MORPHISMS FOR SEMI–DYNAMICAL SYSTEMS

BY

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Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday

Introduction. The semi–dynamical systems were introduced in the framework of the axiomatic potential theory by H m i s s i [5] and, in the more general form used in this note, by B u c u r – B e z z a r g a [1], [2].

The present paper is a natural continuation of [8] and represents the printed form of the technical report [9]. The notion of morphism was introduced in potential theory in [6], see also [3]. Since not every semi–dynamical system is associated with a H–cone, the definitions of the notions of a morphism and of a H–aplication must be adapted accordingly. The main result of this paper (Th. 4.1.) characterizes the H–aplications among the morphisms of semi–dynamical systems. Of course, this characterization is different from the one given in the framework of H–cones, as the minimum of two excessive functions is replaced with their product.

1. Preliminaries. We recall the following notions and results from [1]. Let \((X, \mathcal{A})\) be a separated and countably generated measurable space [4]. \(\omega \in X\) is a fixed element; we denote by \(X_0 = X \setminus \{\omega\}\).

A semi–dynamical system is a measurable map\[
\Phi : [0, +\infty) \times X \to X
\]
which has the following properties:

1. \(\Phi(0, x) = x, \forall x \in X\).
2. \(\Phi(s + t, x) = \Phi(s, \Phi(t, x)), \forall s, t \in [0, +\infty), \forall x \in X\).
3. $\Phi(t, x) = \omega$ and $t' \geq t \Rightarrow \Phi(t', x) = \omega$.

4. $\Phi(t, x) = \Phi(t, y)$, $\forall t > 0 \Rightarrow x = y$.

If $\Phi(t_0, x) = \Phi(t_0, y)$, then:

$$\Phi(s + t_0, x) = \Phi(s, \Phi(t_0, x)) = \Phi(s, \Phi(t_0, y)) = \Phi(s + t_0, y)$$

hence $\Phi(t, x) = \Phi(t, y)$, $\forall t \geq t_0$.

We call the life time (of the system $\Phi$) the map $\zeta : X \to [0, +\infty]$, defined by

$$\zeta(x) = \inf\{t \geq 0 | \Phi(t, x) = \omega\}$$

Eliminating those points $x \in X_0$ for which $\zeta(x) = 0$, we may and do suppose that $\zeta(x) > 0$, $\forall x \in X_0$.

If $\zeta(x) < +\infty$, then $\Phi(\zeta(x), x) = \omega$. Indeed, if $x_0 = \Phi(\zeta(x), x) \in X_0$ then

$$\forall t > 0 : \Phi(t, x_0) = \Phi(t, \Phi(\zeta(x), x)) = \Phi(t + \zeta(x), x) = \omega$$

hence $\zeta(x_0) = 0$, contradiction. So, for each $x \in X_0$ we have either

$$\zeta(x) = +\infty \quad \text{or} \quad \Phi(\zeta(x), x) = \omega$$

For each $x \in X_0$, we denote by

$$\Gamma_x = \{\Phi(t, x) \mid t \in [0, \zeta(x))\}$$

(and call the trajectory starting from the point $x$).

The semi-dynamical system $\Phi$ is called transient if there exists $A_n \in \mathcal{X}$ such that $X_0 = \bigcup_{n \in \mathbb{N}} A_n$ and:

$$\forall x \in X_0 : m(\{t \in [0, +\infty) | \Phi(t, x) \in A_n\}) < +\infty$$

($m$ denoting the Lebesgue measure).

We denote $x \leq_\Phi y$ if there exists $t \in [0, +\infty)$ such that $y = \Phi(t, x)$.

If the system under consideration is transient, then $x \leq_\Phi y$ is an order relation.

The fine topology, denoted by $\tau_\Phi$ consists of the sets $D \subseteq X_0$ for which:

$$\forall x \in D \exists \tau \in (0, \zeta(x)) \text{ such that } \Phi(t, x) \in D, \forall t \in (0, \tau)$$
With every semi-dynamical system $\Phi$ one associates the semi-group $P = (P_t)_{t \geq 0}$ (markovian, of kernels on $(X, X)$), defined by $P_t f(x) = f(\Phi(t, x))$. The same definition gives a sub-markovian semi-group on $X_0$: if $f$ is defined, numerical, positive on $X_0$, then we extend it to $X$ by $f(\omega) = 0$.

