Boundary feedback stabilization of periodic fluid flows in a magnetohydrodynamic channel

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Problem presentation

We consider a two-dimensional incompressible electrically conducting fluid driven by a pressure gradient and affected by a constant transverse magnetic field $B_0$, evolving in a channel of the form

$$(x, y) \in \mathbb{R} \times (-1, 1)$$
Problem presentation

We consider a two-dimensional incompressible electrically conducting fluid driven by a pressure gradient and affected by a constant transverse magnetic field $B_0$, evolving in a channel of the form

$$(x, y) \in \mathbb{R} \times (-1, 1)$$

The MHD equations in a channel

\[
\begin{align*}
  u_t - \nu \Delta u + uu_x + vv_y + CC_x - CB_y &= p_x, \\
  v_t - \nu \Delta v + uv_x + vv_y + BB_y - BC_x &= p_y, \\
  B_t - \eta \Delta B + uB_x + vB_y - Bu_x - Cu_y &= 0, \\
  C_t - \eta \Delta C + uC_x + vC_y - Bv_x - Cv_y &= 0,
\end{align*}
\]

\(1\)

and initial data $u_o, v_o, B_o, C_o$. 
Introduction

Problem presentation

$2\pi -$periodic conditions in the $x$ coordinate

\[
\begin{align*}
  u(t, x + 2\pi, y) &= u(t, x, y), \\
  v(t, x + 2\pi, y) &= v(t, x, y), \\
  B(t, x + 2\pi, y) &= B(t, x, y), \\
  C(t, x + 2\pi, y) &= C(t, x, y), \\
  p(t, x + 2\pi, y) &= p(t, x, y),
\end{align*}
\]

$t \geq 0, \ x \in \mathbb{R}, \ y \in (-1, 1).$
Introduction

Problem presentation

$2\pi$–periodic conditions in the $x$ coordinate

$u(t, x + 2\pi, y) = u(t, x, y), \ v(t, x + 2\pi, y) = v(t, x, y),$

$B(t, x + 2\pi, y) = B(t, x, y), \ C(t, x + 2\pi, y) = C(t, x, y),$

$p(t, x + 2\pi, y) = p(t, x, y), \ t \geq 0, \ x \in \mathbb{R}, \ y \in (-1, 1).$

Boundary conditions on the walls $y = -1, 1$

$u(t, x, -1) = u(t, x, 1) = v(t, x, -1) = 0, \ v(t, x, 1) = \Psi(t, x),$

$B(t, x, -1) = B(t, x, 1) = C_y(t, x, -1) = C_y(t, x, 1) = 0,$

$C(t, x, 1) = \Xi(t, x), \ t \geq 0, \ x \in \mathbb{R}.$
Previous results in the literature

The most important stabilization results are obtained in the case where the magnetic resistivity $\eta$ is large by Krstic et al. [2] and Munteanu [3].
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In this case, the MHD eqs. (1) transform into the so-called simplified magnetohydrodynamic eqs. (SMHD)

$$
\begin{align*}
\left\{ 
& u_t - \nu \Delta u + uu_x + vu_y + Nu = p_x, \\
& v_t - \nu \Delta v + uv_x = p_y, \\
& u_x + v_y = 0, \quad t \geq 0, \quad x \in \mathbb{R}, \quad y \in (-1, 1) \\
& u(t, 2\pi + x, y) = u(t, x, y), \quad v(t, 2\pi + x, y) = v(t, x, y), \\
& u(t, x, -1) = v(t, x, -1) = 0, \quad u(t, x, 1) = U_c, \quad v(t, x, 1) = V_c.
\end{align*}
$$

