MORPHISMS OF SEMI–DYNAMICAL SYSTEMS
ON ALGEBRAIC STRUCTURES

BY

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Introduction. The notion of semi–dynamical system arose from the study of ordinary differential equations and was successively generalised by several authors [12], [1]. BUCUR–BEZZARGA [4] seem to be the first to accept the ramification points. In this way, some specific notions from potential theory were introduced within the framework of semi–dynamical systems (for example, excessive functions). My papers [14], [15], [16], continued the research in this direction, introducing and studying further notions, such as: entrance and exit boundaries, harmonic functions and potentials, morphisms, Green function, time changes and so on. Even after these developments, it seems a surprising result that this theory, with a purely deterministic character (at least at first sight), by its starting point, includes all the situations which appear in the axiomatic potential theory (hence essentially probabilistic). I defined the notion of semi–dynamical system on an algebraic structure (s.d.s.a.s.) \((X, \Phi)\), imposing that the set \(X\) possess a structure of ordered convex cone, and the map \(\Phi : [0, +\infty) \times X \to X\) has some natural properties of compatibility with the algebraic structure. A good duality theory can be developed (this was not available within the theory of general semi–dynamical systems, see [5]), each such system possessing a canonical dual of the same type. An abstract version for the potential theory associated with a kernel or with a resolvent family of kernels, is proposed in [8]. The definition of s.d.s.a.s. develops the analogous abstract theory for the semi–groups of kernels. It was showed that the following situations are subsumed to this notion: the semi–groups of kernels (on measurable spaces) [9], as well as of linear, positive operators (on linear ordered spaces). Particularly, with each semi–dynamical system one associates, using this procedure,
a semi–dynamical system on an algebraic structure. The notion of excessive point was also considered [18], in analogy (and generalizing), the notion of excessive function.

The notion of a morphism between harmonic structures was already studied by many authors. The firsts to consider this notions were CONSTANTINESCU and CORNEA [7]; its study was continued, among others, by: SIBONY [19], FUGLEDE [11], BOBOC–BUCUR–CORNEA [3], ERIKSSON [10]. In 1980 [13], I begin the study of the notion of morphisms between standard H–cones of functions.

The framework considered in this paper belongs to the axiomatic potential theory; however, it is not contained in any of the previous theories, having its own characteristics. We present various classes and examples of morphisms between semi–dynamical systems, such as: the structure morphisms, the elements from the dual, the intertwining morphisms. We consider as well the adjoint of a morphism.

The set of morphisms between two systems receives naturally the structure of a bi–system; from this structure can be derived next various system structures. A canonical structure of a system exists on the set of intertwining morphisms.

Once the morphisms are defined, one can consider various categorical constructions. Thus, we study: direct sums and products; sub–systems and quotient systems (i.e. equivalence relations on systems); projective and inductive limits of systems (the latter exist only under additional assumptions).

1. Preliminaries.

**Definition 1.** ([17]) A semi–dynamical system on an algebraic structure (s.d.s.a.s. for short) is an application $\Phi : [0, +\infty) \times X \to X$ with the following properties:

- $\Phi(s, \Phi(t, x)) = \Phi(s + t, x), \forall x \in X, \forall s, t \geq 0$
- $\Phi(t, x + x') = \Phi(t, x) + \Phi(t, x'), \forall x, x' \in X, \forall t \geq 0$
- $\Phi(t, \lambda x) = \lambda \Phi(t, x), \forall x \in X, \forall \lambda \geq 0$
- $x \leq x' \implies \Phi(t, x) \leq \Phi(t, x'), \forall t \geq 0$
- $x_n \nrightarrow x \implies \Phi(t, x_n) \nrightarrow \Phi(t, x), \forall t \geq 0$

With $\Phi$ we associate a pre–order relation on $X$, denoted by $\leq_{\Phi}$ and defined as:

$$x \leq_{\Phi} y \iff \exists t \geq 0 \text{ such that } y = \Phi(t, x)$$
**Definition 2.** ([17]) \( x \in X \) is called supermedian if \( \Phi(t, x) \leq x, \forall t \geq 0. \)

\( X_s \) will denote the set of supermedian elements.