The resolvent $V = (V_\alpha)_{\alpha > 0}$ considered in [1] is that associated with this semi-group:

$$V_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt$$

(a) $\Phi$ is transient if and only if $V_0$ is a proper kernel.

(b) The excessive functions associated to the semi-group $P$ (or with the resolvent $V$) are exactly those functions $f : X_0 \to [0, +\infty]$ which are: measurable, decreasing (with respect to the associated order relation $\leq_\Phi$) and right continuous (i.e. $\lim_{t \downarrow 0} f(\Phi(t, x)) = f(x), \forall x \in X_0$ or equivalently, continuous in the fine topology).

We denote by $E_\Phi$ the set of excessive functions with respect to the semi-group $P$ on $X_0$.

Hm i s s i [5] considers a locally compact space $X$, with countable base. $\Phi$ is supposed to be continuous and with properties 1) and 2). In this case, the associated semi-group is Feller, but not strongly Feller.

The fine topology $\tau_\Phi$ is separated.

If the system is transient, then every trajectory $\Gamma_x$ is both open and closed in the fine topology.

The natural topology, denoted by $\tau_\Phi^0$ is formed of the sets $D \subseteq X_0$ such that:

$$\forall x \in X_0 \text{ and } t_0 \in [0, \zeta(x)) \text{ for which } \Phi(t_0, x) \in D, \exists \varepsilon > 0$$

such that: $\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap [0, \zeta(x)) \Rightarrow \Phi(t, x) \in D$

For each $x \in X_0$, the map $t \mapsto \Phi(t, x)$ is continuous on $[0, \zeta(x))$, in the natural topology.

2. Some problems of measurability. The following notions were introduced in [8].

**Definition 2.1.** Let $T$ denote the set of all parts of $X$, which are totally ordered for the order relation $\leq_\Phi$. Clearly, each $\Gamma_x$ belongs to $T$. By Zorn’s lemma, there exist maximal elements in $T$ (which will be called maximal trajectories or initial points for the system $\Phi$). The set of all initial
points is called the entrance boundary of the semi–dynamical system $\Phi$ and will be denoted by $I$.

Each $s \in \mathcal{E}_\Phi$ extends to $I$ by: $\tilde{s}(\alpha) = \sup_{x \in \alpha} s(x)$ for $\alpha \in I$.

**Definition 2.2.** For each $x_0 \in X_0$ we denote by $p_{x_0} : X_0 \to [0, +\infty)$ the function defined as follows:

$$p_{x_0}(x) = \begin{cases} 
1 & \text{, } x \leq_{\Phi} x_0, \ x \neq x_0 \\
0 & \text{, if not}
\end{cases}$$

**Proposition 2.3.** Each $\alpha \in I$ is a measurable part. Especially, each $p_x$ is a measurable function, hence $p_x \in \mathcal{E}_\Phi$, $\forall x \in X_0$.

**Proof.** Indeed, each $\alpha \in I$ may be written as:

$$\alpha = \bigcup_{x \in \alpha} \Gamma_x.$$

Let us fix $x_0 \in \alpha$.
If $\alpha \equiv \Gamma_{x_0}$, then the proof is ended.
If not, let us denote $t_x \in (0, +\infty)$, for each $x \in \alpha$, $x < x_0$ such that:

$$\Phi(t_x, x) = x_0.$$

If $\sup_{x \in \alpha} t_x \in (0, +\infty)$ is attained, the proof is ended.
If not, we can choose a sequence, for which:

$$t_{x_n} \not\to \sup_{x \in \alpha} t_x.$$

It results that:

$$\alpha = \bigcup_{n \in \mathbb{N}} \Gamma_{x_n}.$$

The measurability is thus proved in all cases.

**Lemma 2.4.** Let $\alpha \in I$. If $\Phi(\varepsilon, x) \in \alpha$, $\forall \varepsilon > 0$, then $x \in \alpha$.

**Proof.** We prove that $x$ is comparable to each $y \in \alpha$. If there exists $\varepsilon_0$ such that $\Phi(\varepsilon_0, x) \leq y$, then $x \leq y$. If not, for each $\varepsilon > 0$ we have $y < \Phi(\varepsilon, x)$. Hence, there exists $t_\varepsilon > 0$ for which $\Phi(t_\varepsilon, y) = \Phi(\varepsilon, x)$. 