(2)
The equilibrium profile is given by

\[ U^e = \frac{1}{B_0 \exp B_0} (\cosh(B_0y) - \cosh B_0) \] and \( V^e \equiv 0 \)

and

\[ B^e = \frac{1}{B_0 \exp B_0} (y \sinh B_0 - \sinh(B_0y)) \] and \( C^e \equiv B_0 \).
The linearized system around the equilibrium profile

\[
\begin{align*}
    u_t - \Delta u + U^e u_x + U^e_y v + B_0 C_x - B_0 B_y - B^e_y C &= p_x, \\
    v_t - \Delta v + U^e v_x + B^e_y B + B^e B_y - B^e C_x &= p_y, \\
    B_t - \Delta B + U^e B_x + B^e_y v - B^e u_x - B_0 u_y - U^e C &= 0, \\
    C_t - \Delta C + U^e C_x - B^e v_x - B_0 v_y &= 0, \\
    u_x + v_y &= 0, \quad B_x + C_y = 0, \quad t \geq 0, \quad x \in \mathbb{R}, \quad y \in (-1, 1), \\
    u(t, x+2\pi, y) &= u(t, x, y), \quad v(t, x+2\pi, y) = v(t, x, y), \\
    B(t, x+2\pi, y) &= B(t, x, y), \quad C(t, x+2\pi, y) = C(t, x, y), \\
    p(t, x+2\pi, y) &= p(t, x, y), \quad t \geq 0, \quad x \in \mathbb{R}, \quad y \in (-1, 1), \\
    u(t, x, -1) &= u(t, x, 1) = v(t, x, -1) = 0, \quad v(t, x, 1) = \Psi(t, x), \\
    B(t, x, -1) &= B(t, x, 1) = C_y(t, x, -1) = C_y(t, x, 1) = 0, \\
    C(t, x, 1) &= \Xi(t, x), \quad t \geq 0, \quad x \in \mathbb{R}.
\end{align*}
\]
Fourier functional settings

Let $L^2(Q)$, $Q = (0, 2\pi) \times (-1, 1)$, be the space of all functions $u \in L^2_{\text{loc}}(\mathbb{R} \times (-1, 1))$, which are $2\pi$-periodic in $x$. These functions are characterized by their Fourier series $u(x, y) = \sum_{k \in \mathbb{Z}} u_k(y) \exp(ikx)$, such that $\sum_{k \in \mathbb{Z}} \int_{-1}^{1} |u_k(y)|^2 \, dy < \infty$. The norm in $L^2(Q)$ is defined as $\|u\| = \left( \frac{2\pi}{\sum_{k \in \mathbb{Z}} |u_k|^2_{L^2((-1,1))}} \right)^{1/2}$. 
Introduction

Fourier functional settings

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$$u(x, y) = \sum_{k \in \mathbb{Z}} u_k(y) \exp(ikx), \ u_k = \overline{u_{-k}}, \ \forall k \in \mathbb{Z},$$
Fourier functional settings

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$$u(x, y) = \sum_{k \in \mathbb{Z}} u_k(y) \exp(i k x), \quad u_k = \overline{u_{-k}}, \quad \forall k \in \mathbb{Z},$$

such that

$$\sum_{k \in \mathbb{Z}} \int_{-1}^{1} |u_k(y)|^2 dy < \infty.$$
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$$u(x, y) = \sum_{k \in \mathbb{Z}} u_k(y) \exp(ikx), \quad u_k = \overline{u_{-k}}, \quad \forall k \in \mathbb{Z},$$

such that

$$\sum_{k \in \mathbb{Z}} \int_0^1 |u_k(y)|^2 dy < \infty.$$

The norm in $L^2(Q)$ is defined as

$$\|u\| := \left(2\pi \sum_{k \in \mathbb{Z}} |u_k|_{L^2(-1, 1)}^2 \right)^{\frac{1}{2}}.$$
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$$H^1_0(-1,1) := \{ v \in H^1(-1,1) : v(-1) = v(1) = 0 \},$$

$$H^2_0(-1,1) := \{ v \in H^2(-1,1) \cap H^1_0(-1,1) : v'(-1) = v'(1) = 0 \}.$$
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$$H^1_0(-1, 1) := \{ \nu \in H^1(-1, 1) : \nu(-1) = \nu(1) = 0 \},$$

$$H^2_0(-1, 1) := \{ \nu \in H^2(-1, 1) \cap H^1_0(-1, 1) : \nu'(-1) = \nu'(1) = 0 \}.$$

