NEW SHARP BOUNDS FOR GAMMA AND DIGAMMA FUNCTIONS

BY

CRISTINEL MORTICI

Abstract. Motivated by Sandor and Debnath, Batir, we prove that a function involving gamma function is completely monotonic. As applications, we establish new upper and lower bounds for the gamma and digamma functions, with sharp constants.

Mathematics Subject Classification 2000: 30E15, 26D07, 41A60.
Key words: factorial function, gamma function, digamma and polygamma functions, completely monotonic function, inequalities, Euler constant.

1. Introduction

We discuss here the approximations of the factorial function of the form

\[ n^{n+1}e^{-n}\sqrt{2\pi} \leq n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}}, \]

where \( \alpha, \beta \) are real parameters. The bounds (1.1) were stated in Sandor and Debnath [9] with \( \alpha = 0 \) and \( \beta = 1 \). Their result was rediscovered by Guo [5]. Very recently, Batir [3] determined the largest number \( \alpha = 1 - \frac{1}{2\pi}e^{-2} \) and the smallest number \( \beta = \frac{1}{6} \) such that the inequalities (1.1) hold for all \( n = 1, 2, 3, \ldots \).

Numerical computations made in [3] show that the upper approximation

\[ n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}} \]

is better than the other lower approximation from (1.1) (with \( \alpha = 1-2\pi e^{-2} \)) and also it is more accurate than other known formulas as Stirling’s formula,
or Burnside’s formula. These facts entitled us to consider the following function associated with the approximation (1.2):

\[ f(x) = \ln \Gamma(x + 1) - (x + 1) \ln x + x - \ln \sqrt{2\pi} + \frac{1}{2} \ln \left( x - \frac{1}{6} \right). \]

We prove that \(-f\) is strictly completely monotonic and as a direct consequence, we establish new double inequalities for \(x \geq 1\):

\[
\omega \cdot \frac{x^{x+1}e^{-x} \sqrt{2\pi}}{\sqrt{x-1/6}} \leq \Gamma(x + 1) < \frac{x^{x+1}e^{-x} \sqrt{2\pi}}{\sqrt{x-1/6}},
\]

where \(\omega = e^{\sqrt{\frac{5}{12\pi}}} = 0.98995\ldots\) is best possible. Moreover, the following double inequality for \(x \geq 1\) is established:

\[
\frac{1}{x} - \frac{1}{2(x - \frac{1}{6})} < \psi(x) - (\ln x - \frac{1}{x}) < \frac{1}{x} - \frac{1}{2(x - \frac{1}{6})} + \zeta,
\]

where \(\zeta = -\gamma + \frac{3}{5} = 0.022785\ldots\) (\(\gamma = 0.577215\ldots\) is the Euler constant).

2. The results

The gamma \(\Gamma\) and digamma \(\psi\) functions are defined by

\[
\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt, \quad \psi(x) = \frac{d}{dx} (\ln \Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}
\]

for all complex numbers \(x\) with \(\text{Re} \, x > 0\), but here we restrict them to positive real numbers \(x\). We also have \(\psi(x + 1) = \psi(x) + \frac{1}{x}\), for all \(x > 0\). The gamma function is an extension of the factorial function, since \(\Gamma(n + 1) = n!\), for \(n = 0, 1, 2, 3\ldots\). The derivatives \(\psi', \psi'', \ldots\), known as polygamma functions, have the following integral representations:

\[
\psi^{(n)}(x) = (-1)^{n-1} \int_0^{\infty} \frac{t^n e^{-xt}}{1 - e^{-t}} dt
\]

for \(n = 1, 2, 3, \ldots\). For proofs and other details, see for example, [2]. We also use the following integral representation

\[
\frac{1}{x^n} = \frac{1}{(n-1)!} \int_0^{\infty} t^{n-1} e^{-xt} dt, \quad n \geq 1.
\]

Recall that a function \(f\) is completely monotonic in an interval \(I\) if \(f\) has derivatives of all orders in \(I\) such that \((-1)^n f^{(n)}(x) \geq 0\), for all \(x \in I\)
and \( n = 0, 1, 2, 3 \ldots \). If this inequality is strict for all \( x \in I \) and all non-negative integers \( n \), then \( f \) is said to be strictly completely monotonic.

