



FACULTY OF MATHEMATICS
"ALEXANDRU IOAN CUZA" UNIVERSITY

**PREPRINT SERIES OF
"ALEXANDRU MYLLER"
MATHEMATICAL SEMINARY**

**THE INFINITESIMAL GENERATOR OF A SEMIDYNAMICAL
SYSTEM**

Eugen Popa

Nr. 02 – 2006

$\int M$ ALEXANDRU MYLLER

Bld. Carol I, 11, 700506 – Iași, ROMÂNIA
<http://www.math.uaic.ro/~sm>

The Infinitesimal generator of a semidynamical system

by Eugen Popa

University Al. I. Cuza Iași, Romania

epopa@uaic.ro

1 Abstract.

The purpose of this paper is to analyze various possibilities for the definition of the infinitesimal generator of a semidynamical system (see [15] and also [4]). A duality framework produces the weak topology, in which one take the limit

$$Ax := \lim_{t \rightarrow 0} \frac{P_t x - x}{t}$$

However, in the case of measures, some weaker topologies seem more appropriate. For example, such a definition allows to consider the generator acting on Dirac measures, in such a way that the power series development holds:

$$P_t f(x) = \sum_{n=0}^{\infty} \frac{A^n \varepsilon_x(f)}{n!} t^n$$

2 Introduction.

Let us recall that, in the definition of the infinitesimal generator of a semi-group of operators on a Banach spaces, one considers the norm limit. Thus, the (C_0) condition is necessary for a good definition of the generator. Preserving the (C_0) property when taking the adjoint imposed the modification

to the so called sun-dual [10]. As the (C_0) requirement is equivalent with the weak form, one can use also the weak limit in the definition of the infinitesimal generator; even so, the domain of the generator may not contain points x , for which the limit

$$\lim_{t \rightarrow 0} x^* \left(\frac{P_t x - x}{t} \right)$$

exists only for *some* functionals x^* .

For semigroups of kernels (on measurable spaces) one can interpret the pointwise limit

$$\lim_{t \rightarrow 0} \frac{P_t f(x) - f(x)}{t}$$

as a weak limit, on atomic measures. In the case of resolvents of kernels on measurable spaces, a thorough study was realized by G. Mokobodzki ([7][XII, 67]). The main result shows the existence (almost everywhere) of that limit, for excessive functions.

Let us cite also the paper of C. Dellacherie [5], where the potential theory is build from an axiomatic notion of generator (not related á priori with a semigroup or a resolvent).

In what follows, we present another possibility, suggested by some examples, namely that the infinitesimal generator A^*y be defined on a smaller domain. Explicitly, starting with the translation semigroup on \mathbb{R} and μ a measure, then $A^*\mu$ is no more considered as a measure, but rather as a distribution. In the case of the brownian semigroup, the infinitesimal generator transform measures into functionals on analytic functions. Moreover, we encounter here the "finite life time" of the semigroup, since the convergence radius of the image function may be $< +\infty$. The main interest of such considerations is in the fact that it allows to recover the usual formula of the exponential for the semigroup:

$$P_t f(x) = \sum_{n=0}^{\infty} \frac{A^n \varepsilon_x(f)}{n!} t^n$$

3 Preliminaries.

I have to recall some abstract considerations from [4], where the potential theory associated with a kernel or with a resolvent family of kernels was developed. Let X be an ordered convex cone of positive elements (but **not** with its own order, see for example the cone of (positive) increasing functions: the natural order is the pointwise one, the own order is: the difference is also

an increasing function). We require that for each increasing and dominated sequence (x_n) from X , there exists $\bigvee x_n$ in X . Thus, we write $x_n \nearrow x$ when (x_n) is an increasing (and dominated) sequence and $x = \bigvee x_n$. Let us stress that such a cone is not necessarily the positive cone of some ordered vector space.