Here, $'$ stands for the partial derivative with respect to the $y$-coordinate, that is $\frac{\partial}{\partial y}$. 
Fourier decomposition of the linearized system

We decompose system (3) in Fourier series, to get

\[
\begin{align*}
(u_k)_t &- (-k^2 u_k + u''_k) \\
&+ i k U^e u_k + (U^e)' v_k + i k B_0 c_k - B_0 b'_k - (B^e)' c_k = i k p_k, \\
(v_k)_t &- (-k^2 v_k + v''_k) + i k U^e v_k + (B^e)' b_k + B^e b'_k - i k B^e c_k = p'_k, \\
(b_k)_t &- (-k^2 b_k + b''_k) \\
&+ i k U^e b_k + (B^e)' v_k - i k B^e u_k - B_0 u'_k - (U^e)' c_k = 0, \\
(c_k)_t &- (-k^2 c_k + c''_k) + i k U^e c_k - i k B^e v_k - B_0 v'_k = 0, \\
i k u_k + v'_k = 0, & i k b_k + c'_k = 0, \quad t \geq 0, \quad y \in (-1, 1), \\
b_k(-1) = b_k(1) = c_k(-1) = 0, & u_k(-1) = u_k(1) = v_k(-1) = 0, \\
v_k(1) = \psi_k, & c_k(1) = \xi_k, \\
\end{align*}
\]

with initial data \( u_k^0, \ v_k^0, \ b_k^0, \ d_k^0 \).
Reducing the complexity of the problem

We set

\[ S_{1k} := u_k + b_k, \quad S_{2k} := v_k + c_k \quad \text{and} \quad D_{1k} := u_k - b_k, \quad D_{2k} := v_k - c_k, \]

(5)
Main results

Reducing the complexity of the problem

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\[ S_{1k} := u_k + b_k, \quad S_{2k} := v_k + c_k \quad \text{and} \quad D_{1k} := u_k - b_k, \quad D_{2k} := v_k - c_k, \]

and, of course,

\[ S_{1k}^0 := u_k^0 + b_k^0, \quad S_{2k}^0 := v_k^0 + c_k^0 \quad \text{and} \quad D_{1k}^0 := u_k^0 - b_k^0, \quad D_{2k}^0 := v_k^0 - c_k^0. \]
Reducing the complexity of the problem

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(5)

and, of course,

\[ S^0_{1k} := u^0_k + b^0_k, \quad S^0_{2k} := v^0_k + c^0_k \quad \text{and} \quad D^0_{1k} := u^0_k - b^0_k, \quad D^0_{2k} := v^0_k - c^0_k. \]

(6)

We summ \((4)_1 \) and \((4)_3, \) \((4)_2 \) and \((4)_4, \)

\[
\begin{cases}
(S_{1k})_t - (-k^2 S_{1k} + S_{1k}'') \
+ ik(U^e - B^e)S_{1k} - B_0 S_{1k}' \
+ ikB_0 c_k + ik B^e b_k + (U^e + B^e)' D_{2k} = i k p_k, \\
(S_{2k})_t - (-k^2 S_{2k} + S_{2k}'') + ik(U^e - B^e)S_{2k} + (B^e b_k)' - B_0 v'_k = p'_k, \\
S_{1k}(-1) = S_{1k}(1) = S_{2k}(-1) = 0, \quad S_{2k}(1) = \psi_k + \xi_k.
\end{cases}
\]

(7)
Reducing the complexity of the problem

Then, reducing the pressure from system (7) and using the divergence free condition, we obtain

\[
\begin{cases}
(-S''_{2k} + k^2 S_{2k})_t + S'''''_{2k} + B_0 S''''_{2k} - [2k^2 + ik(U^e - B^e)]S''_{2k} \\
- [ik(U^e - B^e)' + k^2 B_0]S'_{2k} \\
+ [(k^4 + ik^3(U^e - B^e)]S_{2k} + ik[(U^e + B^e)'D_{2k}]
\end{cases}
\]

\[= 0, \ y \in (-1, 1), \]

\[S'_{2k}(-1) = S'_{2k}(1) = S_{2k}(-1) = 0, \ S_{2k}(1) = \psi_k + \xi_k. \quad (8)\]
Reduction of the complexity of the problem

