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Tensor Product of Kernels acting on Positive Measures.

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1 Abstract.

The purpose of this paper is to present the construction of a tensor product of kernels (and of semigroups of kernels) acting on (positive, σ–finite) measures. The first step is to extend such a kernel, acting on one factor, to a kernel acting on the product. A commutativity problem arise (unlike the case of kernels on functions, but similar to the non linear case). The solution is to consider a smaller cone of measures on the product space. On this cone, the tensor product always makes sense and has the expected properties.

2 Introduction.

R. Cairoli [1] seems to be the first to construct the tensor product of kernels and of semigroups, with applications to processes. Later, this construction served as the basis for the tensor product of harmonic structures (harmonic spaces or standard H–cones) see [5]. As a natural continuation of the theory developed by A. Cornea and G. Licea [2], general semigroups of kernels, acting on (abstract) ordered convex cones were considered in several papers [9].

The most important example of such a cone, after that of positive functions, is the cone of positive measures. This example was studied for example in [8].

The main reason for this study is the hope that the property of absolute continuity (for the kernels composing a semi-group) may be reformulated, in such a way to obtain an additional condition on a basic resolvent which ensures that (an) associated semi-group is also absolutely continuous. This property appears in the construction of a tensor product: if we start with two semi-groups, one of which is absolutely continuous and the other has absolutely continuous resolvent, then the tensor product still has absolutely continuous resolvent (equivalent: the excessive functions form a standard H–cone).

Another reason is as follows. When trying to define the tensor product of two ordered convex cones, the underlying algebraic object may be defined in the usual way, however the *correct* order relation has to be found; and a completion in needed. Hence, it is natural to connect this subject with:

(i) representing an ordered convex cone as a cone of measurable functions/measures; and (ii) the completion of an ordered convex cone.

Several difficulties appeared during the construction. First, in the extension of the kernel, from X to $X \times Y$: we obtain first the definition only on sets $A \times B$; in order to extend it to all measurable sets for the product $\mathcal{X} \otimes \mathcal{Y}$, a Caratheodory type result seems unavoidable. This fact requires some regularity assumption. However, the most important is the problem of commutation that appears now. I have no reasons to believe that this commutativity holds for all measures and for all kernels.

3 Preliminaries.

Let $(X, \mathcal{X})$ be a measurable space. If X is a topological space, then $\mathcal{X}$ will always be the Borel σ-algebra. The measures will always be positive and σ-finite; moreover, the measures will be supposed regular on topological spaces:

$$m(A) = \sup_{\substack{K \subset A \\ K \text{ compact}}} m(K)$$

$\mathcal{M}(X)$ will denote the set of all those measures; it is an ordered convex cone.

Let us recall [3][III, T38] that a finite, positive mesure on $(X, \mathcal{X})$ is automatically regular if E is a metrizable, separable space which is either: lusinian, or suslinian, or cosuslinian.

We will need the following result of Morando [4]:

Theorem. *Let X, Y be metrizable and separable spaces. Let*

$$\mu : \mathcal{X} \times \mathcal{Y} \longrightarrow [0, 1]$$

be a map with the properties:

$\mu(X, Y) = 1$*;*

$\forall A \in \mathcal{X}$*,* $\mu(A, \cdot)$ *is a regular measure on* $(Y, \mathcal{Y})$*;*

$\forall B \in \mathcal{Y}$*,* $\mu(\cdot, B)$ *is a regular measure on* $(X, \mathcal{X})$*;*

There exists then a unique regular measure $\tilde{\mu}$ *on* $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$*, such that*

$$\tilde{\mu}(A \times B) = \mu(A, B),\ \forall A \in \mathcal{X}, B \in \mathcal{Y}$$

The extension is done by Caratheodory's theorem.

We consider next $\mathcal{M}$ a generic convex cone of (positive) measures, with the property that for any increasing and dominated sequence (μ_n) from $\mathcal{M}$ we have $\sup_n \mu_n \in \mathcal{M}$.

