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Existence results in relaxed variational calculus

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Abstract. We give some existence results for relaxed solutions in the calculus of variations.

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1 Introduction

In the last period, the technological progress has generated a huge need of complex mathematical models. Many problems in science can be formulated in the language of optimization theory where an optimal solution or the best answer to a particular situation is sought. Under situations of interest, they all lack classical optimal solutions, or at least, the existence of such solutions is far from being straightforward. Non-convex optimization problems (arising, e.g., in variational calculus, optimal control, nonlinear evolutions equations) may not possess a classical solution because approximate solutions show typically rapid oscillations. This phenomenon requires the extension of the notion of solution of such problems, often constructed by means of Young measures; the asymptotic oscillatory behavior is well controlled with the Young measure technique developed by L. C. Young [11], J. Warga [10] and recently by E.J. Balder [1], M. Valadier [9], P. Pedregal [7] and others.

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Many authors wrote on Young measures in Control and Calculus of Variations: Castaing, Ch., Raynaud de Fitte, P. [2], Castaing, Ch., Raynaud de Fitte, P. Valadier, M. [3].

The Young measures generalize the measurable functions. Thus, a Young measure is herself a measurable application that, to every point t of Ω, associates a probability τ_t on a topological space X; for all Borel set A of X, $\tau_t(A)$ maybe interpreted as the probability that the value in t of the "function" $\tau_.$ to belong to A. In the particular case, a measurable application $u : \Omega \to X$ is a Young measure, where, for all $t \in \Omega$, $\tau_t = \delta_{u(t)}$($\delta_{u(t)}$ indicates the masse of Dirac in $u(t)$).

In the present work we review some results of relaxed existence for the problem of the variational calculus using the Young measures.

Section 2 is concerned with an abstract approach to optimisation and relaxation as is it sketch in [7].

In section 3 we give some existence results in the case where the sequence of gradients for a minimizing sequence is tight (see [1], [3], [7], [8] and [9]).

The particular case where the sequence of gradients is Jordan finite tight is treat in 4. The Jordan finite tight is an excellent tool in the study of behavior of minimizing sequences. So, if H is a tight in $W^{1,1}(\Omega, \mathbb{R}^m)$ for which the set of its gradients ∇H is Jordan finite tight then H is relatively compact in measure. This result offers very good conditions to use fiber product lemma (see [2]) for obtain relaxed minimizers in the calculus of variations. The Jordan finite tight condition is a natural condition of accumulation of big growths for the gradients of ∇H on the sets of small Jordan measure. A first attempt to study finite tights was made in [4] in the particular case $\Omega =]0, 1[$ and $X = \mathbb{R}^m$; in [5] this notion is extended to a bounded, open, convex set $\Omega \subseteq \mathbb{R}^d$.

2 Relaxed variational calculus - an abstract framework

The standard problem of the calculus of variations is to find minimizers of the functional

$$I(u) = \int_\Omega F(t, u(t), \nabla u(t))dt,$$

where Ω is an open bounded set in $\mathbb{R}^d$, the mappings $u : \Omega \to \mathbb{R}^m$ belong to $W^{1,p}(\Omega), p \geq 1$, they all satisfy Dirichlet boundary conditions $u - u_0 \in W_0^{1,p}(\Omega)$ and the integrand $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{md} \to \mathbb{R}$ is assumed to be measurable on t and continuous in $(u, \nabla u)$ (a Carathéodory integrand).

Let H be the set of competing objects u; under suitable technical assumptions the problem:

$$(m) \qquad \text{Find } u_0 \in H \text{ such that } I(u_0) = m = \inf_{u \in H} I(u) \in \mathbb{R}$$

has a solution.

This method is is fruitful when some "convexity" conditions in the gradient variable is assumed on the integrand F.

In non-convex optimization problems (arising, e.g., in calculus of variations, optimal control, nonlinear evolutions equations related to elastodynamics and analysis of microstructure) there usually does not exist any classical solution (i.e. does not exists $u \in H$ for which $I(u) = m$). To exemplify, we can recall the next one problem of Bolza that doesn't admit a classical solution.

Example 2.1. Let

$$I(u) = \int_0^1 \left[\left((u'(t)^2 - 1 \right)^2 + u^2(t) \right] dt$$

where $u \in W^{1,4}(]0,1[)$ satisfies the boundary conditions $u(0) = u(1) = 0$. The sequence $(u_n)_n$

$$u_n(t) = \sum_{k=0}^{2^{n-1}-1} \left[\left(t - \frac{2k}{2^n} \right) \cdot \mathbb{1}_{\left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)}(t) + \left(\frac{2k+2}{2^n} - t \right) \cdot \mathbb{1}_{\left[\frac{2k+1}{2^n}, \frac{2k+2}{2^n}\right)}(t) \right]$$

has as gradients $(\nabla u_n)_n$, where $\nabla u_n(t) = u_n'(t) = \text{sign}\left[\sin(2^n \pi t)\right], \forall t \in]0,1[, \forall n \in \mathbb{N}$ (the Rademacher' sequence). We remark that $(u_n')^2 = 1$ almost everywhere and that $I(u_n) \to 0$.