Let us denote $t_0 = \inf t \geq 0$. From the injectivity of the function $t \mapsto \Phi(t, y)$, it follows that $t_\varepsilon = \varepsilon + t_0$, hence:

$$\Phi(\varepsilon, x) = \Phi(t_\varepsilon, y) = \Phi(\varepsilon + t_0, y) = \Phi(t_0, y)$$

which means: $x = \Phi(t_0, y)$.

**Lemma 2.5.** If $x_1 \not\leq x_2$, then there exists $s \in \mathcal{E}_\Phi$ finite, such that $s(x_1) < s(x_2)$.

**Proof.** The choice

$$s(x) = \begin{cases} 0 , & \text{if } x_1 < x \\ 1 , & \text{if not} \end{cases}$$

is convenient.

3. Morphisms of semi-dynamical systems. While $\mathcal{E}_\Phi$ is not always a $H$–cone, in analogy with [6] we define the notion of a morphism of semi–dynamical systems as follows.

**Definition 3.1.** Let $\Phi$ and $\Psi$ be semi–dynamical systems. We call morphism (of semi–dynamical systems), any application:

$$\varphi : \mathcal{E}_\Phi \to \mathcal{E}_\Psi$$

possessing the properties:

- $\varphi(s + t) = \varphi(s) + \varphi(t)$, $\forall s, t \in \mathcal{E}_\Phi$
- $s \leq t \implies \varphi(s) \leq \varphi(t)$
- $s_i \nearrow s \implies \varphi(s_i) \nearrow \varphi(s)$

The morphism $\varphi$ is called finite if $s < +\infty \implies \varphi(s) < +\infty$. The finite morphisms extend naturally to linear applications on the linear space (denoted by $[\mathcal{E}_\Phi]$) of differences of finite functions from $\mathcal{E}_\Phi$.

Let us mention that the notion of morphism of semi–dynamical systems (in a particular case), was considered also by H m i s i [5].

An example of morphism of semi–dynamical systems is the following one. Let $\Phi$ be semi–dynamical system, $Y$ a set, and $\varphi : Y \to X$ a bijection. Then: $\Psi(t, y) = \varphi^{-1}[\Phi(t, \varphi(y))]$ defines a semi–dynamical system on $Y$, while between $\mathcal{E}_\Phi$ and $\mathcal{E}_\Psi$ there exists a canonical identification.
More generally, if $\Phi$ and $\Psi$ are semi–dynamical systems and $\varphi : Y \to X$ and $\psi : X \to Y$ are measurable applications, such that $\psi \circ \varphi = 1_Y$, then $s \mapsto s \circ \psi$ defines a canonical morphism $E_\Psi \to E_\Phi$.

An important example of morphisms of semi–dynamical systems is given by the analogue of the $H$–aplications [6], [7].

**Proposition 3.2.** Let $\Phi$ and $\Psi$ be semi–dynamical systems on the sets $X$ and $Y$. Let $\varphi : Y \to X$ be an application, which is: measurable, monotone (i.e. $y_1 \leq_Y y_2 \implies \varphi(y_1) \leq_\Phi \varphi(y_2)$) and finely continuous.

Then: $s \in E_\Phi \mapsto \varphi(s) \circ \psi$ defines a canonical morphism $E_\Psi \to E_\Phi$.

**Proof.** Indeed, the function $s \circ \varphi$ is measurable, descreasing and finely continuous, hence $s \circ \varphi \in E_\Psi$. Moreover, if $s$ is finite and $s \circ \varphi$ is finite. The rest of the properties are immediate.

**Proposition 3.3.** If $\varphi : E_\Phi \to E_\Psi$ is a finite morphism, then the following properties are equivalent:

1. $\varphi(s.t) = \varphi(s).\varphi(t)$, $\forall s, t \in E_\Phi$
2. $\varphi(u.v) = \varphi(u).\varphi(v)$, $\forall u, v \in [E_\Phi]$
3. $\varphi(s^2) = \varphi^2(s)$, $\forall s \in E_\Phi$
4. $\varphi(u^2) = \varphi^2(u)$, $\forall u \in [E_\Phi]$
5. $\varphi(\sqrt{s}) = \sqrt{\varphi(s)}$, $\forall s \in E_\Phi$

**Proof.** The implications $2 \Rightarrow 1$, $4 \Rightarrow 3$, $2 \Rightarrow 4$, $1 \Rightarrow 3$ are clear.

1$\Rightarrow 2$ Let us denote: $u = u' - u''$, $v = v' - v''$ with $u', u'', v', v'' \in E_\Phi$.