Let us remark that, for \( x \in X_s \), the map \( t \mapsto \Phi(t, x) \) is decreasing. Indeed, for \( t \leq t' \), we may write \( t' = t + h \), where \( h \geq 0. \) It follows that:

\[
\Phi(t', x) = \Phi(t, \Phi(h, x)) \leq \Phi(t, x)
\]

Hence, for any decreasing sequence \( t_n \downarrow 0 \), the sequence \( (\Phi(t_n, x))_n \) is increasing and dominated by \( x \), hence there exists \( \bigvee_n \Phi(t_n, x) \). For any other decreasing sequence \( s_n \downarrow 0 \), \( \bigvee_n \Phi(s_n, x) \) is the same, since \( \forall n \exists m \) such that \( s_m \leq t_n \) (and conversely). Hence, we can use the notation:

\[
\tilde{x} := \bigvee_{t \geq 0} \Phi(t, x)
\]

Clearly \( \tilde{x} \leq x \). We have \( \tilde{x} \in X_s \). Indeed:

\[
\Phi(s, \tilde{x}) = \bigvee_n \Phi(s, \Phi(t_n, x)) = \bigvee_n \Phi(t_n, \Phi(s, x)) \leq \bigvee_n \Phi(t_n, x) = \tilde{x}, \ \forall s \geq 0
\]

In the same way, we can consider:

\[
\tilde{\tilde{x}} = \bigvee_n \Phi(t_n, \tilde{x}) = \bigvee_n \Phi(t_n, \bigvee_m \Phi(s_m, x)) = \bigvee_{n,m} \Phi(t_n + s_m, x) = \tilde{x}
\]

since \( t_n + s_m \downarrow 0. \)

**Definition 3.** ([17]) \( x \in X \) is called excessive if \( x \in X_s \) and \( x \leq \tilde{x}. \)

We denote \( X_e \) the set of excessive elements.

\( X_s \) and \( X_e \) are convex subcones, in which exists the upper bound for increasing and dominated sequences. Moreover, these are absorbent parts for the relation \( \leq \Phi \) (that means: \( x \in X_s \) resp. \( X_e \Longrightarrow \Phi(t, x) \in X_s \) resp. \( X_e \)).

**Example.** Let \( X \) be an ordered convex cone, in which:

\[
(*) \ a_n \not\rightarrow a \Longrightarrow \bigvee_n (a_n, x) = a x, \ \forall x \in X
\]
For each $\alpha \in \mathbb{R}$ we define $\Phi(t, x) := e^{\alpha t} x$. In this way, we obtain a s.d.s.a.s.
If $\alpha \leq 0$, then $X = X_s = X_e$ while, for $\alpha > 0$, $X_s = X_e = \emptyset$.

2. Morphisms of s.d.s.a.s.: definitions and examples.

Definition 4. Let $X, Y$ be s.d.s.a.s. We call a pre–morphism (of s.d.s.a.s.) any map $\varphi : Y \to X$, with the properties:

$\varphi(0) = 0$
$\varphi(y + y') = \varphi(y) + \varphi(y')$
$y \leq y' \implies \varphi(y) \leq \varphi(y')$
$y_n \uparrow y \implies \varphi(y_n) \uparrow \varphi(y)$

Any pre–morphism $\varphi$ has an adjoint, denoted $\varphi^* : X^* \to Y^*$, (for the notation $X^*$ see [17]) defined as: for $\mu \in X^*$ one defines

$\varphi^*(\mu) : Y \to [0, +\infty]$ by $\varphi^*(\mu)(y) = \mu(\varphi(y))$

The definition is correct; $\varphi^*(\mu) \in Y^*$, and $\varphi^* : X^* \to Y^*$ is a pre–morphism (called the canonical adjoint).

More generally, if $X, X_1$, resp. $Y, Y_1$ are in duality, and $\varphi : Y \to X$ is a pre–morphism, we say that $\varphi^* : X_1 \to Y_1$ is an adjoint for $\varphi$, if:

$[\varphi(y), x] = [y, \varphi^*(x)], \forall y \in Y, x \in X_1$

If the duality between $X$ and $X_1$ is separated, then the adjoint, if it exists, is unique.

Definition 5. A pre–morphism $\varphi$ is called a morphism (of s.d.s.a.s.), if it has the properties:

$y \in Y_e \implies \varphi(y) \in X_e$ and $\mu \in (X^*)_e \implies \varphi^*(\mu) \in (Y^*)_e$

The composition of two (pre–)morphisms is clearly a (pre–)morphism.