It follows that $\inf_{u \in H} I(u) = 0$ but the problem (m) does not admit any minimizer (it will be necessary that such a competitor $u_0 \in H$ be null a.e. and with the derivative ± 1 a.e.).

We note that the integrand $F(t,u,v) = u^2 + (v^2 - 1)^2$ is not convex in the variable v. Of course, one can take the convex envelope of F (the largest convex function with respect to variable v which is smaller than F):

$$F^{**}(t,u,v) = \begin{cases} F(t,u,v) & , |v| \geq 1, \\ u^2 & , |v| < 1. \end{cases}$$

The problem gotten thus:

$$\text{Find } u \text{ such that } m = 0 = \min \int_0^1 F^{**}(t, u(t), u'(t)) dt$$

has a minimal solution $\bar{u} = \underline{0}$.

It is obvious that such a solution doesn't contain any information on the fast oscillations of the minimizing sequence $(u_n)_n$.

In order to obtain a satisfactory minimizer we will enlarge the set of competitors H to a set $\bar{H}$ and will extend I to $\bar{I}$ defined on $\bar{H}$ to obtain the relaxed problem:

$$(\bar{m}) \qquad \text{Find } \bar{u}_0 \in \bar{H} \text{ s.t. } \bar{I}(\bar{u}_0) = \min_{\bar{u} \in \bar{H}} \bar{I}(\bar{u}) = m.$$

This enlargement is consistent if we can endow $\bar{H}$ with a topology in relation to which H is a dense subset of $\bar{H}$, $\bar{I}$ is a continuous mapping and so that the problem (m) have a relatively compact minimizing sequence.

3 Existence results in tight conditions

Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set and let $\mathcal{M}(\mathbb{R}^m)$ be the set of all measurable mappings $u : \Omega \to \mathbb{R}^m$; many of the following results can be proved in the general case where Ω is an abstract measure space and $\mathbb{R}^m$ is replace by a separable Banach space or more general by a Suslin regular space.

Definition 3.1. A **Young measure** on $\mathbb{R}^m$ is a positive measure $\mathcal{T}$ on $\Omega \times \mathbb{R}^m$ such that, $\mathcal{T}(A \times \mathbb{R}^m) = \mu(A)$, for every measurable set $A \subseteq \Omega$ (μ denote here the Lebesgue measure on Ω). Thanks to a theorem of disintegration one can identify each Young measure $\mathcal{T}$ with an application $\tau_\cdot$ ($\mathcal{T} \equiv \tau_\cdot$) which, to each $t \in \Omega$, associates a probability τ_t on $\mathbb{R}^m$ ($\tau_t \in \mathcal{P}_{\mathbb{R}^m}$) with the measurability condition: the application $t \mapsto \tau_t(C)$ is measurable, for any borelian set $C \subseteq \mathbb{R}^m$.

$\mathcal{Y} = \mathcal{Y}(\mathbb{R}^m)$ will denote the space of all Young measures on $\mathbb{R}^m$.

For every bounded (or positive) integrand $\Psi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ and for every $\mathcal{T} \equiv \tau_\cdot \in \mathcal{Y}$,

$$\int_{\Omega\times\mathbb{R}^m} \Psi(t,x)d\mathcal{T}(t,x) = \int_\Omega \left(\int_{\mathbb{R}^m} \Psi(t,x)d\tau_t(x) \right) d\mu(t).$$

Definition 3.2. A sequence $(u_n)_n \subseteq \mathcal{M}(\mathbb{R}^m)$ is **stable convergent** to a Young measure $\mathcal{T} \equiv \tau_\cdot \in \mathcal{Y}(\mathbb{R}^m)$ ($u_n \xrightarrow{\mathcal{S}} \mathcal{T}$) if, for every measurable subset $A \subseteq \Omega$ and every real bounded continuous mapping $f : \mathbb{R}^m \to \mathbb{R}$,

$$\int_{\Omega\times\mathbb{R}^m} \mathbb{1}_A(t) \cdot f(x)d\mathcal{T}(t,x) = \int_A \left(\int_{\mathbb{R}^m} f(x)d\tau_t(x) \right) d\mu(t) =$$

$$= \lim_n \int_A f(u_n(t))d\mu(t),$$

or, equivalently, if, for every bounded below l.s.c. integrand $\Psi : \Omega\times\mathbb{R}^m \to \mathbb{R}$,

$$\int_{\Omega\times\mathbb{R}^m} \Psi(t,x)d\mathcal{T}(t,x) \leq \liminf_n \int_\Omega \Psi(t,u_n(t))d\mu(t).$$

An useful property of stable convergence is given in the corollary 2.3.2 of [3]:

Theorem 3.1. *Let $\mathcal{T} \in \mathcal{Y}$ and $(u_n)_n \subseteq \mathcal{M}(\mathbb{R}^m)$ such that $u_n \xrightarrow{\mathcal{S}} \mathcal{T}$. Then, for every $\mathcal{T}$-integrable Carathéodory integrand $\Psi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ for which $\{\Psi(\cdot, u_n(\cdot)) : n \in \mathbb{N}\}$ is a uniformly integrable subset in $\mathcal{L}^1(\mu)$ we have:*

$$\mathcal{T}(\Psi) = \int_{\Omega\times\mathbb{R}^m} \Psi(t,x)d\mathcal{T}(t,x) = \lim_{n\to\infty} \int_\Omega \Psi(t,u_n(t))d\mu(t).$$

Particularly, if for any $t \in \Omega, \tau_t = \delta_{u(t)}$ (where $u \in \mathcal{M}(\mathbb{R}^m)$ and $\delta_{u(t)}$ indicates the mass of Dirac in $u(t)$), then $u_n \xrightarrow{\mathcal{S}} \mathcal{T} \equiv \delta_{u(.)}$ if and only if $(u_n)_n$ is convergent in measure to u (see the definition of Hoffmann-Jørgensen in [6]); we say in this case that the Young measure $\mathcal{T}$ is a degenerate Young measure.

The mapping $u \mapsto \delta_{u(.)}$ is an embedding of $\mathcal{M}(\mathbb{R}^m)$ in $\mathcal{Y}(\mathbb{R}^m)$.

Furthermore $\mathcal{M}(\mathbb{R}^m)$ is a dense subspace of $\mathcal{Y}(\mathbb{R}^m)$ in relation to the stable topology.

Definition 3.3. A subset $H \subseteq \mathcal{M}(\mathbb{R}^m)$ is **tight** if and only if, for any $\varepsilon > 0$ there is a $k > 0$ such that $\mu(\{t \in \Omega : \|u(t)\| > k\}) < \varepsilon$, for every $u \in H$.

Every bounded set $H \subseteq L^1(\Omega, \mathbb{R}^m)$ is tight; indeed, if $M = \sup_{u \in H} \|u\|_1 < +\infty$ then, for every $k > 0, \mu(\|u\| > k) \leq \frac{M}{k}$.

The interest for tights consists in the following theorem (see the theorem 4.3.5 of [3]).

Theorem 3.2. (Prohorov). *A set $H \subseteq \mathcal{M}(\mathbb{R}^m)$ is relatively stable compact (sequentially stable compact) in $\mathcal{Y}(\mathbb{R}^m)$ if and only if it is tight.*

Particularly, every bounded set $H \subseteq L^1(\Omega, \mathbb{R}^m)$ is tight and thus each net (sequence) has a subnet (subsequence) stable convergent to a Young measure.

Let the Young measures $\mathcal{T} \equiv \tau_. \in \mathcal{Y}(\mathbb{R}^m)$ and $\boldsymbol{\sigma} \equiv \sigma_. \in \mathcal{Y}(\mathbb{R}^p)$; the fiber product of $\mathcal{T}$ and $\boldsymbol{\sigma}$ is the Young measure $\mathcal{T} \otimes \boldsymbol{\sigma} \equiv (\tau \otimes \sigma)_. \in \mathcal{Y}(\mathbb{R}^m \times \mathbb{R}^p)$ defined by $(\tau \otimes \sigma)_t = \tau_t \otimes \sigma_t$, for every $t \in \Omega$. In the case where $\mathcal{T}$ and $\boldsymbol{\sigma}$ are degenerates Young measures (there exist $u \in \mathcal{M}(\mathbb{R}^m), v \in \mathcal{M}(\mathbb{R}^p)$ such that $\tau_t = \delta_{u(t)}$ and $\sigma_t = \delta_{v(t)}$, for every $t \in \Omega$), then $(\tau \otimes \sigma)_t = \delta_{(u(t),v(t))}$, for every $t \in \Omega$.

In the relaxed control theory an important tool is the fiber product lemma (see the theorem 2.3.1 from [2]):

Theorem 3.3. (fiber product lemma). *Let $(u_n)_n \subseteq \mathcal{M}(\mathbb{R}^m), (v_n)_n \subseteq \mathcal{M}(\mathbb{R}^p), u \in \mathcal{M}(\mathbb{R}^m), \mathcal{T} \equiv \tau_. \in \mathcal{Y}(\mathbb{R}^p)$. Assume that*

(i) $(u_n)_n$ is convergent in measure to u;

(ii) $v_n \xrightarrow{\mathcal{S}} \mathcal{T}$.

Then $(u_n, v_n) \xrightarrow{\mathcal{S}} \delta_{u(.)} \otimes \tau_. \in \mathcal{Y}(\mathbb{R}^m \times \mathbb{R}^p)$.