The conclusion follows from:

$$u.v = (u'v' + u''v'') - (u'v'' + u''v')$$

3$\Rightarrow 4$ Follows by particularisation from the previous implication.

3$\Rightarrow 1$ and 4$\Rightarrow 2$ are obtained from:

$$s.t = \frac{1}{2} [(s + t)^2 - s^2 - t^2]$$

3$\Rightarrow 5$ Follows from:

$$\varphi(s) = \varphi((\sqrt{s})^2) = \varphi^2(\sqrt{s})$$
$$\forall s \in E_\Phi$$

$$\varphi(s) = \varphi(\sqrt{s^2}) = \sqrt{\varphi(s^2)}$$
$$\forall s \in E_\Phi$$
4. The characterization of the morphisms between semi-dynamical systems.

**Theorem 4.1.** Let us suppose that \((X, \mathcal{X})\) is a measurable, suslinean space. Let \(\Phi\) and \(\Psi\) be semi-dynamical systems (transient) on \(X\) and \(Y\). Let \(\varphi: E_\Phi \to E_\Psi\) be a morphism, having the following properties:

- \(\varphi(s \cdot t) = \varphi(s) \cdot \varphi(t), \ \forall s, t \in E_\Phi\)
- \(\varphi(1) = 1\)

Then, there exists a measurable application: \(\tilde{\varphi}: Y_0 \to X_0 \cup I\), which is monotone and finely continuous, uniquely defined, such that:

\[\varphi(s) = s \circ \tilde{\varphi}, \ \forall s \in E_\Phi\]

**Proof.** For each \(y_0 \in Y_0\), let us define:

\[\mu(s) = \varphi(s)(y_0), \ \forall s \in E_\Phi\]

In such a way, we obtain an application: \(\mu: E_\Phi \to [0, +\infty]\), having the next properties:

- \(\mu(s + t) = \mu(s) + \mu(t), \ \forall s, t \in E_\Phi\)
- \(s \leq t \implies \mu(s) \leq \mu(t)\)
- \(s_i \not\sim s \implies \mu(s_i) \not\sim \mu(s)\)
- \(\mu(s \cdot t) = \mu(s) \cdot \mu(t), \ \forall s, t \in E_\Phi\)
- \(\mu(1) = 1\)

Since for each \(x \in X_0\) we have \(p_x^2 = p_x\), it follows that \(\mu(p_x) \in \{0, 1\}\). If we suppose that \(\mu(p_x) = 0, \ \forall x\), then the contradictory relation would follow:

\[1 = \mu(1) = \mu(\sup_x p_x) = \sup_x \mu(p_x) = 0\]

Because:

\[p_x \cdot p_{x'} \not= 0 \iff \exists x_0 \text{ such that } p_x(x_0) = p_{x'}(x_0) = 1 \iff \exists x_0 \text{ such that } x_0 < x \text{ and } x_0 < x' \iff x, x' \text{ are comparable.}\]

From: \(\mu(p_x) = \mu(p_{x'}) = 1\) it results: \(x\) and \(x'\) are comparable. Hence, the set:

\[\alpha_1 = \{x \in X_0 \mid \mu(p_x) = 1\}\]
is non–void and totally ordered.

Let $\alpha \in \mathcal{I}$ be such that $\alpha_1 \subseteq \alpha$. We consider next two cases.