Examples. a) A pre–morphism $\varphi$ is called intertwining [2], if it satisfies the relation:

(I) $\Phi(t, \varphi(y)) = \varphi[\Psi(t, y)], \forall t > 0, y \in Y$
This property is inherited by the adjoint:

$$
\Psi^*(t, \varphi^*(\mu))(y) = \varphi^*(\mu) [\Psi(t, y)] = \mu [\varphi(\Psi(t, y))] = \mu [\Phi(t, \varphi(y))] = \\
= \Phi^*(t, \mu)(\varphi(y)) = \varphi^* [\Phi^*(t, \mu)](y)
$$

From (1), it follows immediately that $y \in Y_s$ resp. $Y_e \implies \varphi(y) \in X_s$ resp. $X_e$.

Hence, any intertwining morphism is a morphism.

Each intertwining morphism induces a morphism of $H$–cones (cf.[15]) from $E_\Phi$ to $E_\Psi$ defined as: $s \mapsto s \circ \varphi$. Indeed, it suffices to show that $s \circ \varphi \in E_\Psi$:

$$
[s \circ \varphi] [\Psi(t, x)] = s [\Phi(t, \varphi(x))] \leq (s \circ \varphi)(x)
$$

b) The structure morphisms. For each $t \geq 0$ let us denote $\varphi_t : X \to X$ defined as: $\varphi_t(x) = \Phi(t, x)$. $\varphi_t$ is an intertwining morphism (see [17] for a verification). The fact that $\varphi_t^*(\mu) = \Phi^*(t, \mu)$ shows that $\varphi_t^*$ is a structure morphism for $X^*$. 

c) Clearly, each $\mu \in X^*$ is a pre–morphism. If we organize $[0, +\infty]$ as a s.d.s.a.s. with: $\Phi(t, x) = e^{\alpha t}.x$ (for fixed $\alpha \leq 0$), then each $\mu \in (X^*)_e$ becomes a morphism. Indeed, it is easy to prove that $[0, +\infty]^*$ identifies canonically with $[0, +\infty]$, and $\mu^*$ is given by: $\mu^*(\lambda) = \lambda.\mu$. However, $\mu$ is an intertwining morphism if and only if $\mu [\Phi(t, x)] = e^{\alpha t}.\mu(x)$.

3. The structure of the space of morphisms. The set $H$ of the morphisms from $X$ to $Y$ is naturally organised in the following way:

$$
F : [0, +\infty) \times [0, +\infty) \times H \to H
$$

where:

$$
F(s, t; \varphi)(x) = \Phi(t, \varphi [\Psi(s, x)])
$$

has the properties of what can be named a bi–system:
\[ F(s + s', t + t'; \varphi) = F(s, t; F(s', t'; \varphi)) \]

\[ F(s, t; \varphi + \varphi') = F(s, t; \varphi) + F(s, t; \varphi') \]

\[ F(s, t; \lambda \varphi) = \lambda F(s, t; \varphi) \]

\[ \varphi \leq \varphi' \implies F(s, t; \varphi) \leq F(s, t; \varphi') \]

\[ \varphi_n \not\leq \varphi \implies F(s, t; \varphi_n) \not\leq F(s, t; \varphi) \]

Indeed:

\[ F(s, t; F(s', t'; \varphi))(x) = \Phi(t, F(s', t'; \varphi)[\Psi(s, x)]) = \]

\[ = \Phi(t, \Phi(t', \varphi[\Psi(s', \Psi(s, x))])) = \]

\[ = \Phi(t + t', \varphi[\Psi(s + s', x)]) = F(s + s', t + t'; \varphi)(x) \]

\[ F \text{ possesses the usual compatibility properties with the algebraic structure.} \]