We remark that, generally, $u_n \xrightarrow{\mathcal{S}} \mathcal{T}$ and $v_n \xrightarrow{\mathcal{S}} \boldsymbol{\sigma}$ don't imply

$$(u_n, v_n) \xrightarrow{\mathcal{S}} \mathcal{T} \otimes \boldsymbol{\sigma}.$$

Particularly, this result is useful in the case where $(u_n)_n$ is a minimizing sequence for an optimization problem and, for every $n \in \mathbb{N}$, $v_n = \nabla u_n$ is the gradient of u_n; if $(u_n)_n$ is convergent in measure to u and $(\nabla u_n)_n$ is stable convergent to $\mathcal{T} \equiv \tau_. \in \mathcal{Y}(\mathbb{R}^{md})$ then, for every bounded below l.s.c. integrand $\Psi : \Omega \times \mathbb{R}^m \times \mathbb{R}^{md} \to \mathbb{R}$,

$$\int_{\Omega \times \mathbb{R}^{md}} \Psi(t, u(t), y) d\mathcal{T}(t, y) \leq \liminf_n \int_\Omega \Psi\left(t, u_n(t), \nabla u_n(t)\right) d\mu(t).$$

For all concepts and results about Young measures used in this paper one can consult [3] and [9].

Now we present a relaxation of the problem (m).

If $(u_n)_n$ is a minimizing sequence for (m) then, in a large measure we can suppose that $(u_n)_n$ and $(\nabla u_n)_n$ are tights. According to the theorem of Prohorov, we can even suppose that $(u_n)_n$ is stable convergent to $\mathcal{T} \equiv \tau_. \in \mathcal{Y}(\mathbb{R}^m)$ and that $(\nabla u_n)_n$ is stable convergent to $\boldsymbol{\sigma} \equiv \sigma_. \in \mathcal{Y}(\mathbb{R}^{md})$; furthermore, we will suppose that

$$(u_n, \nabla u_n) \xrightarrow{\mathcal{S}} \tau_. \otimes \sigma_. \in \mathcal{Y}(\mathbb{R}^m \times \mathbb{R}^{md}).$$

According to the fiber product lemma, that will always happen if $\tau_.$ is a degenerate Young measure, $\delta_{u(.)}$, because, in this case $(u_n)_n$ is convergent in measure to u.

We will define then the set of relaxed competitors:

$$\bar{H} = \left\{ (\tau_., \sigma_.) \in \mathcal{Y}(\mathbb{R}^m \times \mathbb{R}^{md}) : \begin{array}{l} \exists (u_n) \subseteq H \text{ s.t. } (u_n, \nabla u_n) \xrightarrow{\mathcal{S}} \tau_. \otimes \sigma_. \text{ and} \\ (F(., u_n, \nabla u_n)) \text{ is U.I. in } L^1(\Omega, \mathbb{R}) \end{array} \right\}$$

where U.I. is the abbreviation of uniformly integrable.

We extend now I to $\bar{H}$ letting

$$\bar{I}(\tau_., \sigma_.) = \int_\Omega \left[\int_{\mathbb{R}^m \times \mathbb{R}^{md}} F(t, x, y) d(\tau_t \otimes \sigma_t)(x, y) \right] dt.$$

Remark 3.1. We remark that, if $(\tau_., \sigma_.) \in \bar{H}$, there exists a sequence $(u_n)_n \subseteq H$ such that $(u_n, \nabla u_n) \xrightarrow{\mathcal{S}} \tau_. \otimes \sigma_.$ and, as $\left(F(., u_n, \nabla u_n)\right)_n$ is uniformly integrable in $L^1(\Omega, \mathbb{R})$, then, from the theorem 3.1 it follows that

$$\bar{I}(\tau_., \sigma_.) = \int_\Omega \left[\int_{\mathbb{R}^m \times \mathbb{R}^{md}} F(t, x, y) d(\tau_t \otimes \sigma_t)(x, y) \right] dt =$$

$$= \lim_{n\to\infty} \int_\Omega F(t, u_n(t), \nabla u_n(t))dt = \lim_{n\to\infty} I(u_n).$$

The relaxed problem is now:

$$(\bar{m}) \qquad \text{Find } (\tau^0_., \sigma^0_.) \in \bar{H} \text{ s.t. } \bar{I}(\tau^0_., \sigma^0_.) = \min_{(\tau_.,\sigma_.)\in\bar{H}} \bar{I}(\tau_., \sigma_.) = m.$$

In the example 2.1, if we denote $\tau^0_. = \delta_{\underline{0}}$ and $\sigma^0_. = \frac{1}{2}(\delta_{\underline{1}} + \delta_{-\underline{1}})$, then $(\tau^0_., \sigma^0_.) \in \bar{H}$. Indeed, the minimizing sequence $(u_n)_n$ given in the quoted example is uniformly convergent to $\underline{0}$ and $\nabla u_n = u'_n \xrightarrow{\mathcal{S}} \sigma^0_.$. According to the fiber product lemma $(u_n, u'_n) \xrightarrow{\mathcal{S}} \tau^0_. \otimes \sigma^0_.$. Furthermore, for every $t \in]0, 1[$, $F(t, u_n(t), u'_n(t)) = u_n^2(t)$ and, because $(u_n^2)_n$ is uniformly bounded, $(F(., u_n, u'_n))_n$ is uniformly integrable.