Main examples of such cones: let (E, \mathcal{E}) be a measurable space. Then $X = \mathcal{F}$ the cone of positive, numerical, measurable functions; or \mathcal{M} the cone of positive measures. These cones are in a canonical duality.

We say that two ordered convex cones X and Y are in duality through: $[\cdot, \cdot] : X \times Y \rightarrow [0, +\infty]$ if the following properties hold:

1. $[x + x', y] = [x, y] + [x', y]$; $[x, y + y'] = [x, y] + [x, y']$
2. $x \leq x' \implies [x, y] \leq [x', y]$; $y \leq y' \implies [x, y] \leq [x, y']$
3. $x_n \nearrow x \implies [x_n, y] \nearrow [x, y]$; $y_n \nearrow y \implies [x, y_n] \nearrow [x, y]$

One defines a kernel (on X) as a map $V : X \rightarrow X$ with the properties:

$$V(x + y) = Vx + Vy$$

$$x \leq y \implies Vx \leq Vy$$

$$x_n \nearrow x \implies V(x_n) \nearrow Vx$$

This definition is inspired from that of a morphism of a H -cone (see [11] and [2]). Replacing the continuity on increasing sequences with some weak continuity results in dramatic changes.

Some points of interest to consider this framework are:

excessive elements can be defined: $Vx \leq x$; the set of excessive elements is a potential cone (see [14]).

allows a good duality theory (the elements of the dual cone being the kernels $\mu : X \rightarrow [0, +\infty]$).

The kernels on \mathcal{F} are usual in probability and (axiomatic) potential theory, [7] and [2]. Each kernel V on \mathcal{F} induces canonically (as adjoint) a kernel on \mathcal{M} :

$$V^* \mu(f) := \mu[Vf]$$

but not conversely; one may try

$$\tilde{V}f(x) := (V\varepsilon_x)(f)$$

but these are not inverse correspondences. Let us recall the following example [4]: let μ be a fixed positive measure; then V_μ associates with each (positive) measure ν its absolutely continuous part with respect to μ . Thus, if μ is the Lebesgue measure, then $V_\mu \varepsilon_x = 0$.

While [4] consider the potential theory associated with one kernel or with a resolvent family of kernels, it is natural to study also the abstract semigroups (or semidynamical systems):

$$\Phi : [0, +\infty) \times X \longrightarrow X$$

with each $\Phi(t, \cdot)$ kernel and the semigroup property

$$\Phi(t, \Phi(s, x)) = \Phi(s + t, x)$$

([15] and [16]).

For such a general semigroup, no C_0 -type requirement is imposed, but some measurability or even weak right continuity on Φ should at some point be imposed.

For the very general case when X is a set see [8]. The most studied case is that X a topological space (even locally compact) (see [3]). The case when X is a measurable space was considered by Bucur and Bezzarga [1]; the remarkable fact is that a potential theory can be developed in this framework.

The case of an ordered convex cone has several advantages, from the potential theory view point. Thus, a good duality theory is available; moreover, we are not restricted to the whole cone of measurable functions or positive measures: other remarkable sub-cones may be considered, as the necessity arises.

The fact that we consider semidynamical systems in duality may also bring some advantages for the definition of the generator.

Let X and Y be in duality. Then $X \times Y$ is in a canonical duality with itself:

$$[(x, y), (x', y')] := [x, y'] + [x', y]$$

Moreover, if Φ is a semidynamical system on X and Ψ on Y , then the product $\Phi \otimes \Psi$ is defined on $X \times Y$ as $(\Phi \otimes \Psi)(t, (x, y)) := (\Phi(t, x), \Psi(t, y))$. This system is autodual, in the following sense:

$$\begin{aligned} [(\Phi \otimes \Psi)(t, (x, y)), (x', y')] &= [(\Phi(t, x), \Psi(t, y)), (x', y')] = \\ &= [\Phi(t, x), y'] + [x', \Psi(t, y)] = [(x, y), (\Phi \otimes \Psi)(t, (x', y'))] \end{aligned}$$

In the case of semidynamical systems Φ and Ψ in duality on X and Y , one can, of course, define a generator for each one. More generally, Ax may be considered as a linear functional $Ax : Y \rightarrow \mathbb{R}$ as

$$(Ax)(y) := \lim_{t \rightarrow 0} \frac{[\Phi(t, x), y] - [x, y]}{t}$$

However, Y^* may be very complicated (see the case of measures).

