CERTAIN SUFFICIENT CONDITIONS
FOR MEROMORPHIC CLOSE–TO–STAR FUNCTIONS

BY

B.A. URALEGADDI and A.R. DESAI

Abstract. Some sufficient conditions for meromorphic close–to–star functions are obtained.

1. Introduction. Let Σ denote the class of functions of the form

\[ f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \]

which are regular in the punctured disk

\[ E = \{ z : 0 < |z| < 1 \} \]

with a simple pole at \( z = 0 \). For functions in \( \Sigma \) let

\[ D^n f(z) = \frac{1}{z(1-z)^{n+1}} \ast f(z) = \frac{1}{z} + (n+1)a_0 + \frac{(n+1)(n+2)}{2!}a_1z + \cdots \]

\[ n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \]

and \( z \in U = \{ z : |z| < 1 \} \) where \( \ast \) denotes the Hadamard product. The symbol \( D^n f(z) \) which is referred as the \( n^{th} \) order Ruscheweyh type derivate of \( f \) was introduced by GANIGI and URALEGADDI [1]. Denote \( \Sigma^* \) and \( \Sigma_S \) the subclasses of \( \Sigma \) consisting of functions which are starlike and univalent respectively.
A function

\[ g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \]

in \( E \) is said to be starlike if \( \text{Re}(zg'(z)/g(z)) < 0 \), for \( z \in U \). A function \( f \in \Sigma \) is said to be close–to–star of order \( \alpha \) \((0 \leq \alpha < 1)\) in \( E \) if there exists a meromorphic starlike function \( g \in \Sigma^* \) such that \( \text{Re}(f(z)/g(z)) > \alpha \) for \( z \in U \), and is said to be close–to–star if \( \text{Re}(f(z)/g(z)) > 0 \) for \( z \in U \). The concept of meromorphic close–to–star was introduced by READE [4].

In this paper we obtain sufficient conditions for functions in \( \Sigma \) to be close–to–star. In [5], [6] SARANGI and SUGUNA B. URALEGGADI have obtained some sufficient conditions for functions in \( \Sigma \) to be close–to–convex.

We need the following lemma due to JACK [2].

**Lemma.** Let \( w \) be nonconstant and regular in the unit disk \( U \), \( w(0) = 0 \). If \( |w| \) attains its maximum value on the circle \( |z| = r < 1 \) at \( z_0 \), we have

\[ z_0 w'(z_0) = kw(z_0) \]

where \( k \) is real number and \( k \geq 1 \).

**2. Some sufficient conditions for close–to–starlikeness.**

**Theorem 1.** If \( f \in \Sigma \) satisfies

\( (1) \quad \text{Re} \frac{zf'(z)}{f(z)} > -3/2, \ z \in U \)

then \( f \) is meromorphically close–to–star of order \( 1/2 \) in \( U \).

**Proof.** It suffices to show that (1) implies

\[ \left| \frac{zf(z) - 1}{zf(z)} \right| < 1 \ \text{in} \ U. \]

Define \( w \) in \( U \) by

\[ w(z) = \frac{zf(z) - 1}{zf(z)} \]

which is same as
(2) \[ zf(z) = \frac{1}{1 - w(z)}. \]

Clearly, \( w(0) = 0 \) and \( w \) is regular in \( U \). Differentiating (2) logarithmically and simplifying we get

(3) \[ \frac{zf'(z)}{f(z)} = \frac{zw'(z)}{1 - w(z)} - 1. \]

Now we claim that \(|w(z)| < 1\). For otherwise by Jack’s lemma there exists \( z_0, |z_0| < 1 \) such that \(|w(z_0)| = 1, z_0w'(z_0) = kw(z_0), k \geq 1\). Then from (3) we have

\[ \frac{w_0f'(z_0)}{f(z_0)} = kw(z_0) - 1. \]

Thus we have

\[ \text{Re}(z_0f'(z_0)/f(z_0)) \leq -3/2 \]

which contradicts (1). Hence the result follows.

In the next theorem we shall show that for \( n \in N_0, p_{n+1} \subset p_n \), where \( p_n \) is the subclass of functions in \( \Sigma \) satisfying the condition

\[ \text{Re}\left\{ \frac{D^{n+2}f(z)}{D^{n+1}f(z)} \right\} > \frac{2n + 3}{2n + 4}, \]

\( z \in U \).

Since \( p_0 \) is the subclass of functions \( f(z) \) satisfying \( \text{Re}(zf'(z)/f(z)) > -3/2 \) it follows from Theorem 1 that all functions in \( p_n \), for \( n \in N_0 \) are close–to–star.