In our case $(\tau^0_., \sigma^0_.)$ is just an optimal pair because $\bar{I}(\tau^0_., \sigma^0_.) = 0 = \inf_{u\in H} I(u)$.

The following result present the conditions in which the problem (m) admits relaxed solutions.

Theorem 3.4. *Let $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{md} \to \mathbb{R}$ be a Carathéodory integrand, let $H \subseteq W^{1,1}(\Omega)$ and*

$$m = \inf_{u\in H} \int_\Omega F(t, u(t), \nabla u(t))dt.$$

We suppose that:

(H_1) there exists a minimizing sequence $(u_n)_n \subseteq H$ convergent in measure to a mapping $u \in \mathcal{M}(\mathbb{R}^m)$ such that $(\nabla u_n)_n$ is tight;

(H_2) $(F(., u_n, \nabla u_n))_{n\in\mathbb{N}}$ is uniformly integrable in $L^1(\Omega, \mathbb{R})$.

Then there exists $\sigma_. \in \mathcal{Y}(\mathbb{R}^{md})$ such that $\left(\delta_{u(.)}, \sigma_.\right) \in \bar{H}$ and

$$\int_\Omega \left(\int_{\mathbb{R}^{md}} F(t, u(t), y) d\sigma_t(y) \right) dt = \inf_{u\in H} \int_\Omega F(t, u(t), \nabla u(t))dt.$$

Furthermore, if the following condition is accomplished

(H_3) $(\nabla u_n)_{n\in\mathbb{N}}$ is uniformly integrable in $L^1(\Omega, \mathbb{R}^{md})$

then, almost for all $t \in \Omega$, there exists $v(t) = \mathrm{bar}(\sigma_t) = \int_{\mathbb{R}^{md}} y d\sigma_t(y)$, $v \in L^1(\Omega, \mathbb{R}^{md})$ and $v = \nabla u$.

Finally, if we have in more the condition:

(H_4) F is convex in the gradient variable then

$$\int_\Omega F(t, u(t), \nabla u(t))dt \leq \inf_{u\in H} \int_\Omega F(t, u(t), \nabla u(t))dt.$$

Proof. According to the Prohorov's theorem $(\nabla u_n)_n$ can be considered stable convergent to a Young measure $\sigma_. \in \mathcal{Y}(\mathrm{I\!R}^{md})$. The fiber product lemma assure us that $(u_n, \nabla u_n) \xrightarrow{\mathcal{S}} \delta_{u(.)} \otimes \sigma_.$ and therefore $(\delta_{u(.)}, \sigma_.) \in \bar{H}$.

Then it follows that $\bar{I}(\delta_{u(.)}, \sigma_.) = \lim_{n\to\infty} I(u_n)$ or

$$\int_\Omega \left(\int_{\mathrm{I\!R}^{md}} F(t, u(t), y) d\sigma_t(y) \right) dt = \lim_{n\to\infty} I(u_n) = m.$$

Furthermore, if we suppose that $(\nabla u_n)_n$ is U.I. in $L^1(\Omega, \mathrm{I\!R}^{md})$ then we can even suppose that $(\nabla u_n)_n$ is weakly convergent in $L^1(\Omega, \mathrm{I\!R}^{md})$; the theorem 9 from [9] assure us that there exists $v(t) = \mathrm{bar}(\sigma_t) = \int_{\mathrm{I\!R}^{md}} y d\sigma_t(y)$, almost for all $t \in \Omega$, $v \in L^1(\Omega, \mathrm{I\!R}^{md})$ and the sequence $(\nabla u_n)_n$ is weakly convergent to v. For every $n \in \mathrm{I\!N}$, let us denote $u_n = \left(u_n^1, ..., u_n^m\right)$ and $u = (u^1, ..., u^m)$; then, for every $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega \phi(t) v(t) dt = \lim_{n\to\infty} \int_\Omega \phi(t) \nabla u_n(t) dt =$$

$$= - \lim_{n\to\infty} \int_\Omega \begin{pmatrix} \frac{\partial \phi}{\partial t_1} \\ \vdots \\ \frac{\partial \phi}{\partial t_d} \end{pmatrix} \cdot (u_n^1 \cdots u_n^m) dt = - \int_\Omega \begin{pmatrix} \frac{\partial \phi}{\partial t_1} \\ \vdots \\ \frac{\partial \phi}{\partial t_d} \end{pmatrix} \cdot (u^1 \cdots u^m) dt$$

what shows that $v = \nabla u$ and so $u \in W^{1,1}(\Omega)$.