If \( \varphi \) is a pre–morphism, then clearly \( F(s, t; \varphi) \) is also a pre–morphism. Moreover, if \( \varphi \) is a morphism, then \( F(s, t; \varphi) \) is also a morphism. Indeed, for \( x \in X_s \) resp. \( X_e \), we have \( \Psi(s, x) \in X_s \) resp. \( X_e \), hence \( \varphi(\Psi(s, x)) \in Y_s \) resp. \( Y_e \). It follows that \( F(s, t; \varphi) = \Phi(t, \varphi(\Psi(s, x))) \in Y_s \) resp. \( Y_e \). The adjoint has the same form:

\[ [F(s, t; \varphi^*)(\mu)](y) = \mu[F(s, t; \varphi)(y)] = \mu[\Phi(t, \varphi(\Psi(s, y)))] = \]

\[ = \Phi^*(t, \mu)(\varphi(\Psi(s, y))) = \varphi^*[\Phi^*(t, \mu)](\Psi(s, y)) = \Psi^*(s, \varphi^*[\Phi^*(t, \mu)])(y) \]

Hence \( F(s, t; \varphi^*) = F(t, s; \varphi^*) \) and, as above, we obtain: \( \mu \in (Y^*)_s \) resp. \( (Y^*)_e \) \( \implies F(t, s; \varphi^*)(\mu) \in (X^*)_s \) resp. \( (X^*)_e \).

There are natural functors for the passage from the bi–systems to s. d. s. a. s., associating to \( F(s, t; x) \) respectively:

\[ \Phi_s(t, x) = F(0, t; x) \]

\[ \Phi_d(t, x) = F(t, 0; x) \]

\[ \Phi_{\Delta}(t, x) = F(t, t; x) \]

The simplest example of a bi–system (obtained in fact as a tensor product), is defined on the set of measurable and positive functions on \( \mathbb{R}^2 \), as:

\[ F(s, t; f)(x, y) = f(x + s, y + t) \]
More generally, this structure appears in the case of the (tensor) product of two s.d.s.a.s. \((X',\Phi')\) and \((X'',\Phi'')\); one defines \((X' \times X'',\Phi)\) as:

\[
\Phi(s, t; (x', x'')) = (\Phi'(s, x'), \Phi''(t, x''))
\]

The set of the intertwining morphisms, denoted by \(\text{Hom}(X, Y)\), is canonically organised as a s.d.s.a.s. Indeed:

\[
(\varphi_1 + \varphi_2)(\Phi(t, x)) = \Psi(t, \varphi_1(x)) + \Psi(t, \varphi_2(x)) = \Psi(t, (\varphi_1 + \varphi_2)(x))
\]

0 is intertwining since \(\Psi(t, 0) = 0, \forall t > 0\).

Then:

\[
(\alpha \varphi)(\Phi(t, x)) = \alpha \Psi(t, \varphi(x)) = \Psi(t, (\alpha \varphi)(x))
\]

\(\bigvee_n \varphi_n\) is intertwining:

\[
(\bigvee_n \varphi_n)(\Phi(t, x)) = \bigvee_n \Psi(t, \varphi_n(x)) = \Psi(t, (\bigvee_n \varphi_n)(x))
\]

If \(\varphi\) is an intertwining morphism, then \(F(s, t; \varphi)\) (denoted for simplicity by \(\psi\)), is also intertwining:

\[
\psi(\Psi(s, x)) = \Phi(t, \varphi(\Psi(s, \Psi(s, x)))) = \Phi(t, \varphi(\Psi(s + \sigma, x))) = \Phi(t, \Phi(s + \sigma, \varphi(x))) = \Phi(t + s + \sigma, \varphi(x))
\]

while:

\[
\Phi(\sigma, \psi(x)) = \Phi(\sigma, \Phi(t, \varphi(\Psi(s, x)))) = \Phi(\sigma + t, \varphi(\Psi(s, x))) = \Phi(\sigma + t, \Phi(s, \varphi(x))) = \Phi(\sigma + t + s, \varphi(x))
\]

For the intertwining morphisms we have:

\[
F(s, t; \varphi)(x) = \Psi(s + t, \varphi(x)) = \varphi(\Phi(s + t, x))
\]