By a *kernel* (cf. [2]) on $\mathcal{M}$ we mean a map $V : \mathcal{M} \longrightarrow \mathcal{M}$ with the properties:

$$V(\mu + \nu) = V\mu + V\nu$$

$$\mu \leqslant \nu \Longrightarrow V\mu \leqslant V\nu$$

$$\mu_n \nearrow \mu \Longrightarrow V\mu_n \nearrow V\mu$$

As a specific property (see [3][II D13]), we will consider:

$$(M)\ \ x \mapsto V(\varepsilon_x)(A) \text{ is measurable } \forall A \in \mathcal{E}$$

A kernel V on measures is called *submarkovian* if $V\mu(1) \leqslant \mu(1)$, for each measure $\mu \in \mathcal{M}$.

More generally, if a kernel V satisfies $V\mu(1) \leqslant C.\mu(1)$, for each measure $\mu \in \mathcal{M}$, then it is norm–continuous.

When V is a (true) kernel on functions, $V1 \leqslant 1$ is equivalent with this property.

With each kernel $V : \mathcal{F} \longrightarrow \mathcal{F}$ on functions, one associates a kernel on measures $\tilde{V} : \mathcal{M} \longrightarrow \mathcal{M}$ defined through $\tilde{V}(\mu)(f) := \mu\,(Vf)$. This kernel has also the property (M). Conversely, let $V : \mathcal{M} \longrightarrow \mathcal{M}$ be a kernel on measures, possessing the property (M). Let us define $\hat{V} : \mathcal{F} \longrightarrow \mathcal{F}$ as: $\hat{V}f(x) := V(\varepsilon_x)(f)$. This definition is correct, while the measurability of the function $\hat{V}(f)$ results easily.

Hence, starting with a kernel on functions $V : \mathcal{F} \longrightarrow \mathcal{F}$, we may consider a new the kernel on functions $\hat{\tilde{V}}$ and we have $\hat{\tilde{V}} = V$. However, starting with a kernel on measures $V : \mathcal{M} \longrightarrow \mathcal{M}$, even with the property (M), we may consider the kernel on measures $\tilde{\hat{V}}$, but the equality $\tilde{\hat{V}}(\mu) = V\mu$ **does not** hold for any measure μ (it holds however for atomic measures). This means that not any kernel on measures is associated with a kernel on functions.

The following example of such a kernel was given [2]. Let $\mu \in \mathcal{M}$ and define:

$$B_\mu(\nu) := \bigvee_n (\nu \wedge n.\mu)$$

($B_\mu \nu$ is the μ–absolutely continuous part ν).

B_μ is a kernel on measures, but there is no kernel on functions V, such that $\tilde{V} = B_\mu$.

If we replace the last requirement from the definition of a kernel on measures:

$$\mu_n \nearrow \mu \Longrightarrow V\mu_n \nearrow V\mu$$

by

$$\mu_i \longrightarrow \mu \Longrightarrow V\mu_i \longrightarrow V\mu$$

(where $\mu_n \rightarrow \mu$ means $\mu_n(f) \longrightarrow \mu(f)$, for any positive, bounded measurable f), then there exists a kernel W on functions, such that $V = \tilde{W}$ (see [10]).

4 Extension of a kernel.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measurable spaces. Let V be a kernel on $\mathcal{M}(X)$; we want to define a kernel, denoted $\tilde{V}$, on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$. Let $m \in \mathcal{M}(X \times Y)$. For each fixed $B \in \mathcal{Y}$ we define

$$m_B(A) := m(A \times B)$$

Clearly m_B is a measure on $(X, \mathcal{X})$. Thus, Vm_B is a measure on $(X, \mathcal{X})$.

Theorem 1. *Let X and Y be metrizable and separable spaces; suppose moreover that Y is lusinian. Let V be a submarkovian kernel on $\mathcal{M}(X)$, with the property:*

for each regular measure μ on $(X, \mathcal{X})$, $V\mu$ is a regular measure

Let m be a finite, positive measure on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$. Then $A \times B \mapsto (Vm_B)(A)$ extends uniquely to a measure on the product space $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$, denoted as $\tilde{V}m$.

