_{s_1}[\hat{K}_{s_2}(\rho_h)\phi](y) = \sqrt{\frac{h}{sh(4\pi hs_1)}} \int_{\mathbb{R}} [\hat{K}_{s_2}(\rho_h)\phi](z) e^{-\frac{\pi h}{2}a_1(z-y)^2} \cdot e^{-\frac{\pi h}{2}\frac{1}{a_1}(z+y)^2} dz$$

$$= \frac{h}{\sqrt{sh(4\pi hs_1)sh(4\pi hs_2)}} \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(x) e^{-\frac{\pi h}{2}(a_1(z-y)^2+\frac{1}{a_1}(z+y)^2+a_2(x-z)^2+\frac{1}{a_2}(x+z)^2)} dxdz$$

$$= C\int_{\mathbb{R}} \phi(x)\cdot e^{\frac{-\pi h}{2}(x^2(a_2+\frac{1}{a_2})+y^2(a_1+\frac{1}{a_1}))}\cdot \int_{\mathbb{R}} e^{-\frac{\pi h}{2}(z^2A-2zB)}dzdx$$

where

$$A = a_1 + \frac{1}{a_1} + a_2 + \frac{1}{a_2} \ ,$$

$$B = y(a_1 - \frac{1}{a_1}) + x(a_2 - \frac{1}{a_2}) \ ,$$

$$C = \frac{h}{\sqrt{sh(4\pi hs_1)sh(4\pi hs_2)}}.$$

Thus,

$$\hat{K}_{s_1}[\hat{K}_{s_2}(\rho_h)\phi](y) = C\int_{\mathbb{R}} \phi(x)\cdot e^{\frac{-\pi h}{2}(x^2(a_2+\frac{1}{a_2})+y^2(a_1+\frac{1}{a_1}))}\cdot f(x,y)dx$$

where

$$f(x,y) = \int_{\mathbb{R}} e^{-\frac{\pi h}{2}(z^2A-2zB)}dz = \int_{\mathbb{R}} e^{\frac{-\pi h}{2}(z\sqrt{A}-\frac{B}{\sqrt{A}})^2}\cdot e^{\frac{\pi h}{2}\cdot\frac{B^2}{A}}dz$$

$$f(x,y) = \sqrt{\frac{2}{hA}}\cdot e^{\frac{\pi h}{2}\cdot\frac{B^2}{A}}$$

But

$$C\cdot\sqrt{\frac{2}{hA}} = \sqrt{\frac{2h}{sh(4\pi hs_1)sh(4\pi hs_2)}}\cdot\frac{1}{\sqrt{\frac{ch(2\pi hs_1)}{sh(2\pi hs_1)}+\frac{sh(2\pi hs_1)}{ch(2\pi hs_1)}+\frac{ch(2\pi hs_2)}{sh(2\pi hs_2)}+\frac{sh(2\pi hs_2)}{ch(2\pi hs_2}}}$$

and because

$$sh^2(x)+ch^2(x) = ch(2x),\ ch^2(x)-sh^2(x) = 1,$$

$$ch(x+y) = chx\cdot chy + shx\cdot shy,\ sh(x+y) = shx\cdot chy + shy\cdot chx$$

we get:

$$C\cdot\sqrt{\frac{2}{hA}} = \sqrt{\frac{h}{sh(4\pi h(s_1+s_2))}}$$

It remains to prove that:

$$x^2(a_2+\frac{1}{a_2})+y^2(a_1+\frac{1}{a_1})-\frac{B^2}{A} = a(x-y)^2+\frac{1}{a}(x+y)^2$$

where $a = ctgh(2\pi h(s_1+s_2))$. For this we will identify the coefficients so we need to show that:

$$for\ x^2: \quad a_2 + \frac{1}{a_2} - \frac{(a_2-\frac{1}{a_2})^2}{a_1+\frac{1}{a_1}+a_2+\frac{1}{a_2}} = a + \frac{1}{a}$$

$$for\ y^2: \quad a_1 + \frac{1}{a_1} - \frac{(a_1-\frac{1}{a_1})^2}{a_1+\frac{1}{a_1}+a_2+\frac{1}{a_2}} = a + \frac{1}{a}$$

$$and\ for\ xy: \quad \frac{(a_1-\frac{1}{a_1})(a_2-\frac{1}{a_2})}{a_1+\frac{1}{a_1}+a_2+\frac{1}{a_2}} = a - \frac{1}{a},$$

things that are easy to verify. Thus we have:

$$\hat{K}_{s_1}[\hat{K}_{s_2}(\rho_h)\phi](y) = \sqrt{\frac{h}{sh(4\pi h(s_1+s_2))}} \int_{\mathbb{R}} \phi(x) e^{-\frac{\pi h}{2}a(x-y)^2} \cdot e^{-\frac{\pi h}{2}\cdot\frac{1}{a}(x+y)^2} dx$$

$= [\hat{K}_{s_1+s_2}\phi](y)$.