Hence, in that case:
\[ F(s, t; \varphi) = F(s + t, 0; \varphi) = F(0, s + t; \varphi) \]

which allows a canonical organization of the set of intertwining morphisms as a s.d.s.a.s., as follows. Let us denote \( \Phi^+(t, \varphi) := F(t, 0; \varphi) \). Then:

\[
\Phi^+(s, \Phi^+(t, \varphi)) (x) = \Phi^+(t, \varphi) [\Phi(s, x)] = \varphi [\Phi(t, \Phi(s, x))] = \\
\varphi [\Phi(s + t, x)] = \Phi^+(s + t, \varphi)(x)
\]

\[
\Phi^+(t, \varphi_1 + \varphi_2)(x) = (\varphi_1 + \varphi_2) [\Phi(t, x)] = \Phi^+(t, \varphi_1)(x) + \Phi^+(t, \varphi_2)(x)
\]

\[
\Phi^+(t, \lambda \varphi)(x) = (\lambda \varphi) [\Phi(t, x)] = \lambda \Phi^+(t, \varphi)(x)
\]

\[
\varphi_1 \leq \varphi_2 \implies \Phi^+(t, \varphi_1)(x) = \varphi_1 [\Phi(t, x)] \leq \varphi_2 [\Phi(t, x)] = \Phi^+(t, \varphi_2)(x)
\]

\[
\varphi_n \not\leq \varphi \implies \Phi^+(t, \varphi_n)(x) = \varphi_n [\Phi(t, x)] \not\leq \varphi [\Phi(t, x)] = \Phi^+(t, \varphi)(x)
\]

We remark the next formula for the structure morphisms:

\[
\Phi^+(t, \varphi + \tau) = \varphi_{t+\tau}
\]

We get a canonical map \( * : \text{Hom}(X, Y) \to \text{Hom}(Y^*, X^*) \). This map is an intertwining morphism, since:

\[
[\Phi^+(t, \varphi)]^*(\mu)(x) = \mu [\Phi^+(t, \varphi)(x)] = \mu [\varphi (\Phi(t, x))] = \mu [\Psi(t, \varphi(x))] = \\
\mu [\Phi^+(t, \varphi^*)(x)] = \Phi^+(t, \varphi^*)(\mu)(x)
\]

If \( Y^* \) separates \( Y \), then the above morphism is injective: \( \varphi^* = \psi^* \iff \mu(\varphi(x)) = \mu(\psi(x)), \forall \mu \in Y^*, \forall x \in X \), hence \( \varphi = \psi \). Next, we have an analogous map:

\[
\text{Hom}(Y^*, X^*) \to \text{Hom}(X^{**}, Y^{**})
\]

Since \( X^{**} \) separates \( X^* \), this map is injective. Hence, if we suppose that \( X^{**} = X \) and \( Y^{**} = Y \), then \( \text{Hom}(X, Y) \) and \( \text{Hom}(Y^*, X^*) \) are canonically isomorphic.

The evaluation map \( E_x : \text{Hom}(X, Y) \to Y \) (where \( x \in X \)), defined as:

\[
E_x(\varphi) = \varphi(x)
\]

is an intertwining morphism:

\[
E_x[\Phi^+(t, \varphi)] = \Phi^+(t, \varphi)(x) = \varphi [\Phi(t, x)] = \Psi(t, \varphi(x)) = \Psi(t, E_x(\varphi))
\]
Let us remark that the intertwining morphism \( \varphi \) is a supermedian element in \( \text{Hom}(X,Y) \), in each of the next two situations:

(a) \( X = X_s \)

(b) \( \varphi(X) \subseteq Y_s \)

Indeed:

\[
\Phi^+(t, \varphi)(x) = \varphi[\Phi(t, x)] \leq \varphi(x)
\]

respectively:

\[
\Phi^+(t, \varphi)(x) = \Psi(t, \varphi(x)) \leq \varphi(x)
\]

**Bilinear forms and morphisms.** Let \( A : X \times Y^* \to [0, +\infty] \) be a map with the properties:

\[
A(x + x', \nu) = A(x, \nu) + A(x', \nu); \quad A(x, \nu + \nu') = A(x, \nu) + A(x, \nu')
\]

\[
A(\alpha x, \nu) = \alpha A(x, \nu); \quad A(x, \alpha \nu) = \alpha A(x, \nu)
\]

\[
x \leq x', \nu \leq \nu' \implies A(x, \nu) \leq A(x', \nu')
\]

\[
x_n \not\to x, \nu_n \not\to \nu \implies A(x_n, \nu_n) \not\to A(x, \nu)
\]

With \( A \) one associates canonically a pre–morphism \( \varphi : X \to Y^{**} \), defined as follows; for \( x \in X \), \( \varphi(x) : Y^* \to [0, +\infty] \) is given by \( \varphi(x)(\mu) = A(x, \mu), \forall \mu \in Y^* \).