**Theorem 2.** \( p_{n+1} \subset p_n, n \in N_0 \).

**Proof.** Let \( f(z) \in p_{n+1} \). Then

(4) \[ \text{Re}\left\{ \frac{D^{n+2}f(z)}{D^{n+1}f(z)} \right\} > \frac{2n + 3}{2n + 4}. \]

We shall show that the inequality (4) implies
Define \( w(z) \) in \( U \) by

\[
\frac{D^{n+1}f(z)}{D^nf(z)} = 2n + 1 + \frac{1}{2n+2} \cdot \frac{1 - w(z)}{1 + w(z)} = \frac{n + 1 + nw(z)}{(n+1)(1 + w(z))}
\]

Clearly \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). Differentiating (5) logarithmically and using the following identity [1]

\[
z(D^n f(z))' = (n + 1)D^{n+1}f(z) - (n + 2)D^n f(z)
\]

We have

\[
\frac{D^{n+1}f(z)}{D^{n+1}f(z)} = \frac{z w'(z)}{n + 1 + nw(z)(1 + w(z))}
\]

Now we claim that \(|w(z)| < 1\). For otherwise by Jack’s lemma there exists \( z_0, |z_0| < 1 \) such that \(|w(z_0)| = 1, z_0 w'(z_0) = kw(z_0), k \geq 1\).

Then, from (7) we have

\[
\frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} = \frac{kw(z_0)}{(n + 1 + nw(z_0))(1 + w(z_0))}
\]

Then we have

\[
\text{Re} \frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} \leq \frac{4n^2 + 8n + 2}{(2n + 4)(2n + 1)} < \frac{2n + 3}{2n + 4}
\]

which contradicts (4). The result follows.
Using the techniques employed by OBRADOVIC [3] we shall prove the following results.

**Theorem 3.** Let \( f \in \Sigma, \frac{\alpha}{2n+4} \geq \frac{\beta}{2n+1} \geq 0, n \in N_0 \) and let

\[
\operatorname{Re} \left\{ \alpha \frac{D^{n+2}f(z)}{D^{n+1}f(z)} + \beta \frac{D^{n}f(z)}{D^{n+1}f(z)} \right\} > \alpha \left( \frac{2n+3}{2n+4} \right) + \beta \left( \frac{2n+2}{2n+1} \right),
\]

\( z \in U, \) then \( f \in p_n. \) Hence \( f \) is meromorphic close-to-star of order \( 1/2. \)

**Proof.** Let \( f \in \Sigma \) satisfy the condition (8). We shall prove that \( f \in p_n, \) i.e.

\[
\operatorname{Re} \frac{D^{n+1}f(z)}{D^{n}f(z)} > \frac{2n+1}{2n+2}, \ z \in U.
\]

Define \( w(z) \) in \( U \) by

\[
D^{n+1}f(z) = 2n+1 + \frac{1}{2n+2} \cdot \frac{1-w(z)}{1+w(z)} = \frac{n+1+nw(z)}{(n+1)(1+w(z))}
\]

Proceeding as in Theorem 2, we obtain

\[
\frac{D^{n+2}f(z)}{D^{n+1}f(z)} =
\]

\[
= \frac{1}{n+2} \left[ \frac{2n+3}{2} + \frac{1}{2} \cdot \frac{1-w(z)}{1+w(z)} - \frac{zw'(z)}{(n+1+nw(z))(1+w(z))} \right]
\]

Hence we have

\[
\alpha \frac{D^{n+2}f(z)}{D^{n+1}f(z)} + \beta \frac{D^{n}f(z)}{D^{n+1}f(z)} =
\]

\[
= \frac{\alpha}{n+2} \left[ \frac{2n+3}{2} + \frac{1}{2} \cdot \frac{1-w(z)}{1+w(z)} - \frac{zw'(z)}{(n+1+nw(z))(1+w(z))} \right] +
\]

\[
+ \beta \left[ \frac{(n+1)(1+w(z))}{(n+1+nw(z))} \right]
\]
Now we claim that $|w(z)| < 1$. For otherwise by Jack’s lemma there exists $z_0$, $|z_0| < 1$ such that $|w(z_0)| = 1$, $z_0 w'(z_0) = kw(z_0), k \geq 1$. Then from (11) we have

$$\frac{\alpha D^{n+2}f(z_0)}{D^{n+1}f(z_0)} + \frac{\beta D^nf(z_0)}{D^{n+1}f(z_0)} =$$