4 The case of measures.

Let us return to a semigroup $(P_t)_{t>0}$ of kernels acting on measures $P_t : \mathcal{M} \rightarrow \mathcal{M}$. When we try to define the infinitesimal generator

$$A\mu := \lim_{t \rightarrow 0} \frac{P_t\mu - \mu}{t}$$

interesting problems appear:

in what sense take the limit?

how to interpret the resulting functional $A\mu$, and consequently the operator A ?

In order to choose a "good" definition for the infinitesimal generator, let us take a look at some examples.

4.1 Examples.

(i) **Translation semigroup.** Let me start with the development of a (real) analytic function:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

After a simple change of notations, one obtains:

$$f(x+t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} t^n$$

which can be interpreted:

$$P_t f(x) = \sum_{n=0}^{\infty} \frac{A^n f(x)}{n!} t^n$$

where $(P_t)_{t>0}$ is the translation semigroup (on \mathbb{R}) having the (right) derivative $Af = f'_d$ as infinitesimal generator.

The translations semigroup has the interesting property to transform polynomial functions into polynomial functions, not only in the variable x , but also in the variable t and the degree is preserved, both in x and in t . Indeed, if $f(x) := a_0 + a_1x + \dots + a_nx^n$, then

$$P_t f(x) = \sum_{k=0}^n a_k \sum_{m=0}^k C_k^m x^m t^{k-m}$$

which can be written as

$$P_t f(x) = \sum_{m=0}^n \left(\sum_{k=m}^n a_k C_k^m t^{k-m} \right) x^m$$

and also as:

$$P_t f(x) = \sum_{m=0}^n \left(\sum_{k=m}^n a_k C_k^m x^{k-m} \right) t^m$$

If m denotes the Lebesgue measure and $\mu = f.m$ is an absolutely continuous measure with respect to m , then $A\mu = f'.m$, as soon as the derivative of the density exists in a suitable sense.

However, for $\mu = \varepsilon_x$ the limit

$$\lim_{t \rightarrow 0} \frac{\varepsilon_{x+t} - \varepsilon_x}{t}$$

does not make much sense. At best, $A\varepsilon_x$ may be considered as some "distribution" (i.e. a functional acting on derivable functions).

Moreover, if we want to consider A^n , then indefinite derivable functions should be considered.

Finally, if we want to recover the semigroup from its infinitesimal generator (Hille–Yosida–type result, see [15]), we look for a formula:

$$\sum_{n=0}^{\infty} \frac{A^{*n} \varepsilon_x}{n!} t^n = P_t \varepsilon_x$$

and this is exactly the starting point.

Hence in this case it seems useful to consider A as acting on (linear) functionals on the space of analytic functions. However, the (next) example of the brownian semigroup shows that probably only a subspace (containing the polynomials) have to be considered.

(ii) Brownian semigroup. This is not a coincidence; as one may guess, the sum:

$$\sum_{n=0}^{\infty} \frac{f^{(2n)}(x)}{n!} t^n = P_t f(x) := \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\frac{(y-x)^2}{4t}} dy$$

should be connected with the brownian semigroup, with the infinitesimal generator $Af(x) = f''(x)$.

Remark. The series has a meaning for $|t| < R$, while the semigroup is defined for $t > 0$! While the series depends only on the local behavior of

the function f (around the point x), the value of $P_t f(x)$ depends on all the values of f .

Also, the series can be considered for any indefinitely derivable function; however, this series may be divergent everywhere (except $t = 0$); or may have a sum different from $P_t f(x)$.