If we suppose further that:

\[
A[\Phi(t, x), \nu] = A[x, \Phi^*(t, \nu)]
\]

then \( A \) induces an intertwining morphism.

Conversely, a pre–morphism \( \varphi : X \to Y \) being given, let us define \( A : X \times Y^* \to [0, +\infty] \) as \( A(x, \nu) = \nu(\varphi(x)) \). \( A \) is clearly a bilinear form.

4. Categorical constructions.

**Structure transfer.** Let \( X \) be a s.d.s.a.s. and \( Y \) be a set. Let \( \varphi : X \to Y \) be a surjective map. We organize next \( Y \) as a s.d.s.a.s.
If we impose the following conditions:
\[ \varphi(x) = \varphi(x') \] and \[ \varphi(x_1) = \varphi(x_1') \implies \varphi(x + x_1) = \varphi(x' + x_1') \]
\[ \varphi(x) = \varphi(x') \] and \[ \alpha \geq 0 \implies \varphi(ax) = \varphi(ax') \]
the transfer on \( Y \) of the two algebraic operations: \( + \) and \( \cdot \) is granted. In order that the relation \( \leq \), defined as:
\[ y \leq y' \text{ if } \forall x \in X \text{ with } \varphi(x) = y, \exists x' \text{ with } \varphi(x') = y' \text{ and } x \leq x', \]
be an order relation on \( Y \), we impose the condition:
\[ x \leq x' \leq x'' \text{ and } \varphi(x) = \varphi(x'') \implies \varphi(x) = \varphi(x') \]
For the existence of the increasing, countable \( \bigvee \), we impose the following conditions:
\[ \text{in } X \text{ there exists the increasing, countable } \bigvee, \text{ while the set } \{ x \mid \varphi(x) = y \} \]
is increasing and closed for \( \bigvee, \forall y \in Y \)
This condition guarantees the existence, in each equivalence class, of a canonical representative: namely the greatest element.
Finally, in order to transfer \( \Phi \), let us suppose that:
\[ \varphi(x) = \varphi(x') \implies \varphi[\Phi(t, x)] = \varphi[\Phi(t, x')], \forall t > 0 \]
Defining now \( \Psi(t, y) = \varphi[\Phi(t, x)] \), where \( \varphi(x) = y \), we get a s.d.s.a.s. \( (Y, \Psi) \), while \( \varphi \) becomes an intertwining morphism.
The difficulties concerning the order relation disappear if we suppose that \( \varphi \) is a bijection:

**Proposition 1.** Let \( X \) be a s.d.s.a.s. and \( Y \) be a set. Let \( \varphi : X \to Y \) be a bijection. If we transfer by \( \varphi \) the structure of ordered convex cone on \( Y \) and define \( \Psi(t, y) := \varphi[\Phi(t, \varphi^{-1}(y))] \), then \( (Y, \Psi) \) becomes a s.d.s.a.s. and \( \varphi \) becomes an intertwining morphism.

**Subsystems.** The notion of a subobject presuppose that \( X' \subseteq X \) be closed for the algebraic operations, to contain the element 0; for any increasing sequence \( (x_n) \) from \( X' \) to have \( \bigvee x_n \in X' \); and to be a absorbent part
for the relation \(\leq_\Phi\). The canonical inclusion becomes then an intertwining
morphism.

**Equivalence relations**

**Theorem 2.** Let \(X\) be a s.d.s.a.s. and let \(A \subseteq X\) have the properties:

1. \(0 \in A\)
2. \(a, b \in A \implies a + b \in A\)
3. \(a \in A, \alpha \geq 0 \implies \alpha a \in A\)
4. \(a \in A \implies \Phi(t, a) \in A, \forall t > 0\)

Then \(x \sim y\) defined as \(\exists a, b, c \in A\) such that \(x + a \leq y + b \leq x + c\) is
an equivalence relation on \(X\).

\(X/\sim\) is canonically organised as a s.d.s.a.s., while the canonical pro-
jecttion \(X \to X/\sim\) is an intertwining morphism.

**Proof.** From (1) it follows the reflexivity of the relation. From (2) we
get:

\[ y + a + b \leq x + a + c \leq y + b + c \]

From the same property, it follows that \(x \sim y\) iff \(\exists a, b \in A\) such that \(x \leq y + b\)
and \(y \leq x + a\); hence, the transitivity is proved. The properties (2) and (3)
show the compatibility with the algebraic operations.