Now, subtracting (4)_3 from (4)_1 and (4)_4 from (4)_2, and reducing the pressure, we get

\[
\begin{cases}
\left(-D''_{2k} + k^2 D_{2k}\right)_t + D''''_{2k} - B_0 D''_{2k} - [2k^2 + ik(U^e + B^e)] D''_{2k} \\
- [ik(U^e + B^e)' - k^2 B_0] D'_{2k} \\
+ [(k^4 + ik^3(U^e + B^e)] D_{2k} + ik[(U^e - B^e)' S_{2k}]
\end{cases} = 0, \ y \in (-1, 1),
\]

\[D'_{2k}(-1) = D'_{2k}(1) = D_{2k}(-1) = 0, \ D_{2k}(1) = \psi_k - \xi_k.\] (9)
We set

\[ S^e := U^e + B^e \quad \text{and} \quad D^e := U^e - B^e. \]  

(10)
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(10)

\[
\begin{cases}
    (-S''_{2k} + k^2 S_{2k})_t + S''''_{2k} + B_0 S''_{2k} - [2k^2 + ikD^e]S''_{2k} \\
    - [ik(D^e)' + k^2 B_0]S'_{2k} \\
    + [(k^4 + ik^3 D^e)S_{2k} + ik[(S^e)'D_{2k}]' \\
    = 0, \ y \in (-1, 1),
\end{cases}
\]

\[
\begin{cases}
    (-D''_{2k} + k^2 D_{2k})_t + D''''_{2k} - B_0 D''_{2k} - [2k^2 + ikS^e]D''_{2k} \\
    - [ik(S^e)' - k^2 B_0]D'_{2k} \\
    + [(k^4 + ik^3 S^e)D_{2k} + ik[(D^e)'S_{2k}]' \\
    = 0, \ y \in (-1, 1),
\end{cases}
\]

\[
\begin{align*}
    S'_{2k}(-1) &= \psi_S^S_{2k}, \\
    S'_{2k}(1) &= \psi_S^{S_{2k}}, \\
    D'_{2k}(-1) &= \psi_D^{D_{2k}}, \\
    D'_{2k}(1) &= \psi_D^{D_{2k}}.
\end{align*}
\]
Main results

We recover $\psi_k$ and $\xi_k$, $\nu_k$ and $c_k$, by using the obvious relations

$$\psi_k = \frac{1}{2}(\psi_k^S + \psi_k^D) \quad \text{and} \quad \xi_k = \frac{1}{2}(\psi_k^S - \psi_k^D),$$

(12)

and

$$\nu_k = \frac{1}{2}(S_{2k} + D_{2k}) \quad \text{and} \quad c_k = \frac{1}{2}(S_{2k} - D_{2k}).$$
Writing (11) in an abstract form

\[ L_k v := -v'' + k^2 v, \quad \mathcal{D}(L_k) = H^2(-1, 1) \cap H^1_0(-1, 1), \quad (13) \]
Writing (11) in an abstract form

\[ L_k v := -v'' + k^2 v, \quad \mathcal{D}(L_k) = H^2(-1, 1) \cap H^1_0(-1, 1), \quad (13) \]

\[ L_k \begin{pmatrix} S \\ D \end{pmatrix} := \begin{pmatrix} L_k S \\ L_k D \end{pmatrix}, \quad \mathcal{D}(L_k) = H^2(-1, 1) \cap H^1_0(-1, 1) \times H^2(-1, 1) \cap H^1_0(-1, 1), \quad (14) \]
Writing (11) in an abstract form

\[ L_k v := -v'' + k^2 v, \quad D(L_k) = H^2(-1, 1) \cap H^1_0(-1, 1), \quad (13) \]

