ANTI-PERIODIC SOLUTIONS TO STRONGLY NONLINEAR EVOLUTION EQUATIONS IN HILBERT SPACES

BY

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Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday

Abstract. We prove the existence of strong solutions to a nonlinear evolution equation of the form

\[
\begin{aligned}
&u'(t) + g(t) = f(t) + h(t) \\
g &\in S_{L^2}\{\partial\varphi(u(\cdot))\} \\
f &\in S_{L^2}\{F(\cdot,u(\cdot))\} \\
u(0) = -u(T),
\end{aligned}
\]

in a Hilbert space \( H \), where \( \varphi : D(\partial \varphi) \subset H \to \mathbb{R}_+ \) is a proper, convex, l.s.c. function of compact type, \( F : [0, T] \times D(\partial \varphi) \to 2^H \) is a multifunction which is demiclosed and dominated in some sense by \( \partial \varphi \) and \( h \in L^2(0, T; H) \) is sufficiently small.

1. Introduction. Let \( H \) be a real Hilbert space of norm \( \| \cdot \| \) and inner product \( \langle \cdot, \cdot \rangle \).

The main goal of the present paper is to prove an existence result for the nonlinear anti-periodic boundary-value problem

\[
\begin{aligned}
&u'(t) + g(t) = f(t) + h(t) \quad \text{a.e. on } [0, T] \\
g &\in S_{L^2}\{\partial\varphi(u(\cdot))\} \\
f &\in S_{L^2}\{F(\cdot,u(\cdot))\} \\
u(0) = -u(T),
\end{aligned}
\]

where \( \varphi : D(\partial \varphi) \subset H \to \mathbb{R}_+ \) is a proper, convex, l.s.c. function of compact type, \( F : [0, T] \times D(\partial \varphi) \to 2^H \) is a multifunction which is demiclosed and
dominated in some sense by $\partial \varphi$ and $h \in L^2(0, T; H)$ is sufficiently small. Here, if $G : [0, T] \to 2^H$ is a multivalued mapping, $S_{L^2}(G)$ denotes the class of all selections of $G$ that belong to $L^2(0, T; H)$.

The study of anti–periodic solutions to abstract evolution equations was initiated by O k o c h i [7]. Subsequently, there has been a great deal of research on anti–periodicity. See, e.g., A i z i c o v i c i–P a v e l [1], A i z i – c o v i c i–R e i c h [2], H a r a u x [6], and the references therein. In particular, the papers [1], [2] and [6] contain existence results for non-monotone first and second order anti–periodic boundary value problems. The present work is a contribution in this direction, as well. Our main result, Theorem 2.1 below, establishes the existence of a solution to Problem (1.1) provided that the $L^2$ norm of the forcing term $h$ is "small enough." Unlike [1] we do not require $F$ to be cyclically maximal monotone. As compared to O t a–n i [8] who studied a related periodic problem, we do not assume that $A$ is coercive, and do not impose any sign condition on $F$. The plan of the paper is the following. The basic existence result is stated and proved in Section 2. Two examples showing how our abstract theory applies are presented in Section 3. For background material on monotone operators, we refer the reader to B r e z i s [5].

2. The main result. We begin with the hypotheses which will be in effect throughout this paper. First we recall that if $B$ is a closed convex subset in $H$, $B^0$ is the element of minimal norm in $B$, while $\|B\| = \sup\{\|y\|; y \in B\}$.

\[ (H_1) \quad \varphi : H \to \mathbb{R}_+ \text{ is a proper, convex, l.s.c. and even function with } \varphi(0) = 0. \text{ In addition, } \varphi \text{ is of compact type, i.e., for each } r > 0 \text{ the level set } \{u \in D(\varphi); \varphi(u) + \frac{1}{2}\|u\|^2 \leq r\} \text{ is compact.} \]

\[ (H_2) \quad F : [0, T] \times D(\partial \varphi) \to 2^H \text{ is a convex valued function satisfying:} \]

(i) for each $u \in W^{1,2}(0, T; H)$ with $u(t) \in D(\partial \varphi)$ a.e. on $[0, T]$ and $S_{L^2}(\partial \varphi(u(\cdot))) \neq \emptyset$, we have $S_{L^2}(F(\cdot, u(\cdot))) \neq \emptyset$.

(ii) if $(u_n)_{n \in \mathbb{N}}$ is in $W^{1,2}(0, T; H)$ with $u_n(t) \in D(\partial \varphi)$ for each $n \in \mathbb{N}$ and a.e. $t \in [0, T]$, $(g_n)_{n \in \mathbb{N}}$ is in $S_{L^2}(\partial \varphi(u_n(\cdot)))$, $(f_n)_{n \in \mathbb{N}}$ is in $S_{L^2}(F(\cdot, u_n(\cdot)))$ and in addition

\[
\begin{align*}
\lim n u_n &= u \text{ strongly in } C([0, T]; H) \\
\lim n g_n &= g \text{ weakly in } L^2(0, T; H) \\
\lim n f_n &= f \text{ weakly in } L^2(0, T; H)
\end{align*}
\]

then $f \in S_{L^2}(F(\cdot, u(\cdot)))$. 