In the condition (H_4) Jensen's inequality applied to the convex function $F(t, u(t), .)$ gives us

$$\int_\Omega F(t, u(t), \nabla u(t)) dt = \int_\Omega F\left(t, u(t), \int_{\mathrm{I\!R}^{md}} y d\sigma_t(y)\right) dt \le$$

$$\le \int_\Omega \left(\int_{\mathrm{I\!R}^{md}} F(t, u(t), y) d\sigma_t(y) \right) dt = m.$$

Remarks 3.2. 1). The uniformly integrability condition for the gradients sequence $(\nabla u_n)_n$ is not too restrictive; it is accomplished under the typical coerciveness assumption

$$c(\|y\| - 1) \le F(t, x, y), \forall (t, x, y) \in \Omega \times \mathrm{I\!R}^m \times \mathrm{I\!R}^{md}$$

2). In the purely vector case ($m > 1, d > 1$) the convexity of integrand in the gradient variable is too restrictive; a more flexible one is that of quasiconvexity introduced by Morrey.

4 Existence results in Jordan tight conditions

According to the existence theorem of the previous section it is necessary to find the conditions in which a problem of calculus of variations can admit minimizing sequences convergent in measure and for which the gradients sequence is tight; such a condition is that of Jordan finite tight, introduced in [5].

First we explain the terms used in this section.

A d-dimensional interval $I \subseteq \mathbb{R}^d$ is a product of bounded closed intervals of $\mathbb{R}$: $I = \prod_{i=1}^{d}[a_i, b_i]$. An elementary set E is an union of a finite family of nonoverlapping d-dimensional intervals, i.e. $E = \cup_{k=1}^{p} I_k$ 'si $\mu(I_k \cap I_l) = 0$ for every $k, l \in \{1, ..., p\}$ with $k \neq l$.

Let us denote by $\mathcal{E}$ the family of all elementary sets; $\mathcal{E}$ is a ring which generate the σ-algebra of Borel sets on $\mathbb{R}^d$.

Let $\Omega \subseteq \mathbb{R}^d$ be a Lebesgue measurable bounded set, let $\mathcal{A}$ be the family of all Lebesgue measurable subsets of Ω and let μ be the Lebesgue measure in Ω.

Definition 4.1.. A set $H \subseteq \mathcal{M}(\mathbb{R}^m)$ is **Jordan finite tight** if, for every $\varepsilon > 0$, there exist $k > 0$ and a finite subfamily $\mathcal{E}_f \subseteq \mathcal{E}$ with $\mu(E) < \varepsilon$, for every $E \in \mathcal{E}_f$, such that, for any $u \in H$, there exists $E_u \in \mathcal{E}_f$ with $\{t \in \Omega : \|u(t)\| > k\} \subseteq E_u$.

A sequence $(u_n)_n \subseteq \mathcal{M}(\mathbb{R}^m)$ is **Jordan finite tight** if the set $H = \{u_n : n \in \mathbb{N}\}$ is Jordan finite tight.

Remark 4.1. As $\mathcal{E} \subseteq \mathcal{A}$, every Jordan finite tight is tight.

The following theorem gives a justification for the naming of *Jordan finite tight*.

For every $A \subseteq \Omega$ let $\mu_J^*(A) = \inf\{\mu(E) : E \in \mathcal{E}, A \subseteq E\}$ be the Jordan outer measure of A; obviously, $\mu_J^*(A) = 0$ if and only if A is a Jordan-negligible set.

Theorem 4.1.

(i) *$H \subseteq \mathcal{M}(\mathbb{R}^m)$ is Jordan finite tight if and only if, for every $\varepsilon > 0$, there exist $k > 0$ and $\{H_1, ..., H_p\}$ - a finite partition of H, such that, for every $i = 1, ..., p$,*

$$\mu_J^* \left(\cup_{u \in H_i} \{t \in \Omega : \|u(t)\| > k\}\right) < \varepsilon.$$

(ii) *For every $H \subseteq \mathcal{M}(\mathbb{R}^m)$ let*

$$A_H(\infty) = \left\{ t \in \bar{\Omega} : \limsup_{s \to t} \sup_{u \in H} \|u(s)\| = +\infty \right\}.$$

A set $H \subseteq \mathcal{M}(\mathbb{R}^m)$ is Jordan finite tight if and only if, for every $\varepsilon > 0$, there exists a finite partition of H, $\{H_1, ..., H_p\}$, such that, for any $i = 1, ..., p$,

$$\mu_J^* \left(A_{H_i}(\infty)\right) < \varepsilon.$$

Proof. (i) Let $H \subseteq \mathcal{M}(\mathbb{R}^m)$ be Jordan finite tight; for every $\varepsilon > 0$, there exist $k > 0$ and $\mathcal{E}_f = \{E_1, ..., E_p\} \subseteq \mathcal{E}$ with $\mu(E_i) < \varepsilon$, for any $i = 1, ..., p$, such that, for any $u \in H$, there exists $E_u \in \mathcal{E}_f$ with $\{t \in \Omega : \|u(t)\| > k\} \subseteq E_u$. For every $i = 1, ..., p$, let $H_i = \{u \in H : \{t \in \Omega : \|u(t)\| > k\} \subseteq E_i\}$; then $\{H_1, ..., H_p\}$ is the required partition.