If $\alpha_1 \equiv \alpha$, then we define:

$$\tilde{\varphi}(y_0) = \alpha.$$  

Of course, if $\alpha \equiv \Gamma_{x_0}$, then the definition amounts to:

$$\tilde{\varphi}(y_0) = x_0.$$  

Anyway, let us prove that:

$$\mu(s) = s(\alpha) \left( = \sup_{x \in \alpha} s(x) \right), \forall s \in \mathcal{E}_{\Phi}$$

Since for each $x \in X_0$ we have $[s - s(x)].p_x \in \mathcal{E}_{\Phi}$ it follows that:

$$0 \leq \mu \left( (s - s(x)).p_x \right) = \mu(s - s(x)).\mu(p_x) = \mu(s) - s(x)$$

hence $\mu(s) \geq s(\alpha)$. In order to prove the converse inequality, let us suppose first that the function $s$ is bounded: $s \leq M$. Since $s \leq s(\alpha) + M.\chi_{X_0 \setminus \alpha}$, while $\mu(p_x) = 0, \forall x \notin \alpha$, and $\chi_{X_0 \setminus \alpha} = \sup_{x \notin \alpha} p_x$, we get $\mu(s) \leq s(\alpha)$. The general case follows from the existence of a sequence of bounded functions $s_n \in \mathcal{E}_{\Phi}$, for which $s_n \not\to s$. We consider next the situation when $\alpha_1 \neq \alpha$. Hence, there exists $x_0 \in \alpha$ such that $\mu(p_{x_0}) = 0$, while $\mu(p_x) = 0 \iff x \leq x_0$; $\mu(p_x) = 1 \iff x_0 < x$. In this situation, let us define

$$\tilde{\varphi}(y_0) = x_0.$$  

Using the same arguments as above, we obtain that:

$$\mu(s) = s(x_0), \forall s \in \mathcal{E}_{\Phi}.$$  

Hence, for any $x > x_0$ we get:

$$0 \leq \mu \left( (s - s(x)).p_x \right) = \mu(s) - s(x)$$

so $\mu(s) \geq s(x_0)$.

For the converse inequality, we consider first the case of a bounded function $s$. We have:

$$s \leq s(x_0) + M.\left( p_{x_0} + \chi_{X_0 \setminus \alpha} \right)$$
and the conclusion follows.

Hence, we defined the function: \( \tilde{\varphi} : Y_0 \to X_0 \cup I \), such that

\[
\varphi(s) = s \circ \tilde{\varphi}, \quad \forall s \in E_\Phi.
\]

We prove next the measurability of the application \( \tilde{\varphi} \).

Let \( \mathcal{X}' \) be the \( \sigma \)-algebra generated by the functions from \( E_\Phi \). Clearly \( \mathcal{X}' \subseteq \mathcal{X} \). Since it contains the \( \sigma \)-algebra generated by the \( \mathcal{X} \)-measurable sets from each trajectory, \( \mathcal{X}' \) results separated. Since \( \mathcal{X} \) is separable, let \( (A_n)_{n} \) denote a family, which generate the \( \sigma \)-algebra \( \mathcal{X} \). Using Hunt’s theorem, \( (V_0(\chi_{A_n}))_{n} \) generates \( \mathcal{X}' \). By Lusin’s theorem [4], it follows now that \( \mathcal{X}' = \mathcal{X} \) hence the measurability of the application \( \tilde{\varphi} \) is proved.

We show next the monotony and the fine continuity for the application \( \tilde{\varphi} \). If \( y_1 \leq y_2 \), but \( \tilde{\varphi}(y_1) < \tilde{\varphi}(y_2) \), then, by lemma 3, there exists \( s \in E_\Phi \) such that \( s (\tilde{\varphi}(y_1)) < s (\tilde{\varphi}(y_2)) \), which is contradictory with the fact that \( s \circ \tilde{\varphi} \in E_\Psi \) is decreasing.

If the application \( \tilde{\varphi} \) is not finely continuous in a point \( y_0 \in Y_0 \), then we could find \( \varepsilon_0 > 0 \) and a sequence \( t_n \searrow 0 \) such that, if we denote by \( y_n = \Psi(y_0, t_n) \), then we obtain: \( \tilde{\varphi}(y_n) = \Phi(s_n, \tilde{\varphi}(y_0)) \), with \( s_n \geq \varepsilon_0 \). Let us denote \( s_0 = \inf s_n \). It follows that \( \tilde{\varphi}(y_n) \to x'_0 = \Phi(s_0, \tilde{\varphi}(y_0)) \).