$\tilde{V}$ is a kernel on $\mathcal{M}(X \times Y)$

Proof. Clearly we can suppose that $m(X \times Y) = 1$; then $m_Y(A) \leqslant 1$.
We apply the theorem of Morando to $\mu(A, B) := Vm_B(A)$. Since we have

$m_\emptyset \equiv 0$

If $B_1 \cap B_2 = \emptyset$ then $m_{B_1 \cup B_2} = m_{B_1} + m_{B_2}$

$B_n \nearrow B \Rightarrow m_{B_n} \nearrow m_B$,

it follows that $B \mapsto Vm_B(f)$ is a measure on $(Y, \mathcal{Y})$, for each measurable, positive f.

If m is regular, then m_B is also regular. Indeed:

$$m_B(A) \geqslant \sup_{\substack{K \subset A \\ K \text{ compact}}} m_B(K) \geqslant \sup_{\substack{K \subset A \\ L \subset B \\ K, L \text{ compacts}}} m_L(K) =$$

$$= \sup_{\substack{K \subset A \\ L \subset B \\ K, L \text{ compacts}}} m(K \times L) = \sup_{\substack{K \subset A \\ K \text{ compact}}} m(K) = m_B(A)$$

as for each compact $K \subset A \times B$ we have $K \subset pr_X K \times pr_Y K \subset A \times B$ and both projections are compact. Hence, by assumption, Vm_B is also regular.

Now, from [3][III T32], it follows that, for each fixed $A \in \mathcal{X}$, $B \mapsto Vm_B(A)$ is regular. Hence, all the assumptions from Morando's theorem are fulfilled. Thus, we denote by $\tilde{V}m$ the measure obtained on $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$.

$\tilde{V}$ is a submarkovian kernel on measures: indeed, $\tilde{V}m(1) = Vm_Y(1) \leqslant m_Y(1)$.

Since $(m + m')_B = m_B + m'_B$, by uniqueness we obtain $\tilde{V}(m + m') = \tilde{V}m + \tilde{V}m'$; also $(\alpha m)_B = \alpha.m_B$ implies $\tilde{V}(\alpha m) = \alpha.\tilde{V}m$. If $m \leqslant m'$ we have $m_B \leqslant m'_B$, hence $\tilde{V}m \leqslant \tilde{V}m'$. Finally, from $m_n \nearrow m$ we get $m_{n,B} \nearrow m_B$ hence $\tilde{V}m_n \nearrow \tilde{V}(m)$.

Remark. If X and Y are locally compact spaces and the measures are Radon, then a similar extension of $f \otimes g \longmapsto Vm_g(f)$ allows the definition of the kernel $\tilde{V}$.

Corollary. *Suppose moreover that X is also lusinian. Then the condition that V preserve regularity may be dropped.*

Direct properties. (i) If V has (M), then $\tilde{V}$ also has (M).

(ii) $\widetilde{V_1 + V_2} = \tilde{V}_1 + \tilde{V}_2$

(iii) $V_1 \leqslant V_2 \Longrightarrow \widetilde{V_1} \leqslant \widetilde{V_2}$

(iv) $V_n \nearrow V \Longrightarrow \widetilde{V_n} \nearrow \tilde{V}$

(v) $\tilde{I} = I$

(vi) $\widetilde{V_1 \circ V_2} = \widetilde{V_1} \circ \widetilde{V_2}$

Examples and particular cases. (i) If $m = \mu \otimes \nu$, then $m_B = \nu(B).\mu$, hence $\tilde{V}m = (V\mu) \otimes \nu$.

(ii) If V is a kernel on functions, then it extends as a kernel on the product space by [1]: $\hat{V}(f \otimes g) := (Vf) \otimes g$. Hence we have:

$$\tilde{V}m(A \times B) = (Vm_B)(A) = \int \chi_A d(Vm_B) = \int (V\chi_A) dm_B =$$

$$= \int (V\chi_A) \otimes \chi_B dm = \int \hat{V}(\chi_A \otimes \chi_B) dm = (m\hat{V})(A \times B)$$