We define next an order relation on \(X/\sim\) as: \(\hat{x} \leq \hat{y}\) iff \(\exists a \in A\) such
that \(x \leq y + a\). It is clearly a well defined order relation, compatible with
the algebraic operations.

Finally, (4) proves that the definition:

\[ \hat{\Phi}(t, \hat{x}) = \{\Phi(t, x) | x \in \hat{x}\} \]

is correct. The canonical projection \(\pi : X \to X/\sim\) is, by construction, an
intertwining morphism.

**Remark.** The set: \(\pi^{-1}(\{0\}) = \{x \in X | \exists a \in A\) such that \(x \leq a\}\)
satisfies the same properties as \(A\) and defines the same equivalence relation.
Examples. One can take as the set $A$:
\[ \{ a \in X | \exists t > 0 \text{ such that } \Phi(t, a) = 0 \} \]
or:
\[ \{ h \in X | \Phi(t, h) = h, \forall t > 0 \} \]
or:
\[ \{ x \in X | \mu[\Phi(t, x)] = 0, \forall t \geq 0 \} \]
(for fixed $\mu \in X^*$).

**Theorem 3.** Let $\varphi : X \rightarrow Y$ be an intertwining morphism.

(a) The relation $x \sim x'$ defined as: $\varphi(x) = \varphi(x')$ is an equivalence relation on $X$. $X/\sim$ is canonically organised as a s.d.s.a.s., and the canonical projection $\pi : X \rightarrow X/\sim$ is an intertwining morphism.

(b) $\varphi(X)$ has a canonical structure of a s.d.s.a.s., with the inclusion $i : \varphi(X) \hookrightarrow Y$ as intertwining morphism. There exists a canonical decomposition

\[ X \overset{\pi}{\longrightarrow} X/\sim \overset{\varphi}{\longrightarrow} \varphi(X) \overset{i}{\hookrightarrow} Y \]

with $\varphi$ isomorphism of s.d.s.a.s.

**Remark.** $X/\sim$ may be organised also with following order relation:
\[ \hat{x} \leq \hat{x}' \text{ if for each } x_1 \in \hat{x} \exists x'_1 \in \hat{x}' \text{ such that } x_1 \leq x'_1 \]
In this case, $\varphi$ is no more an isomorphism, since the implication:
\[ \varphi(\hat{x}) \leq \varphi(\hat{x}') \implies \hat{x} \leq \hat{x}' \]
is no more valid in general.

Another example of an equivalence relation is obtained as follows:

**Proposition 4.** Let $A \subseteq X^*$ and define the relation $x \sim x'$ if:
\[ \mu[\Phi(t, x)] = \mu[\Phi(t, x')], \forall \mu \in A, \forall t > 0 \]
Defining $\hat{x} \leq \hat{x}'$ if $\mu[\Phi(t, x)] \leq \mu[\Phi(t, x')], \forall \mu \in A, \forall t > 0$ and $\hat{\Phi}(t, \hat{x}) = \Phi(t, x)$, one obtains a s.d.s.a.s., and the canonical projection is an intertwining morphism.
Remarks. The equivalence relation induced by an intertwining morphism \( \varphi^* \) on \( Y^* \) corresponds to the choice \( A = \varphi(X) \).

If \( A \) is supposed to be an absorbent part for the relation \( \leq \Phi \) (let us remark that there always exists a smallest absorbent part, which contains \( A \)), then the equivalence relation may be described also in the next way: \( \mu(x) = \mu(x') \), \( \forall \mu \in A \).

The relation \( \sim \) becomes an equality iff the absorbent part \( A \) separates \( X \).

Projective and inductive limits.

Proposition 5. Let \((X_i)_{i \in I}\) be a family of s.d.s.a.s. Then \( \bigoplus_{i \in I} X_i \) and \( \prod_{i \in I} X_i \) are s.d.s.a.s. with canonical, intertwining morphisms:

\[ \varepsilon_i : X_i \to \bigoplus_{i \in I} X_i \text{ resp. } \pi_i : \prod_{i \in I} X_i \to X_i \]

and with the classical universality properties.