Using again Lemma 3, we can find \( s \in E_\Phi \) such that \( s(x'_0) \neq s(x_0) \). Hence \( (\varphi s)(y_n) \to (\varphi s)(y_0) \), since \( \varphi s \in E_\Psi \) is finely continuous. So:

\[
s(\tilde{\varphi}y_n) \to s(\tilde{\varphi}y_0).
\]

But \( s \in E_\Phi \) is also finely continuous, hence \( s(\tilde{\varphi}y_n) \to s(x'_0) \), which is a contradiction.

The other assertions are clear.

Remarks.

a) The properties:

- \( \varphi(\min(s, t)) = \min(\varphi s, \varphi t), \quad \forall s, t \in E_\Phi \)
- \( \varphi(\min(u, v)) = \min(\varphi u, \varphi v), \quad \forall u, v \in [E_\Phi] \)
- \( \varphi(|u|) = |\varphi(u)|, \quad \forall u, v \in [E_\Phi] \)

are equivalent, and by the theorem are implied by any of the other properties from Prop. 6.

I do not know if any of the converse implications hold.

b) If one gives up the hypothesis ”\( \varphi(1) = 1 \)”, then \( \varphi(1) \) is a function which takes the only values 0 and 1. Considering the semi–dynamical
system $\Psi$ reduced to the set where $\varphi(1) \equiv 1$, we obtain still a $H$–
aplication. The same effect is obtained if $\tilde{\varphi}$ is defined as $\omega$ in all points where $\varphi(1) = 0$.

The restriction of the semi–dynamical system $\Psi$ pe $Y$ to a subset $A \subseteq Y$ is obtained as follows. One defines

$$
\Psi_1 : [0, +\infty) \times A \rightarrow A
$$

as:

$$
\Psi_1(t, x) = \begin{cases} 
\Psi(t, x), & \text{if } \Psi(s, x) \in A, \forall s \in [0, t] \\
\omega, & \text{if not}
\end{cases}
$$

We need to eliminate from $A$ the "final points", which ammounts to suppose that $\forall x \in A \exists \varepsilon > 0$ such that $\forall t \in [0, \varepsilon) \implies \Psi(t, x) \in A$. In order to prove the fact that $\Psi_1$ is a semi–dynamical system, we remark that, if $\Psi(u, x) \in A$, $\forall u \in [0, s + t]$, then $\Psi(v, x) \in A$, $\forall v \in [0, t]$ and $\Psi(w, \psi(t, x)) \in A$, $\forall w \in [0, s]$. Hence, in this case, the equality 2 holds. If not, let $\Psi_1(t, x) = \omega$. In this case, the proof is ended; $\Psi(w, \Psi(t, x))$ cannot be in $A$ for any $w \in [0, s]$, hence we have

$$
\Psi_1(s, \Psi_1(t, x)) = \Psi_1(s, \Psi(t, x)) = \omega
$$

We can extend the notion of time–change [8] as follows.

c) Any $H$–aplication $\tilde{\varphi} : Y_0 \rightarrow X_0$ extends canonically to the boundaries $\mathcal{I}$ and $\mathcal{O}$ as follows. For each $\alpha' \in \mathcal{I}_Y$, the set $\tilde{\varphi}(\alpha')$ is totally ordered. Hence, either there exists $x_0 \in X_0$ such that $x_0 = \sup \tilde{\varphi}(\alpha')$, in which case we define $\tilde{\varphi}(\alpha') = x_0$; or $\tilde{\varphi}(\alpha')$ is contained in a unique set $\alpha \in \mathcal{I}_X$. Indeed, if we suppose that there exists two sets $\alpha$ and $\alpha_1$, then there exists a ramification point, common for $\alpha$ and $\alpha_1$, which was discussed in the first situation. Let us define $\tilde{\varphi}(\alpha') = \alpha$.

For $\beta' \in \mathcal{O}_Y$, the points from $\tilde{\varphi}(\beta')$ are pairwise equivalent, hence there exists a unique $\beta \in \mathcal{O}_X$, for which $\tilde{\varphi}(\beta') \subseteq \beta$. We define either $\tilde{\varphi}(\beta') = \inf_{y \in \beta'} \tilde{\varphi}(y)$ or $\tilde{\varphi}(\beta') = \beta$, depending on the fact that inf exists or not. By construction, the extension

$$
Y_0 \cup \mathcal{I}_Y \cup \mathcal{O}_Y \rightarrow X_0 \cup \mathcal{I}_X \cup \mathcal{O}_X
$$

transports the extensions of the functions from $\mathcal{E}_\varphi$ into the extensions of the functions from $\mathcal{E}_\psi$.