$$= \frac{\alpha}{n+2} \left[ \frac{2n+3}{2} + \frac{1}{2} \cdot \frac{1-w(z_0)}{1+w(z_0)} \right] +$$

$$+ \beta \left[ \frac{(n+1)(1+w(z_0))}{(n+1+nw(z_0))} \right]$$

Thus we have

$$\Re \left\{ \frac{\alpha D^{n+2}f(z_0)}{D^{n+1}f(z_0)} + \frac{\beta D^nf(z_0)}{D^{n+1}f(z_0)} \right\} \leq$$

$$\leq \frac{\alpha}{n+2} \frac{4n^2 + 8n + 2}{(2n+4)(2n+1)} + \frac{\beta}{2n+1} \frac{2n+2}{2n+1} < \frac{2n+3}{2n+4} \geq 0$$

which contradicts (8). It follows that $f \in p_n$. Hence $f$ is close–to–star of order $1/2$. Since for $z = 0$ the left hand side of (8) have the value $\alpha + \beta$, the condition $\frac{\alpha}{2n+4} \geq \frac{\beta}{2n+1} \geq 0$ is necessary.

**Theorem 4.** Let $f \in \Sigma$, $\alpha \geq 0$, $\beta \geq 0$, $n \in N_0$. If

$$\left| \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 \right|^\alpha \left| \frac{D^{n+1}f(z)}{D^nf(z)} - 1 \right|^\beta <$$

$$< \left( \frac{n+1}{(n+2)(2n+1)} \right)^\alpha 2^{-\beta(n+1)}$$

then $f \in p_n$. Hence $f$ is meromorphic close–to–star of order $1/2$.

**Proof.** Let $f \in \Sigma$ satisfy the inequality (12). Proceeding as in Theorem 3, from (9) and (10) we have
\[
\left| \frac{D^{n+2} f(z)}{D^{n+1} f(z)} - 1 \right|^\alpha \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right|^\beta = (n+2)^{-\alpha(n+1)} - \beta \left| \frac{1}{2} \frac{1 - w(z)}{1 + w(z)} + \frac{zw'(z)}{(n+1+nw(z))(1+w(z))} \right|^\alpha \frac{w(z)}{1+w(z)} \left| w(z) \right|^{\beta}
\]

(13)

Now we claim that \(|w(z)| < 1\). For otherwise by Jack's lemma there is a \(z_0\), \(|z_0| < 1\) such that \(|w(z_0)| = 1\), \(z_0 w'(z_0) = kw(z_0), k \geq 1\). Then from (13) we have

\[
\left| \frac{D^{n+2} f(z_0)}{D^{n+1} f(z_0)} - 1 \right|^\alpha \left| \frac{D^{n+1} f(z_0)}{D^n f(z_0)} - 1 \right|^\beta = (n+2)^{-\alpha(n+1)} - \beta \left| \frac{1}{2} \frac{1 - w(z_0)}{1 + w(z_0)} + \frac{kw(z_0)}{(n+1+nw(z_0))(1+w(z_0))} \right|^\alpha \frac{w(z_0)}{1+w(z_0)} \left| w(z_0) \right|^{\beta}
\]

(14)

\[
\text{Re} \left\{ \frac{1}{2} - \frac{1}{2} \frac{1 - w(z_0)}{1 + w(z_0)} + \frac{kw(z_0)}{(n+1+nw(z_0))(1+w(z_0))} \right\} \geq \frac{1}{2} + \frac{1}{2(2n+1)} = \frac{n+1}{2n+1}
\]

and

\[
\text{Re} \left\{ \frac{w(z_0)}{1 + w(z_0)} \right\} = \frac{1}{2}
\]

Since \(|z| \geq |\text{Re} z|\) for all \(z\), we have from (14)

\[
\left| \frac{D^{n+2} f(z_0)}{D^{n+1} f(z_0)} - 1 \right|^\alpha \left| \frac{D^{n+1} f(z_0)}{D^n f(z_0)} - 1 \right|^\beta = (n+2)^{-\alpha(n+1)} - \beta \left( \frac{n+1}{2n+1} \right)^\alpha 2^{-\beta} = \left( \frac{n+1}{(n+1)(2n+1)} \right)^\alpha 2^{-\beta(n+1)^{-\beta}}.
\]

which contradicts (12). Hence \(|w(z)| < 1\) and \(f \in p_n\).
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*Department of Mathematics
Karnatak University
Dharwad – 580 003
INDIA

*Department of Mathematics
S.D.M. College of Engg.& Tech.
Dharwad – 580 002
INDIA