Inverse, for every $i = 1, ..., p$, there exists $E_i \in \mathcal{E}$ such that $\mu(E_i) < \varepsilon$ and $\cup_{u \in H_i} \{t \in \Omega : \|u(t)\| > k\} \subseteq E_i$. Then $\mathcal{E}_f = \{E_1, ..., E_p\}$ is the finite subfamily of elementary sets required in the definition 4.1.

(ii) ($\Longrightarrow$): At first we remark that $\limsup_{s \to t} \sup_{u \in H} \|u(s)\| = +\infty$ if and only if there exist a sequence $(u_n)_{n \in \mathbb{N}} \subseteq H$ and a sequence $(s_n)_{n \in \mathbb{N}} \subseteq \Omega$ with $s_n \to t$ such that $\|u_n(s_n)\| \to +\infty$.

Let us suppose that $H \subseteq \mathcal{M}(\mathbb{R}^m)$ is Jordan finite tight; for every $\varepsilon > 0$, there exist $k > 0$ and a finite subfamily $\mathcal{E}_f \subseteq \mathcal{E}$ with $\mu(E) < \varepsilon$, for every $E \in \mathcal{E}_f$, such that, for all $u \in H$, there exists $E_u \in \mathcal{E}_f$ with $(\|u\| > k) \subseteq E_u$. For every $E \in \mathcal{E}_f$ let $H_E = \{u \in H : (\|u\| > k) \subseteq E\}$; then $\{H_E : E \in \mathcal{E}_f\}$ is a finite partition of H.

Let us show that, for all $E \in \mathcal{E}_f, A_{H_E}(\infty) \subseteq E$.

Indeed, for every $t \in A_{H_E}(\infty), \limsup_{s \to t} \sup_{u \in H_E} \|u(s)\| = +\infty$ and then there exist a sequence $(s_n)_{n \in \mathbb{N}} \subseteq \Omega$ with $s_n \to t$ and a sequence

$(u_n)_{n\in\mathbb{N}} \subseteq H_E$ such that $\|u_n(s_n)\| \to +\infty$; we can suppose that, for any $n \in \mathbb{N}, \|u_n(s_n)\| > k$ and then $(s_n)_{n\in\mathbb{N}} \subseteq (\|u_n\| > k) \subseteq E$. As E is closed, $t \in E$.

Then $\mu_J^*(A_{H_E}(\infty)) < \varepsilon$, for every $E \in \mathcal{E}_f$.

($\Longleftarrow$): For every $\varepsilon > 0$, let $\{H_1, ..., H_p\}$ a finite partition of H such that, for any $i = 1, ..., p, \mu_J^*(A_{H_i}(\infty)) < \varepsilon$ and let $E_i \in \mathcal{E}$ such that $A_{H_i}(\infty) \subseteq E_i$ et $\mu(E_i) < \varepsilon$. Obviously, we can suppose that every $A_{H_i}(\infty)$ is contained in the interior $\mathring{E}_i$ of E_i. Then, for any $i = 1, ..., p$, there exists $k_i > 0$ such that, for every $t \in \Omega \setminus \mathring{E}_i$ and for every $u \in H_i, \|u(t)\| \leq k_i$ (if we suppose on the contrary that there exist a sequence $(t_n)_{n\in\mathbb{N}} \subseteq \Omega \setminus \mathring{E}_i$ and a sequence $(u_n)_{n\in\mathbb{N}} \subseteq H_i$ with $\|u_n(t_n)\| > n$, for any $n \in \mathbb{N}$, then $(t_n)_{n\in\mathbb{N}}$ has a subsequence $(t_{k_n})_{n\in\mathbb{N}}$ convergent to $t \in \bar{\Omega} \setminus \mathring{E}_i$ where from $t \in A_{H_i}(\infty)$; but $A_{H_i}(\infty) \subseteq \mathring{E}_i$).

Let $k = \max\{k_1, ..., k_p\}$. For every $u \in H$, there exists $i \in \{1, ..., p\}$ such that $u \in H_i$ thus $(\|u\| > k) \subseteq (\|u\| > k_i) \subseteq E_i$.

Remarks 4.2. (i) Let $u \in \mathcal{M}(\mathbb{R}^m)$; then, according to (ii) of the previous theorem, $\{u\}$ is Jordan finite tight if and only if $A_u(\infty) = \{t \in \bar{\Omega} : \limsup_{s\to t} \|u(s)\| = +\infty\}$ is a Jordan-negligible set. We notice that $A_u(\infty)$ is the set of points in neighborhood of which u is not bounded.