ii) Let now ϕ_1 and $\phi_2 \in L^2(\mathbb{R})$. We have:

$$< \hat{K}_s(\rho_h)\phi_1, \phi_2 >=$$

$$= \sqrt{\frac{h}{sh(4\pi hs)}} \cdot \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_1(x) e^{-\frac{\pi h}{2}a(x-y)^2} \cdot e^{-\frac{\pi h}{2}\frac{1}{a}(x+y)^2} dx \right) \phi_2(y) dy$$

$$= \sqrt{\frac{h}{sh(4\pi hs)}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \phi_2(y) e^{-\frac{\pi h}{2}a(y-x)^2} \cdot e^{-\frac{\pi h}{2}\frac{1}{a}(y+x)^2} dy \right) \phi_1(x) dx$$

$$=< \phi_1, \hat{K}_s(\rho_h)\phi_2 >$$

which means that $\hat{K}_s(\rho_h)$ is self-adjoint.

iii) Indeed, we have:

$$< \hat{K}_s(\rho_h)\phi, \phi >=< \hat{K}_{\frac{s}{2}}(\rho_h) \circ \hat{K}_{\frac{s}{2}}(\rho_h)\phi, phi >=$$

$$=< \hat{K}_{\frac{s}{2}}(\rho_h)\phi, \hat{K}_{\frac{s}{2}}(\rho_h)\phi >= \|\hat{K}_{\frac{s}{2}}(\rho_h)\phi\|^2 \geq 0.$$

Q.E.D.

Proposition 3.3 *For each $h \neq 0$ the family $(\hat{K}_s(\rho_h))_{s>0}$ is a C_0 compact semigroup of contractions on $L^2(\mathbb{R}^n)$.*

Proof We know that the operators $\hat{K}_s(\rho_h)$ are continuous. Moreover, by their form these operators are all Hilbert-Schmidt operators and hence compact.

Since $(K_s d\lambda)_{s>0}$ where λ is the Lebesgue measure on H_1, is a convolution semigroup, it follows that $K_s d\lambda$ tend to ε_0 vaguely and then $\hat{K}_s(\rho_h)\phi(y) \to \phi(y)$ as $s \to 0$ for any $\phi \in L^2(\mathbb{R})$. Q.E.D.

Remark 3.4 The last relation of the proof of prop 3.2 shows that

$$\|\hat{K}_s(\rho_h)\| = \|\hat{K}_{\frac{s}{2}}(\rho_h)\|^2, \quad \forall s > 0$$

and thus

$$\|\hat{K}_s(\rho_h)\| = \|\hat{K}_{\frac{s}{2^n}}(\rho_h)\|^{2^n}, \quad \forall s > 0, \ n \in \mathbb{N}.$$

Estimating the norm of $\hat{K}_s(\rho_h)$ we get:

$$|\hat{K}_s(\rho_h)\phi(y)| \leq \sqrt{\frac{h}{sh(4\pi hs)}} \int_{\mathbb{R}} |\phi(x)| \cdot e^{-\frac{\pi h}{2}a(x-y)^2} \cdot e^{-\frac{\pi h}{2}\frac{1}{a}(x+y)^2} dx$$

$$\leq \sqrt{\frac{h}{sh(4\pi hs)}} \|\phi\|_{L^2(\mathbb{R})} \cdot \left(\int_{\mathbb{R}} e^{-\pi ha(x-y)^2} \cdot e^{-\pi h\frac{1}{a}(x+y)^2} dx \right)^{\frac{1}{2}}$$

Thus,

$$\|\hat{K}_s(\rho_h)\phi\|^2_{L^2(\mathbb{R})} \leq \frac{h}{sh(4\pi hs)} \|\phi\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi ha(x-y)^2} \cdot e^{-\pi h\frac{1}{a}(x+y)^2} dxdy$$

$$\leq \frac{h}{2sh(2\pi hs)} \cdot \sqrt{\frac{1}{ha}} \cdot \sqrt{\frac{a}{h}},$$

so the operators $\hat{K}_s(\rho_h)$ are continuous and

$$\|\hat{K}_s(\rho_h)\| \leq \frac{1}{\sqrt{2sh(4\pi hs)}}.$$

which show that

$$\lim_{s\to\infty} \|\hat{K}_s(\rho_h)\| = 0.$$

But, for a fixed $s > 0$ we have

$$\|\hat{K}_{2^n s}(\rho_h)\| = \|\hat{K}_s(\rho_h)\|^{2^n}, \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ we get, necessarily,

$$\|\hat{K}_s(\rho_h)\| < 1, \quad \forall s > 0$$

and because

$$\|\hat{K}_{s_1}(\rho_h)\hat{K}_{s_2}(\rho_h)\| \leq \|\hat{K}_{s_1}(\rho_h)\| \cdot \|\hat{K}_{s_2}(\rho_h)\|$$

it follows that $s \mapsto \|\hat{K}_s(\rho_h)\|$ is decreasing.

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