\[ \mathbf{L}_k \left( \begin{array}{c} S \\ D \end{array} \right) := \left( \begin{array}{c} L_k S \\ L_k D \end{array} \right), \quad D(\mathbf{L}_k) = H^2(-1, 1) \cap H^1_0(-1, 1) \times H^2(-1, 1) \cap H^1_0(-1, 1), \quad (14) \]

and

\[ \mathbf{F}_k \left( \begin{array}{c} S \\ D \end{array} \right) := \left( \begin{array}{c} S'''' + B_0 S''' - [2k^2 + ikD^e]S'' \\ - [ik(D^e)' + k^2 B_0]S' + [(k^4 + ik^3 D^e]S + ik[(S^e)' D]' \\ D'''' - B_0 D''' - [2k^2 + ikS^e]D'' \\ - [ik(S^e)' - k^2 B_0]D' + [(k^4 + ik^3 S^e]D + ik[(D^e)' S]' \end{array} \right), \quad (15) \]
Writing (11) in an abstract form

\[ \mathcal{D}(\mathbf{F}_k) = H^4(-1, 1) \cap H^2_0(-1, 1) \times H^4(-1, 1) \cap H^2_0(-1, 1). \]

respectively.
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respectively. Moreover, we define the operator

\[ A_k := F_k L_k^{-1}, \quad \mathcal{D}(A_k) = \left\{ \begin{pmatrix} S \\ D \end{pmatrix} : L_k^{-1} \begin{pmatrix} S \\ D \end{pmatrix} \in \mathcal{D}(F_k) \right\}. \quad (16) \]
Writing (11) in an abstract form

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System (11) can be written as

\[ \frac{d}{dt} L_k \begin{pmatrix} S_{2k} \\ D_{2k} \end{pmatrix} + A_k \left( L_k \begin{pmatrix} S_{2k} \\ D_{2k} \end{pmatrix} \right) = 0. \] (17)
Main results

**Lemma**

The operator $-A_k$ generates a $C_0$- analytic semigroup on $H \times H$ and for each $\lambda \in \rho(-A_k)$ (the resolvent set of $-A_k$), $(\lambda I + A_k)^{-1}$ is compact. Moreover, there exists $L > 0$ such that

$$\sigma(-A_k) \subset \{ \lambda \in \mathbb{C} : \Re \lambda \leq 0 \} , \forall |k| > L.$$ 

Here $\sigma(-A_k)$ is the spectrum of $-A_k$. 

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For $|k| > L$, the solution $S_{2k}, D_{2k}$ to the system (11), with zero boundary control (that is, $\psi^S_k = \psi^D_k = 0, |k| > L$) satisfies

$$\|S_{2k}\|^2 + \|D_{2k}\|^2 \leq C_2 \exp(\gamma_2 t)(\|S^0_{2k}\|^2 + \|D^0_{2k}\|^2), \ t \geq 0, \ (18)$$

for some $C_2, \gamma_2 > 0$. 
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For $|k| > L$, the solution $S_{2k}, D_{2k}$ to the system (11), with zero boundary control (that is, $\psi^S_k = \psi^D_k = 0, \ |k| > L$) satisfies

$$\| S_{2k} \|^2 + \| D_{2k} \|^2 \leq C_2 \exp(\gamma_2 t)(\| S_{0k}^0 \|^2 + \| D_{0k}^0 \|^2), \ t \geq 0, \quad (18)$$

for some $C_2, \gamma_2 > 0$.

Hence, it remains to control system (11) for $0 < |k| \leq L$ only.
Spectral properties

$-A_k$ has a countable set of eigenvalues, denoted by
\[
\{\lambda_j^k\}_{j=1}^{\infty}.
\]
Spectral properties

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\[ \{ \lambda_j^k \}_{j=1}^\infty. \]
Besides, there is only a finite number \(N_k\) of eigenvalues
for which \(\Re \lambda_j^k \geq 0, j = 1, \ldots, N_k\), the unstable eigenvalues.
Spectral properties