We can extend the notion of time–change [8] as follows.
Definition 4.2. Let $\Phi$ be a semi–dynamical system on un $X$ and $\Psi$ be semi–dynamical system on $Y$. Let $\varphi : X \to Y$ be a $H$–aplication between the two semi–dynamical systems. For each $(t, y) \in [0, +\infty) \times Y$ there exists a unique real positive number, denoted $\tau(t, y) \in [0, +\infty)$ such that:

$$\varphi (\Psi(t, y)) = \Phi (\tau(t, y), \varphi(y))$$

The main property of the application $\tau$ (which is the natural generalization of the corresponding property for time–changes) reads as follows:

Proposition 4.3.

$$\tau(s + t, y) = \tau(t, y) + \tau(s, \Psi(t, y))$$

Proof. We can write:

$$\Phi (\tau(s + t, y), \varphi(y)) = \varphi (\Psi(s + t, y)) = \varphi (\Psi(s, \Psi(t, y))) =$$

$$\Phi (\tau(s, \Psi(t, y)), \varphi (\Psi(t, y))) = \Phi [\tau(s, \Psi(t, y)), \Psi(\tau(t, y), \varphi(y))]$$

$$= \Phi [\tau(t, y) + \tau(s, \Psi(t, y)), \varphi(y)]$$

The same result is obtained if we write that the definition is valid also for $y \mapsto \Psi(s, y)$.

Remark. It is a remarkable fact that $\tau$ appears to be related only on $\Psi$ and not on $\Phi$: one could consider a relation among morphisms inducing the same $\tau$.

For the composition of two morphisms: $\varphi_1 \circ \varphi$, the rule for the considered change is: $(t, y) \mapsto \tau_1 (\tau(t, y), \varphi(y))$.

The canonical inclusion $A \to Y$ is a morphism of semi–dynamical systems $E_{\Psi} \to E_{\Psi_1}$. The time–change corresponding to this morphism is simply: $\tau(t, x) = t$, if $\Psi(s, x) \in A$, $\forall s \in [0, t]$ and $\tau(t, x) = \zeta(x)$ if not.

5. Operations in $E_{\Phi}$. In order to characterize the case when $E_{\Phi}$ is an $H$–cone, we study the various operations in that cone. As a cone of excessive functions, with respect to a resolvent, which has proper cogenerator (see [3]), for any increasing and dominated sequence $(s_n)_n$ from $E_{\Phi}$, $\bigvee_{n \in N} s_n$ exists and equals sup $s_n$ (pointwise). The results holds also for increasing and dominated families $(s_i)_i$, on condition that sup $s_i$ be a measurable function.
The notation \( s_i \uparrow s \) will be used to mark the fact that \( (s_i)_i \) is an increasing family from \( \mathcal{E}_\Phi \) and \( \sup_i s_i \in \mathcal{E}_\Phi \).

For any countable family \( (s_n)_n, \bigwedge_{n \in \mathbb{N}} s_n \) exists and equals the right continuous regularisation of the function: \( \inf_{n \in \mathbb{N}} s_n \). Indeed, \( f = \inf_{n \in \mathbb{N}} s_n \) is measurable and decreasing. Defining:

\[
\hat{f}(x) = \lim_{t \searrow 0} s_n(\Phi(t, x))
\]

one obtains a measurable, decreasing and right continuous function. Hence \( \hat{f} \in \mathcal{E}_\Phi \) and clearly \( \hat{f} = \bigwedge_{n \in \mathbb{N}} s_n \).

Let us define next:

\[
f_n(x) = \begin{cases} f \left( \Phi\left(\frac{1}{n}, x\right)\right), & \text{if } \zeta(x) > \frac{1}{n} \\ 0, & \text{if not}\end{cases}
\]

The functions \( f_n \) are measurable, hence \( \hat{f} = \sup_{n \in \mathbb{N}} f_n \) is also measurable. \( \hat{f} \) is clearly decreasing. At last:

\[
\hat{f}(\Phi(\varepsilon, x)) = \lim_{t \searrow 0} f [\Phi(t, \Phi(\varepsilon, x))] = \lim_{t \searrow 0} f [\Phi(t + \varepsilon, x)]
\]

proves that \( \lim_{t \searrow 0} \hat{f} [\Phi(\varepsilon, x)] = \hat{f}(x) \).