Let W be a kernel on $\mathcal{M}(Y)$; then the construction of $\tilde{W}$ works as well; however, it is a kernel on $\mathcal{M}(Y \times X)$. In order to obtain a kernel on $\mathcal{M}(X \times Y)$, we define the map $\varepsilon : \mathcal{M}(X \times Y) \longrightarrow \mathcal{M}(Y \times X)$, induced by $(x, y) \mapsto (y, x)$. Now $\varepsilon^{-1} \circ \tilde{W} \circ \varepsilon$ is a kernel on $\mathcal{M}(X \times Y)$. We will continue to write $\tilde{V} \circ \tilde{W}$ for $\tilde{V} \circ (\varepsilon^{-1} \circ \tilde{W} \circ \varepsilon)$. In the same manner, we will write $\tilde{W} \circ \tilde{V}$ instead of $(\varepsilon^{-1} \circ \tilde{W} \circ \varepsilon) \circ \tilde{V}$. In this way, all kernels are on $\mathcal{M}(X \times Y)$.

The commutativity on product measures holds, since $\varepsilon(\mu \otimes \nu) = \nu \otimes \mu$, hence $\tilde{W}(\nu \otimes \mu) = W\nu \otimes \mu$ so that:

$$(\tilde{V} \circ \tilde{W})(\mu \otimes \nu) = V\mu \otimes W\nu = (\tilde{W} \circ \tilde{V})((\mu \otimes \nu)$$

5 Tensor product of kernels and semigroups.

In this final part, we suppose that all spaces are metrizable, separable and lusinian; and all kernels (on measures) are submarkovian.

Once we have extended V from $(X, \mathcal{X})$ to $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$ and accepted the symmetric extension of a kernel W from $(Y, \mathcal{Y})$ to $(X \times Y, \mathcal{X} \otimes \mathcal{Y})$, we can define:

$$V \otimes W := \tilde{V} \circ \tilde{W}$$

It is natural to expect that $\tilde{V}$ and $\tilde{W}$ commute; in fact, it is even compulsory, if we start with semi-groups $(P_t)_{t>0}$ and $(Q_t)_{t>0}$ and want to obtain a semi-group $(P_t \otimes Q_t)_{t>0}$, since:

$$(P_t \otimes Q_t)(P_s \otimes Q_s) = P_{t+s} \otimes Q_{t+s}$$

requires that $\tilde{Q}_t \circ \tilde{P}_s = \tilde{P}_s \circ \tilde{Q}_t$.

We have remarked two cases when such a commutativity holds:

1) If both V and W are kernels on functions;

2) On measures of the form $\sum_{n=1}^{\infty} \mu_n \otimes \nu_n$.

We show next that there exists a convex subcone, on which the commutativity holds.

Theorem 2. *Let V be a kernel on $(X, \mathcal{X})$ and W be a kernel on $(Y, \mathcal{Y})$. There exists a convex cone $\mathcal{M}_0 \subseteq \mathcal{M}(X \times Y)$ with the properties:*

1) $\omega \in \mathcal{M}(X \times Y)$ such that $\omega = \sum_{n=1}^{\infty} \mu_n \otimes \nu_n$, with $\mu_n \in \mathcal{M}(X)$, $\nu_n \in \mathcal{M}(Y) \Rightarrow \omega \in \mathcal{M}_0$

2) $\omega \in \mathcal{M}(X \times Y)$ such that $\omega_n \nearrow \omega$ with $\omega_n \in \mathcal{M}_0 \Rightarrow \omega \in \mathcal{M}_0$

3) $\omega_n \in \mathcal{M}_0$ and $\|\omega_n - \omega\| \to 0 \Rightarrow \omega \in \mathcal{M}_0$

4) $\omega \in \mathcal{M}_0 \Rightarrow \tilde{V}\omega$ and $\tilde{W}\omega \in \mathcal{M}_0$

5) $\omega \in \mathcal{M}_0 \Rightarrow (\tilde{V} \circ \tilde{W})\omega = (\tilde{W} \circ \tilde{V})\omega$

Proof. It is clear that the choice:

$$\mathcal{M}_0 := \{\omega \in \mathcal{M}(X \times Y) | (\tilde{V}^n \circ \tilde{W}^m)\omega = (\tilde{W}^m \circ \tilde{V}^n)\omega, \forall n, m \in \mathbb{N}\}$$

is good

Corollary. *Let $\mathcal{P} = (P_t)_{t>0}$ be a semigroup of kernels on $(X, \mathcal{X})$ and $\mathcal{Q} = (Q_t)_{t>0}$ be a semigroup of kernels on $(Y, \mathcal{Y})$. There exists a convex cone $\mathcal{M}_0 \subseteq \mathcal{M}(X \times Y)$ with the properties:*