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Besides, there is only a finite number $N_k$ of eigenvalues for which $\Re \lambda_j^k \geq 0$, $j = 1, \ldots, N_k$, the unstable eigenvalues. Finally, let

$$\left\{ \phi_j^k := \left( \begin{array}{c} \phi_{1j}^k \\ \phi_{2j}^k \end{array} \right) \right\}_{j=1}^{\infty} \quad \text{and} \quad \left\{ \phi_j^{k*} := \left( \begin{array}{c} \phi_{1j}^{k*} \\ \phi_{2j}^{k*} \end{array} \right) \right\}_{j=1}^{\infty} \quad (19)$$

the corresponding eigenfunctions of the operator $-\mathbf{A}_k$ and its dual $-\mathbf{A}_k^*$, respectively.
Main results

Decomposition of the system

\[ z_k := L_k \begin{pmatrix} S_{2k} \\ D_{2k} \end{pmatrix}, \quad B_k := (\theta_k + \tilde{F}_k)D_k. \] (20)
Decomposition of the system

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With these notations, the system becomes

\[
\begin{cases}
\frac{d}{dt} z_k + A_k z_k = B_k \begin{pmatrix} \psi_k^S \\ \psi_k^D \end{pmatrix}, & t > 0, \\
z_k(0) = z_{0k} := L_k \begin{pmatrix} S_{2k}^0 \\ D_{2k}^0 \end{pmatrix}.
\end{cases}
\quad (21)
\]
Decomposition of the system

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z_k(0) &= z_{0k} := L_k \begin{pmatrix} S_{2k}^0 \\ D_{2k}^0 \end{pmatrix}.
\end{aligned}
\]  

We shall apply the technique developed in Barbu et al. [1].
Decomposition of the system

\[ z_k := L_k \begin{pmatrix} S_{2k} \\ D_{2k} \end{pmatrix}, \quad B_k := (\theta_k + \tilde{F}_k)D_k. \quad (20) \]

With these notations, the system becomes

\[
\begin{cases}
\frac{d}{dt} z_k + A_k z_k = B_k \begin{pmatrix} \psi_S^k \\ \psi_D^k \end{pmatrix}, \quad t > 0, \\
z_k(0) = z_{0k} := L_k \begin{pmatrix} S_{0k}^0 \\ D_{0k}^0 \end{pmatrix}.
\end{cases} \quad (21)
\]

We shall apply the technique developed in Barbu et al. [1].

We denote by \( Z_{u_k}^u := \text{linspan} \left\{ \phi_j^k \right\}_{j=1}^{M_k} \), and

\( Z_{s_k}^s := \text{linspan} \left\{ \phi_j^k \right\}_{j=M_k+1}^{\infty} \).
Decomposition of the system

Denote by $P_{N_k} : H \times H \rightarrow Z_{N_k}^u$, the projection, defined by

$$P_{N_k} := -\frac{1}{2\pi i} \int_{\Gamma} (\lambda I + A_k)^{-1} d\lambda,$$

where $\Gamma$ separates the unstable spectrum from the stable one of $-A_k$. 
Decomposition of the system

We set

\[-A_{N_k}^u := P_{N_k}(-A_k), \quad -A_{N_k}^s := (I - P_{N_k})(-A_k),\]

for the restrictions of \(-A_k\) to \(Z_{N_k}^u\) and \(Z_{N_k}^s\), respectively.

\[(22)\]
Decomposition of the system

We set

\[-A^u_{N_k} := P_{N_k}(-A_k), \quad -A^s_{N_k} := (I - P_{N_k})(-A_k), \quad \text{(22)}\]

for the restrictions of \(-A_k\) to \(Z^u_{N_k}\) and \(Z^s_{N_k}\), respectively. The system (21) can accordingly be decomposed as

\[z_k = z_{N_k} + \zeta_{N_k}, \quad \text{where} \quad z_{N_k} := P_{N_k}z_k, \quad \zeta_{N_k} := (I - P_{N_k})z_k, \quad \text{(23)}\]

where applying \(P_{N_k}\) and \((I - P_{N_k})\) on (21), we obtain
Decomposition of the system