**Example.** Let \( x_0 \in X_0 \) and \( x_n \) be such that \( \Phi\left(\frac{1}{n}, x_n\right) = x_0 \). Then

\[
\bigwedge_{n \in \mathbb{N}} p_{x_n} = p_{x_0}, \text{ while } \inf_{n \in \mathbb{N}} p_{x_n}(x_0) = 1 > 0 = p_{x_0}(x_0).
\]

From the above considerations, we obtain:

**Theorem 5.1.** \( \mathcal{E}_\Phi \) is a (standard) \( H \)-cone if and only if \( \mathcal{I} \) is at most countable.

**Proof.** It is known (cf. [3]) that \( \mathcal{E}_\Phi \) is \( \sigma-H \)-cone. If the set \( \mathcal{I} \) is at most countable, then the fact that \( \mathcal{E}_\Phi \) is a standard \( H \)-cone results from the corresponding property for the uniform translation on \( \mathbb{R} \). Conversely, if the set \( \mathcal{I} \) is not at most countable, then there cannot exist a set of universally
continuous elements, which is at most countable and which approximate any element of the form \( p_x \) in \( E_\phi \).

The results proved above allows the verification of the fact that the morphisms of semi–dynamical systems commute with countable \( \wedge \).

**Proposition 5.2.** If \( \varphi : Y_0 \to X_0 \) is a morphism of semi–dynamical systems, then for any sequence \((s_n)_n\) the next relation holds:

\[
\tilde{\varphi} \left( \bigwedge_n s_n \right) = \bigwedge_n \tilde{\varphi}(s_n).
\]

**Proof.** On one hand:

\[
\tilde{\varphi} \left( \bigwedge_n s_n \right) = \left( \bigwedge_n s_n \right) (\varphi y) = \inf_n s_n(\varphi y) = \lim_{t \downarrow 0} \inf_n s_n [\Phi(t, \varphi y)].
\]

On the other hand:

\[
\left[ \bigwedge_n \tilde{\varphi}(s_n) \right] (y) = \lim_{s \downarrow 0} \inf_n \tilde{\varphi}(s_n) [\Psi(s, y)] = \lim_{s \downarrow 0} \inf_n s_n [\varphi (\Psi(s, y))].
\]

Introducing the time change \( \tau \) associated with the morphism \( \varphi \), we have:

\[
\varphi (\Psi(s, y)) = \Phi (\tau(s, y), \varphi y),
\]

hence we can write further:

\[
\left[ \bigwedge_n \tilde{\varphi}(s_n) \right] (y) = \lim_{s \downarrow 0} \inf_n s_n [\Phi (\tau(s, y), \varphi y)] = \lim_{s \downarrow 0} \inf_n s_n [\Phi (t, \varphi y)].
\]

Hence, it remains to prove the right continuity of the time changes.

Let \( s \downarrow s_0 \): by the definition itself of the (fine or right) topology, it follows that \( \Psi(s, y) \downarrow \Psi(s_0, y) \). From the fine continuity of the morphism \( \varphi \) it follows that \( \varphi [\Psi(s, y)] \downarrow \varphi [\Psi(s_0, y)] \), that is:

\[
\Phi (\tau(s, y), \varphi y) \downarrow \Phi (\tau(s_0, y), \varphi y).
\]

Since the function \( s \mapsto \tau(s, y) \) is decreasing, let \( \tau(s, y) \downarrow \alpha \). Then

\[
\Phi (\tau(s, y), \varphi y) \downarrow \Phi (\alpha, \varphi y).
\]
From the injectivity of the function \( t \mapsto \Phi(t, y) \) it results \( \alpha = \tau(s_0, y) \) hence
\[
\lim_{s \searrow s_0 \geq 0} \tau(s, y) = \tau(s_0, y).
\]
Particularly:
\[
\lim_{s \searrow 0} \tau(s, y) = 0.
\]

**Remark.** Equivalent conditions to that considered in the theorem above, but for morphisms of standard \( H \)-cones, are established in [7].

**REFERENCES**


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