Let us suppose that f is analytic in \mathbb{C} . Since

$$f(x+t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} t^k$$

has infinite radius of convergence, we get that

$$a_k := \sqrt[k]{\frac{|f^{(k)}(x)|}{k!}} \longrightarrow 0$$

Now, writing:

$$\sqrt[k]{\frac{|f^{(2k)}(x)|}{k!}} = a_{2k}^2 \sqrt[k]{\frac{(2k)!}{k!}}$$

we obtain the radius of convergence for the associated series:

$$R = \frac{e}{2 \limsup_{k \rightarrow \infty} (k a_{2k}^2)}$$

Hence, if $a_k = o\left(\frac{1}{\sqrt{k}}\right)$, then the series has also infinite radius of convergence.

For $f(x) = x^n$, $n \in \mathbb{N}$:

$$P_t f(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} (x+y)^n e^{-\frac{y^2}{4t}} dy = \sum_{k=0}^n \frac{1}{\sqrt{4\pi t}} C_n^k x^{n-k} \int_{\mathbb{R}} y^k e^{-\frac{y^2}{4t}} dy$$

Now:

$$\begin{aligned} \int_{\mathbb{R}} y^{2m} e^{-\frac{y^2}{4t}} dy &= - \int_{\mathbb{R}} t y^{2m-1} d\left(e^{-\frac{y^2}{4t}}\right) dy = -t y^{2m-1} e^{-\frac{y^2}{4t}} \Big|_{-\infty}^{+\infty} + \\ &+ (2m-1)t \int_{\mathbb{R}} y^{2m-2} e^{-\frac{y^2}{4t}} dy = (2m-1)t \int_{\mathbb{R}} y^{2m-2} e^{-\frac{y^2}{4t}} dy \end{aligned}$$

By recurrence:

$$\int_{\mathbb{R}} y^{2m-2} e^{-\frac{y^2}{4t}} dy = \frac{(2m)!}{2^m m!} t^m \sqrt{4\pi t}$$

hence:

$$P_t f(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^k k! (n-2k)!} x^{n-2k} t^k$$

Since $f^{(2k)}(x) = n(n-1)\dots(n-2k+1)x^{n-2k}$, we have:

$$P_t f(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{f^{(2k)}(x)}{2^k k!} t^k$$

hence:

$$\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\frac{(y-x)^2}{4t}} dy = \sum_{n=0}^{\infty} \frac{A^n f(x)}{n!} t^n$$

holds at least for polynomial functions f .

The brownian semigroup shares the same property to transform polynomials into polynomials both in the variable x and in t . While the degree in x is preserved, the degree in t is now $\lfloor \frac{n}{2} \rfloor$.

Some other computations: let $f(x) = e^{cx}$; we have $f^{(2k)}(x) = c^{2k} e^{cx}$ hence

$$\sum_{n=0}^{\infty} \frac{c^{2n} e^{cx}}{n!} t^n = e^{cx} e^{c^2 t}$$

while:

$$\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\frac{(y-x)^2}{4t}} dy = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{cy} e^{-\frac{(y-x)^2}{4t}} dy = e^{cx+c^2 t}$$

since:

$$cy - \frac{y^2 - 2xy + x^2}{4t} = -\frac{1}{4t} (y - 2ct - x)^2 + cx + c^2 t$$

hence the equality holds for this function and the radius is ∞ .

Let now choose $f(x) = \sin x$; we have $f^{(2k)}(x) = (-1)^k \sin x$ hence:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin x}{n!} t^n = e^{-t} \sin x$$

while:

$$\begin{aligned} \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\frac{(y-x)^2}{4t}} dy &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \sin(y+x) e^{-\frac{y^2}{4t}} dy = \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} (\sin y \cos x + \sin x \cos y) e^{-\frac{y^2}{4t}} dy = \end{aligned}$$

$$= \sin x \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4t}} \cos y \, dy = e^{-t} \sin x$$

passing in the complex:

$$\int_{\mathbb{R}} e^{-x^2} e^{i\alpha x} \, dx = \int_{\mathbb{R}} e^{-\frac{\alpha^2}{4}} e^{-(x-\frac{i\alpha}{2})^2} \, dt = \sqrt{\pi} e^{-t}$$

and again the equality holds everywhere.