Verification. \( \prod_{i \in I} X_i \) is organised with the algebraic operations, the order relation and \( \Phi \) on components. Thus, each \( \pi_i((x_i)) = x_i \) becomes an intertwining morphism:

\[ \pi_i[\Phi(t, (x_i))] = \Phi_i(t, \pi_i((x_i))) \]

Let us recall that:

\[ \bigoplus_{i \in I} X_i := \left\{ (x_i) \in \prod_{i \in I} X_i \mid \{i \mid x_i \neq 0\} \text{ is finite} \right\} \]

is a closed part for the algebraic operations and is absorbent for the order relation \( \leq \Phi \), while the canonical inclusion \( i : \bigoplus_{i \in I} X_i \to \prod_{i \in I} X_i \) is an intertwining morphism. However, the increasing, countable \( \bigvee \) exists only for dominated sequences.

Let \((X_i, \psi_{ji})\) be a projective system of s.d.s.a.s. We suppose that each morphism \( \psi_{ji} : X_j \to X_i \) is intertwining. Let us denote by \( X \subseteq \prod_{i \in I} X_i \) the
subset of those elements \((x_i)\) with the property that: \(\psi_{ji}(x_j) = x_i, \forall i \leq j\).
Here \(\psi : X \to X_i\) will denote the restrictions of the usual projections, hence the diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\psi} & X_i \\
\downarrow{\psi_j} & & \nearrow{\psi_{ji}} \\
X_j & & \\
\end{array}
\]

is commutative. Now all the properties hold. Moreover, there exists increasing, countable \(\bigvee\) (resp. arbitrary \(\bigvee\)) and is computed by components, if each \(\psi\) commutes with these operations. The universality property is easily proved, hence the projective limit \(\lim_{\leftarrow} X_i\) exists in the category of intertwining morphisms.

Let \((X_i, \varphi_{ij})\) be an inductive system of s.d.s.a.s. On the disjoint sum \(\bigsqcup_{i \in I} X_i\) let us consider the preorder relation \(x_i \ll x_j\): if there exists \(k \geq i, k \geq j\) such that \(\varphi_{ik}(x_i) \leq \varphi_{jk}(x_j)\). Let \(\sim\) be the associated equivalence relation, i.e. \(x_i \sim x_j\) if there exists \(k \geq i, k \geq j\) for which \(\varphi_{ik}(x_i) = \varphi_{jk}(x_j)\). Let us remark that, if \(\varphi_{ij}\) are bimonotone, then in the definition of the preorder relation \(\ll\) one can use any \(k \geq i, k \geq j\). Let us denote now \(X = \bigsqcup_{i \in I} X_i/\sim\) and let us organize \(X\) as follows. \(\varphi_i : X_i \to X\) is given by \(\varphi_i(x_i) = \hat{x}_i\). It is clear that the following diagram:

\[
\begin{array}{ccc}
X_i & \xrightarrow{\psi_{ij}} & X_j \\
\downarrow{\varphi_i} & & \nearrow{\varphi_j} \\
X & & \\
\end{array}
\]

is commutative. We define the operation \(+\) on \(X\) as:

\[
\hat{x}_i + \hat{x}_j = \varphi_{ik}(x_i) + \varphi_{jk}(x_j)
\]

(the sum is computed for \(k\) large enough). One sees easily that the definition is correct and satisfies the necessary properties. We define: \(\lambda \hat{x}_i = \hat{\lambda x}_i\). The definition is correct and the properties are fulfilled. We define next the order relation associated with \(\ll\), that means: \(\hat{x}_i \leq \hat{x}_j\) if there exists \(k \geq i\),
$k \geq j$ for which $\varphi_{ik}(x_i) \leq \varphi_{jk}(x_j)$. The definition is correct and satisfies the necessary properties, except the existence of countable increasing $\bigvee$. As in [6], we impose the condition that each $\varphi(X_i)$ be solid in $X$. Finally, let us define $\Phi(t,\hat{x}_i) = \hat{\Phi}(t,x_i)$. We get in this way the inductive limit of the considered system: $X = \lim X_i$.

REFERENCES

14. POPA, E., POPA, L. – An example in the axiomatic potential theory: the semi–
383–396.
Cuza", Iaşi, s.I-a Mat., t. XLIV (1998), f.2, 335–349.
18. POPA, E. – Excessive elements in semi–dynamical systems on algebraic structures,

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