We set

$$-A_{N_k}^u := P_{N_k}(-A_k), \quad -A_{N_k}^s := (I - P_{N_k})(-A_k),$$

(22)

for the restrictions of $-A_k$ to $Z_{N_k}^u$ and $Z_{N_k}^s$, respectively. The system (21) can accordingly be decomposed as

$$z_k = z_{N_k} + \zeta_{N_k} \quad \text{where} \quad z_{N_k} := P_{N_k}z_k, \quad \zeta_{N_k} := (I - P_{N_k})z_k,$$

where applying $P_{N_k}$ and $(I - P_{N_k})$ on (21), we obtain

on $Z_{N_k}^u$: \[
\frac{d}{dt}z_{N_k} + A_{N_k}^u z_{N_k} = P_{N_k}B_k \begin{pmatrix}
\psi_k^S \\
\psi_k^D
\end{pmatrix}, \quad z_{N_k}(0) = P_{N_k}z_{0k},
\]

(23)
Decomposition of the system

We set

\[-A_{N_k}^u := P_{N_k}(-A_k), \quad -A_{N_k}^s := (I - P_{N_k})(-A_k), \quad (22)\]

for the restrictions of \(-A_k\) to \(Z_{N_k}^u\) and \(Z_{N_k}^s\), respectively.

The system (21) can accordingly be decomposed as

\[z_k = z_{N_k} + \zeta_{N_k} \text{ where } z_{N_k} := P_{N_k}z_k, \quad \zeta_{N_k} := (I - P_{N_k})z_k,\]

where applying \(P_{N_k}\) and \((I - P_{N_k})\) on (21), we obtain

on \(Z_{N_k}^u\):

\[\frac{d}{dt}z_{N_k} + A_{N_k}^u z_{N_k} = P_{N_k}B_k \begin{pmatrix} \psi_k^S \\ \psi_k^D \end{pmatrix}, \quad z_{N_k}(0) = P_{N_k}z_{0k}, \quad (23)\]

on \(Z_{N_k}^s\):

\[\frac{d}{dt}\zeta_{N_k} + A_{N_k}^s \zeta_{N_k} = (I - P_{N_k})B_k \begin{pmatrix} \psi_k^S \\ \psi_k^D \end{pmatrix}, \quad \zeta_{N_k}(0) = (I - P_{N_k})z_{0k}, \quad (24)\]
The unique continuation property

Let us denote by $\lambda := \lambda^k_j$, and the corresponding eigenfunction $\phi^* = \phi^k_j$. 
The unique continuation property

Let us denote by $\lambda := \lambda^k_j$, and the corresponding eigenfunction

$$
\phi^* = \phi^*_j.
$$

$$
\begin{align*}
(\phi^*_1)''' - B_0(\phi^*_1)''' & - [2k^2 - ikD^e + \bar{\lambda}](\phi^*_1)'' + [ik(D^e)' + k^2B_0](\phi^*_1)' \\
& + [k^4 - ik^3D^e + k^2\bar{\lambda}]\phi^*_1 + ik(D^e)'(\phi^*_2)' = 0,
\end{align*}
$$

$$
\begin{align*}
(\phi^*_2)''' + B_0(\phi^*_2)''' & - [2k^2 - ikS^e + \bar{\lambda}](\phi^*_2)'' + [ik(S^e)' - k^2B_0](\phi^*_2)' \\
& + [k^4 - ik^3S^e + k^2\bar{\lambda}]\phi^*_2 + ik(S^e)'(\phi^*_1)' = 0, \quad y \in (-1, 1)
\end{align*}
$$

$$
\begin{align*}
\phi^*_1(-1) = \phi^*_1(1) = 0, & \quad \phi^*_2(-1) = \phi^*_2(1) = 0, \quad (\phi^*_1)'(-1) = (\phi^*_1)'(1) = 0 \\
(\phi^*_2)'(-1) = (\phi^*_2)'(1) = 0,
\end{align*}
$$

(25)
The unique continuation property

Let us denote by \( \lambda := \lambda_j^k \), and the corresponding eigenfunction \( \phi^* = \phi_j^{k*} \).

\[
\begin{align*}
\left\{ \begin{array}{l}
(\phi_1^*)''' - B_0 (\phi_1^*)'' - [2k^2 - i k D^e + \overline{\lambda}](\phi_1^*)'' + [i k (D^e)' + k^2 B_0](\phi_1^*)' \\
+ [k^4 - i k^3 D^e + k^2 \overline{\lambda}] \phi_1^* + i k (D^e)'(\phi_2^*)' = 0,
\end{array} \right.