However, strange phenomena occur in this case. For example, taking $f(x) := e^{-cx^2}$ for $c > 0$ we have:

$$\begin{aligned} P_t f(x) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-cy^2} e^{-\frac{(y-x)^2}{4t}} \, dy = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{1+4ct}} \int_{\mathbb{R}} e^{-\frac{1+4ct}{4t} (y-\frac{x}{1+4ct})^2} \, dy = \\ &= \frac{1}{\sqrt{1+4ct}} e^{-\frac{cx^2}{1+4ct}} \end{aligned}$$

which certainly has a power series development, but the radius of convergence is $R = \frac{1}{4c}$.

(iii) Cauchy semigroup. Let us consider the Cauchy semigroup on \mathbb{R} :

$$P_t f(x) := \int_{-\infty}^{\infty} f(x+y) \frac{1}{\pi} \frac{t}{t^2 + y^2} \, dy$$

Taking $f(x) = \frac{1}{a+x^2}$ (with $a > 0$), a computation using residue theorem gives:

$$P_t f(x) = \frac{t^3 - t^2 \sqrt{a} + t(x^2 - a) + \sqrt{a}(x^2 + a)}{\sqrt{a}(t^4 + 2t^2(x^2 - a) + (x^2 + a)^2)}$$

with radius of convergence $\sqrt{x^2 + a} \geq \sqrt{a}$.

A curious phenomenon occurs if we take now $f(x) = \chi_{[a,b]}$. We get

$$\begin{aligned} P_t f(x) &= \frac{1}{\pi} \left[\arctan \frac{b-x}{t} - \arctan \frac{a-x}{t} \right] = \\ &= \frac{1}{\pi} \left[\arctan \frac{t}{a-x} - \arctan \frac{t}{b-x} \right] \end{aligned}$$

which has a "Laurent" development for $t > \frac{a+b}{2} + |x - \frac{a+b}{2}|$, but also a power series with radius $R = \min(|a-x|, |b-x|)$ (which may be 0).

4.2 Operations.

(i) **Product with e^{-ct} .** If

$$P_t f(x) = \sum_{n=0}^{\infty} a_n(x) t^n, \quad 0 < t < R$$

then

$$e^{-ct} P_t f(x) = \sum_{n=0}^{\infty} b_n(x) t^n, \quad 0 < t < R$$

where

$$b_n(x) = \frac{a_n(x)}{0!} - \frac{ca_{n-1}(x)}{1!} + \dots + (-1)^k \frac{c^k a_{n-k}(x)}{k!} + \dots + (-1)^n \frac{c^n a_0(x)}{n!}$$

This corresponds to the idea that the generator for the semigroup $(e^{-ct} P_t)$ is $A - cI$.

(ii) **Tensor product.** Let us suppose that:

$$P_t f(x) = \sum_{n=0}^{\infty} a_n(x) t^n, \quad 0 < t < R$$

and

$$Q_t f(x) = \sum_{n=0}^{\infty} b_n(x) t^n, \quad 0 < t < R$$

then:

$$(P_t \otimes Q_t) F(x, y) = \sum_{n=0}^{\infty} c_n(x, y) t^n, \quad 0 < t < R$$

where $F(x, y) := f(x).g(y)$ and $c_n(x, y) = a_n(x)b_0(y) + \dots + a_{n-k}(x).b_k(y) + \dots + a_0(x).b_n(y)$ This corresponds to the idea that the generator for the semigroup $(P_t \otimes Q_t)$ is $A_1 \oplus A_2$.

As a conclusion for the definition of the dual infinitesimal generator, we may consider the following situation.