(ii) Let $Q\cap]0, 1[= \{q_0, q_1, ..., q_n, ...\}$ be all the rational points of $]0, 1[$ and let $u :]0, 1[\to \mathbb{R}, u = \sum_{n=0}^{\infty} n \cdot \mathbb{1}_{\{q_n\}}$. Then $H = \{u\}$ is tight. $A_u(\infty) = [0, 1]$ and thus, according to the previous remark, H is not Jordan finite tight.

We can notice that $u = \underline{0}$ almost everywhere and that $H_1 = \{\underline{0}\}$ is a Jordan finite tight set.

Thus being, Jordan finite tight's property rises as a property of sets of measurable functions and not as a property of the set of classes of functions equal almost everywhere.

(iii) For every bounded sequence $(u_n)_n$ in $(L^\infty(\Omega, \mathbb{R}^m), \|\cdot\|_\infty)$, there exists a sequence $(v_n)_n$ Jordan finite tight such that, for any $n \in \mathbb{N}$, $u_n = v_n$ almost everywhere. Indeed, let $k > 0$ such that, for every $n \in \mathbb{N}, \|u_n\|_\infty \leq k$; then, if we define $v_n(t) = \begin{cases} u_n(t), & \|u_n\|_B \leq k \\ 0_B, & \|u_n\|_B > k \end{cases}$, then $(v_n)_n$ is a sequence uniformly bounded and thus $(v_n)_n$ is Jordan finite tight and, for any $n \in \mathbb{N}$, $v_n = u_n$ almost everywhere.

Definition 4.2. For every mapping $u : \Omega \to \mathbb{R}^m$, let us denote

$$L(u,\Omega) = \sup\left\{\frac{\|u(t)-u(s)\|}{\|t-s\|} : t, s \in \Omega, t \neq s\right\};$$

if $L(u,\Omega) < +\infty$, u is a **Lipschitz function** on Ω.

We recall that, if $\Omega\,' \subseteq \mathbb{R}^d$ is a bounded open convex set and if $u \in L^1(\Omega\,', \mathbb{R}^m)$ is a continuous mapping with

$$\nabla u = (\nabla_j u_i)_{\substack{1\le i\le m\\ 1\le j\le d}} \in L^\infty(\Omega\,', \mathbb{R}^{md})$$

then u is a Lipschitz function on $\Omega\,'$ and

$$L(u,\Omega\,') \le \left(\sum_{i=1}^m \sum_{j=1}^d \|\nabla_j u_i\|^2_{L^\infty(\Omega',\mathbb{R})}\right)^{\frac{1}{2}} < +\infty.$$

The following two results are proved in [5].

Proposition 4.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open convex set and let $H \subseteq W^{1,1}(\Omega, \mathbb{R}^m) \cap C(\Omega, \mathbb{R}^m)$ be a tight.*

If, for every $u \in H$, there exists a gradient ∇u such that $\nabla H = \{\nabla u : u \in H\} \subseteq L^1(\Omega, \mathbb{R}^{md})$ is Jordan finite tight, then H is Jordan finite tight also.

Theorem 4.2. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open convex set and let $H \subseteq W^{1,1}(\Omega, \mathbb{R}^m) \cap C(\Omega, \mathbb{R}^m)$ be a tight such that there exists ∇H - Jordan finite tight; then H is relatively compact in the topology of convergence in measure on $\mathcal{M}(\mathbb{R}^m)$.*

The last theorem offers the setting in which we can verify the hypothesis (H_1) of the theorem 3.4. Then we obtain:

Corollary. *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded and convex set , $F : \Omega \times \mathbb{R}^m \times \mathbb{R}^{md} \to \mathbb{R}$ a Carathéodory integrand, $H \subseteq W^{1,1}(\Omega) \cap C(\Omega, \mathbb{R}^m)$ and*

$$m = \inf_{u\in H} \int_\Omega F(t, u(t), \nabla u(t))dt.$$

We suppose that:

(H_1') There exists a minimizing sequence $(u_n)_n$ which is tight such that $(\nabla u_n)_n$ is Jordan finite tight;

(H_2) $(F(., u_n, \nabla u_n))_{n\in\mathbb{N}}$ *is uniformly integrable in* $L^1(\Omega, \mathbb{R})$.

Then there exist $u \in \mathcal{M}(\mathbb{R}^m)$ *and* $\sigma_. \in \mathcal{Y}(\mathbb{R}^{md})$ *such that* $\left(\delta_{u(.)}, \sigma_.\right) \in \bar{H}$ *and*

$$\int_\Omega \left(\int_{\mathbb{R}^{md}} F(t, u(t), y) d\sigma_t(y) \right) dt = \inf_{u\in H} \int_\Omega F(t, u(t), \nabla u(t)) dt.$$

Remark 4.3. We remark that, in the hypothesis that there is a Jordan finite tight application $\varphi : \Omega \to \mathbb{R}_+$ such that

$$\|\nabla u_n(t)\| \leq \varphi(t), \text{ for almost every } t \in \Omega,$$

$(\nabla u_n)_n$ is Jordan finite tight.

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