1) $\omega \in \mathcal{M}(X \times Y)$ such that $\omega = \sum_{n=1}^{\infty} \mu_n \otimes \nu_n$, with $\mu_n \in \mathcal{M}(X)$, $\nu_n \in \mathcal{M}(Y) \Rightarrow \omega \in \mathcal{M}_0$

2) $\omega \in \mathcal{M}(X \times Y)$ *such that* $\omega_n \nearrow \omega$ *with* $\omega_n \in \mathcal{M}_0 \Rightarrow \omega \in \mathcal{M}_0$

3) $\omega_n \in \mathcal{M}_0$ *and* $\|\omega_n - \omega\| \to 0 \Rightarrow \omega \in \mathcal{M}_0$

4) $\omega \in \mathcal{M}_0 \Rightarrow \tilde{P}_t\omega$ *and* $\tilde{Q}_t\omega \in \mathcal{M}_0$

5) $\omega \in \mathcal{M}_0 \Rightarrow (P_t \otimes Q_s)\omega = (Q_s \otimes P_t)\omega,\ \forall t, s > 0.$

Proof. Let us choose now:

$$\mathcal{M}_0 := \{\omega \in \mathcal{M}(X \times Y) | (\tilde{P}_t \circ \tilde{Q}_s)\omega = (\tilde{Q}_s \circ \tilde{P}_t)\omega, \forall t, s > 0\}$$

However, the convex cone above depends on the semigroups. We prove finally that we can choose a unique convex cone, with the same properties, on which any tensor product can be constructed:

Theorem 3. *There exists a convex cone* $\mathcal{M}_0 \subseteq \mathcal{M}(X \times Y)$*, with the properties:*

1) $\omega \in \mathcal{M}(X \times Y)$ *such that* $\omega = \sum_{n=1}^{\infty} \mu_n \otimes \nu_n$*, with* $\mu_n \in \mathcal{M}(X)$*,* $\nu_n \in \mathcal{M}(Y) \Rightarrow \omega \in \mathcal{M}_0$

2) $\omega \in \mathcal{M}(X \times Y)$ *such that* $\omega_n \nearrow \omega$ *with* $\omega_n \in \mathcal{M}_0 \Rightarrow \omega \in \mathcal{M}_0$

3) $\omega_n \in \mathcal{M}_0$ *and* $\|\omega_n - \omega\| \to 0 \Rightarrow \omega \in \mathcal{M}_0$

4) For any (submarkovian) kernel V *on* X*,* $\omega \in \mathcal{M}_0 \Rightarrow \tilde{V}\omega \in \mathcal{M}_0$

such that, for any semigroups of kernels $\mathcal{P} = (P_t)_{t>0}$ *on* $(X, \mathcal{X})$ *and* $\mathcal{Q} = (Q_t)_{t>0}$ *on* $(Y, \mathcal{Y})$*, the tensor product semigroup* $\mathcal{P} \otimes \mathcal{Q} = (P_t \otimes Q_t)_{t>0}$ *is well defined on* $\mathcal{M}_0$*.*

Proof. Let us denote now:

$$\mathcal{M}_0 := \{\omega \in \mathcal{M}(X \times Y) | \forall V, W \text{submarkovian kernels } (\tilde{V} \circ \tilde{W})\omega = (\tilde{W} \circ \tilde{V})\omega\}$$

Let S be any other kernel. For any $\omega \in \mathcal{M}_0$ we have:

$$(\tilde{V} \circ \tilde{W})(\tilde{S}\omega) = \tilde{V}\left((\tilde{S} \circ \tilde{W})\omega\right) = (\tilde{S} \circ \tilde{W})(\tilde{V}\omega) = (\tilde{W} \circ \tilde{V})(\tilde{S}\omega)$$

thus $\tilde{S}\omega \in \mathcal{M}_0$

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