Let $\mathcal{P} = (P_t)_{t>0}$ be a semigroup of kernels on the measurable space (E, \mathcal{E}) . Let $x \in E$ and $R > 0$ be fixed. A (measurable, positive) function f will be called $\mathcal{P} - R - x$ -analytic if $\lim_{t \rightarrow 0} P_t f(x) = f(x)$ and there exists $(a_n)_{n \geq 0}$, such that

$$P_t f(x) = \sum_{n=0}^{\infty} a_n t^n$$

for any $0 < t < R$.

Now $A^*\varepsilon_x$ may be considered as a linear functional on the space of $\mathcal{P} - R - x$ -analytic functions. Since:

$$\frac{1}{t} [P_t f(x) - f(x)] = \sum_{n=1}^{\infty} a_n t^{n-1}$$

we have

$$A^*\varepsilon_x(f) = Af(x) = \lim_{t \rightarrow 0} \frac{1}{t} [P_t f(x) - f(x)] = a_1$$

5 ACKNOWLEDGMENT

The author acknowledges support from: the grant CEx05-D11-23/2005; and partial support from CNCSIS through grant GR214/20.09.2006; 2-CEx06-11-10/25.07.2006

References

- [1] Gh. Bucur, M. Bezzarga: Théorie du potentiel pour les systèmes semi-dynamiques. Rev. Roumaine Math. Pures Appl. 39 (1994), no. 5, 439–456.
- [2] N. Boboc, Gh. Bucur, A. Cornea **Order and Convexity in Potential Theory: H-Cones**, Lecture Notes in Math. vol. 853, Springer-Verlag Berlin 1981
- [3] N. P. Bhatia, O. Hájek: **Local semi-dynamical systems**, Lecture Notes in Mathematics, Vol. 90 Springer-Verlag, Berlin-New York 1969 ii+157 pp.
- [4] A. Cornea, G. Licea **Order and Potential. Resolvent Families of Kernels**, Lecture Notes in Math. vol. 494, Springer-Verlag 1975
- [5] C. Dellacherie: Théorie générale du potentiel. I. Hommage à P. A. Meyer et J. Neveu. Astérisque No. 236 (1996), 109–124.
- [6] C. Dellacherie: Non linear Dirichlet problem and non linear integration, Picco, Pierre (ed.) et al., From classical to modern probability. CIMPA summer school 2001, Temuco, Chile, January 8–26, 2001. Basel: Birkhäuser. Prog. Probab. 54, 83-92 (2003).
- [7] C. Dellacherie, P.-A. Meyer **Probabilités et Potentiel**, Act. Sci. Ind. Hermann, Paris 1987

-
- [8] O. Hájek: **Dynamical systems in the plane**, Academic Press, London-New York 1968 viii+235 pp
- [9] J. van Neerven: **The adjoint of a semigroup of operators**, LNM 1519, Springer-Verlag, Berlin Heidelberg, 1992
- [10] R. S. Phillips: The adjoint semigroup. *Pac. J. Math.* 5 (1955), 269–283.
- [11] E. Popa: Morphisms of H-cones. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi, N. Ser., Sect. Ia* 29, No.2, Suppl., 53-62 (1983).
- [12] E. Popa: Semi-dynamical systems on algebraic structures. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat.* 44 (Suppl.), 585-596 (1998).
- [13] E. Popa: Excessive elements in semi-dynamical systems. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat.* 45, No.2, 367-378 (1999).
- [14] E. Popa: On Riesz' splitting property. *Libertas Math.* 22, 87-104 (2002).
- [15] E. Popa: A Hille–Yosida type theorem in ordered convex cones, *Positivity Publisher Birkhäuser Basel* Volume 10, Number 3 / September, 2006 Pages 555-571
- [16] E. Popa: **Potential Theory on Semi–dynamical Systems**, (186 pg) Casa Editoriala DEMIURG Iaşi, 2005. ISBN 973-7603-13-3