Completely monotonic functions involving \( \ln \Gamma(x) \) are important because they produce bounds for the polygamma functions. A consequence of the famous Hausdorff-Bernstein-Widder theorem states that \( f \) is completely monotonic on \([0, \infty)\) if and only if \( f(x) = \int_0^\infty e^{-xt} \varphi(t) \, dt \), where \( \varphi \) is a non-negative function on \([0, \infty)\) such that the integral converges for all \( x > 0 \), see [10, p. 161].

**Lemma 2.1.** For the sequence \( x_n = \frac{1}{2}(\frac{7^{n-1}}{6^{n-1}}) + \frac{1}{n} - 1 \) we have \( x_n > 0 \), for all \( n \geq 4 \).

**Proof.** First note that \( x_4 = \frac{1}{21} \) and \( x_5 = \frac{17}{135} \), so we are concentrated to show that \( x_n > 0 \), for all \( n \geq 6 \).

The function \( g(x) = (7^x - 1)/6^x \) is strictly increasing, since \( g'(x) = \frac{1}{6^x} (\ln 6 + 7^x \ln \frac{7}{6}) > 0 \). Then for all \( n \geq 6 \), we have \( x_n > \frac{1}{2}(\frac{7^{n-1}}{6^{n-1}}) - 1 \geq \frac{17}{5} (\frac{7^3 - 1}{6^3}) - 1 > 0 \) and the conclusion follows. \( \Box \)

Now we are in position to prove the following

**Theorem 2.1.** Let \( f : (1/6, \infty) \to \mathbb{R} \), given by \( f(x) = \ln \Gamma(x + 1) - (x + 1) \ln x + x - \ln \sqrt{2\pi} + \frac{1}{2} \ln (x - \frac{1}{6}) \). Then \(-f\) is strictly completely monotonic.

**Proof.** We have \( f'(x) = \psi(x) - \ln x + \frac{1}{2(x - \frac{1}{6})} \) and \( f''(x) = \psi'(x) - \frac{1}{x} - \frac{1}{2(x - \frac{1}{6})^2} \). Using the representations (2.1)-(2.2), we obtain \( f''(x) = \int_0^\infty e^{-xt} \varphi(t) \, dt \), where \( \varphi(t) = te^t - (e^t - 1) - \frac{1}{2}(e^{\frac{2}{3}t} - e^{\frac{1}{3}t}) \), or \( \varphi(t) = -\sum_{n=4}^{\infty} \frac{x_n}{(n-1)!} t^n \), where \( (x_n)_{n \geq 4} \) is defined in Lemma 2.1. According to Lemma 2.1, we have \( \varphi < 0 \) and then, \(-f''\) is strictly completely monotonic.

Now, \( f' \) is strictly decreasing, since \( f'' < 0 \). But we have \( \lim_{x \to \infty} f'(x) = 0 \), so \( f'(x) > 0 \) and consequently, \( f \) is strictly increasing. Using the fact that \( \lim_{x \to \infty} f(x) = 0 \), we deduce that \( f < 0 \). Finally, \(-f\) is strictly completely monotonic. \( \Box \)

As a direct consequence of the fact that \( f \) is strictly increasing, we have \( f(1) \leq f(x) < \lim_{x \to \infty} f(x) = 0 \), for all \( x \geq 1 \). As \( f(1) = 1 + \ln \sqrt{\frac{5}{12\pi}} \), we derive

\[
\omega \cdot \frac{x^{x+1}e^{-x}\sqrt{2\pi}}{\sqrt{x-1/6}} \leq \Gamma(x + 1) < \frac{x^{x+1}e^{-x}\sqrt{2\pi}}{\sqrt{x-1/6}},
\]

where \( \omega \) is a constant.
where $\omega = e^{\sqrt{\frac{3}{12\pi}}}$ is best possible.

Using the fact that $f'$ is strictly decreasing, we have $\lim_{x \to \infty} f'(x) = 0 < f'(x) < f'(1)$, for all $x \geq 1$. As we have $f'(1) = -\gamma + \frac{3}{5} = 0.022785 \ldots$, we obtain $-\frac{1}{2(x-\frac{1}{4})} < \psi(x) - \ln x < -\frac{1}{2(x-\frac{1}{8})} + \zeta$, with best possible constant $\zeta = -\gamma + \frac{3}{5} = 0.022785 \ldots$, which improve other results of the form $\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}$, $x > 1$, see [1, 4, 6, 7, 8].

REFERENCES


Received: 19.V.2009

Valahia University of Târgovişte,
Department of Mathematics,
Bd. Unirii 18, 130082 Târgovişte,
ROMANIA
cmortici@valahia.ro