(\phi_2^*)''' + B_0 (\phi_2^*)'' - [2k^2 - i k S^e + \overline{\lambda}](\phi_2^*)'' + [i k (S^e)' - k^2 B_0](\phi_2^*)' \\
+ [k^4 - i k^3 S^e + k^2 \overline{\lambda}] \phi_2^* + i k (S^e)'(\phi_1^*)' = 0, \quad y \in (-1, 1),
\end{align*}
\]

\[
\begin{align*}
\phi_1^*(-1) = \phi_1^*(1) = 0, & \quad \phi_2^*(-1) = \phi_2^*(1) = 0, \quad (\phi_1^*)'(-1) = (\phi_1^*)'(1) = 0, \\
(\phi_2^*)'(-1) = (\phi_2^*)'(1) = 0, & \quad \text{If } (\phi^*)'''(1) = 0 \text{ then } \phi^* \equiv 0.
\end{align*}
\]
The stabilization result

**Theorem**

There exist finite-dimensional feedback controllers $\Psi$ and $\Xi$, of the form

$$\Psi(t, x) = -\sum_{0 < |k| \leq L} \left( L_k^{-2} R_k L_k \left( \begin{array}{c} v_k + c_k \\ v_k - c_k \end{array} \right)(t) \right)^{'''} (1) \cdot \left( \frac{1}{2} \right) \exp(i k x),$$

and

$$\Xi(t, x) = -\sum_{0 < |k| \leq L} \left( L_k^{-2} R_k L_k \left( \begin{array}{c} v_k + c_k \\ v_k - c_k \end{array} \right)(t) \right)^{'''} (1) \cdot \left( \frac{1}{2} \right) \exp(i k x).$$

(26) and (27)
Boundary feedback stabilization of periodic fluid flows in a magnetohydrodynamic channel

Main results

The stabilization result

Theorem

where

\[ v_k(t, y) = \int_0^{2\pi} v(t, x, y) \exp(-ikx) dx , \quad c_k(t, y) = \int_0^{2\pi} C(t, x, y) \exp(-ikx) dx , \]

such that, once inserted into equation (3), the corresponding solution of the closed-loop system (3) satisfies

\[ \| (u(t), v(t), b(t), c(t)) \|^2 \leq C \exp(-\alpha t) \| (u^0, v^0, b^0, c^0) \|^2 , \quad t \geq 0 , \]

for some \( C, \alpha > 0 \).
The stabilization result

**Theorem**

Here, $R_k : \mathcal{X} \to \mathcal{X}$ are linear self-adjoint operators such that

1. $R_k : H \times H \to H \times H$,
2. $R_k$ satisfy Riccati algebraic equations in $H \times H$

\[
L_k^{-1} R_k z_{0k}, L_k^{-1} A_k z_{0k} + \frac{1}{2} |(L_k^{-2} R_k z_{0k})'''(1)|^2 = \frac{1}{2} \| L_k^{-1} z_{0k} \|^2, \forall z_{0k} \in H
\]

for all $0 < |k| \leq L$, $L$ given in Lemma 1. $A_k = F_k L_k^{-1}$, where $F_k$ is given by (15) and $L_k$ is given by (14).

$\mathcal{X} = (H^2(-1, 1) \cap H^1_0(-1, 1)) \times H^2(-1, 1) \cap H^1_0(-1, 1)$. Finally, $\cdot$ stands for the scalar product in $\mathbb{C}^2$. 
References


- I. Munteanu, *Feedback stabilization and observer design for linearized MHD channel flow at low magnetic Reynolds number* (sent for publication).
Thank You!