(iii) there exist \( M > 0 \) and \( k > \frac{1}{2} \) such that, for each \( \varepsilon > 0 \)

\[
\| F(t, u) \| \leq \varepsilon \| \partial \varphi^0(u) \| + \frac{M}{\varepsilon} (\varphi(u))^k
\]

for all \( u \in D(\partial \varphi) \).

Our main result is

Theorem 2.1. If \((H_1)\) and \((H_2)\) hold then, for each \( T > 0 \), there exists \( \rho > 0 \) such that, for each \( h \in L^2(0, T; H) \) with \( \| h \|_{L^2(0, T; H)} \leq \rho \), the problem \((1.1)\) has at least one solution \( u \in W^{1, 2}(0, T; H) \) satisfying \( u(t) \in D(\partial \varphi) \) for a.a. \( t \in [0, T] \).

Proof. The basic idea consists in showing that a suitably defined multivalued operator has at least one fixed point whose existence is equivalent to that of at least one solution to \((1.1)\).

More precisely, let \( r > 0 \), let \( K = \{ f \in L^2(0, T; H); \| f \|_{L^2(0, T; H)} \leq r \} \) and define the operator \( Q : K \to 2L^2(0, T; H) \) by

\[
Q(f) = SL^2(0, T; H) F(\cdot, u_f(\cdot)) + h,
\]

where \( u_f \) is the unique solution of the problem

\[
\begin{cases}
    u'(t) + \partial \varphi(u(t)) \ni f(t) \\
    u(0) = -u(T).
\end{cases}
\]

By \([1]\) we know that, for each \( f \in L^2(0, T; H) \), the problem \((2.2)\) has a unique strong solution \( u_f : [0, T] \to D(\partial \varphi) \). We will show that, for a suitably chosen small \( r > 0 \), the operator \( Q \) maps \( K \) into \( K \) and satisfies all the hypotheses of a multivalued variant of Theorem 1 in \([3]\). Consequently it has at least one fixed point \( f \in K \). But \( f \) is a fixed point for \( Q \) if and only if \( u_f \) is a strong solution of \((1.1)\) and this will complete the proof.

In order to carry out this program let us observe first that, from \((2.2)\), multiplying both sides by \( u' \) and integrating over \([0, T]\), we obtain

\[
\| u' \|_{L^2(0, T; H)} \leq \| f \|_{L^2(0, T; H)}.
\]

Similarly we get

\[
\| \partial \varphi^0(u) \|_{L^2(0, T; H)} \leq \| f \|_{L^2(0, T; H)}.
\]
Since $u(0) = -u(T)$, we have

(2.5) \[ \|u(t)\| \leq \frac{T^{1/2}}{2} \|f\|_{L^2(0,T;H)} \]

for every $t \in [0, T]$. Next, from the definition of a subdifferential and $(H_1)$, we get

\[ \varphi(u(t)) \leq \langle u(t), \partial \varphi^0(u(t)) \rangle \]

a.e. on $[0, T]$ and consequently

(2.6) \[ \int_0^T \varphi(u(t)) \, dt \leq \frac{T}{2} \|f\|_{L^2(0,T;H)}^2. \]

Next multiplying both sides in (2.2) by $tu'$ and integrating over $[0, T]$ we get

\[ T\varphi(u(T)) \leq \int_0^T \varphi(u(t)) \, dt + T\|f\|_{L^2(0,T;H)}^2. \]

Since $\varphi(u(0)) = \varphi(u(T))$, from (2.6) and the last inequality, we deduce

(2.7) \[ \varphi(u(0)) \leq \frac{3}{2} \|f\|_{L^2(0,T;H)}^2. \]

Now, multiplying both sides in (2.2) by $u'$ and integrating over $[0, T]$ we get

\[ \varphi(u(t)) \leq \varphi(u(0)) + \int_0^t \langle u'(s), f(s) \rangle \, ds \]

and therefore, by (2.3), (2.7)

(2.8) \[ \varphi(u(t)) \leq \frac{5}{2} \|f\|_{L^2(0,T;H)}^2. \]

Next let $r > 0$ and assume that $\|f\|_{L^2(0,T;H)} \leq r$ and $\|h\|_{L^2(0,T;H)} \leq \frac{r}{2}$.

From (2.1), (2.4) and (2.8) we then have (for $0 < \varepsilon < \frac{1}{2}$)

\[ \|Qf\|_{L^2(0,T;H)} \leq \varepsilon r + C \frac{T^{1/2}}{\varepsilon} r^{2k} + \frac{r}{2}, \]

where $C = M \left( \frac{5}{2} \right)^k$. Since $k > \frac{1}{2}$, the last inequality shows that for a sufficiently small $r > 0$ we have

\[ \|Qf\|_{L^2(0,T;H)} \leq r \]
for each \( f \in L^2(0, T; H) \) with \( \|f\|_{L^2(0, T; H)} \leq r \) if \( \|h\|_{L^2(0, T; H)} \leq \frac{r}{2} \). Thus \( Q \) maps \( K \) into itself. By \((H_2)\) we conclude that \( Q \) is closed and convex valued.

We will show next that the graph of \( Q \) is weakly \( \times \) weakly sequentially closed. To this aim let \( ((p_n, q_n))_{n \in \mathbb{N}} \) be a sequence in the graph of \( Q \) such that

\[
\lim_{n \to \infty} p_n = p \quad \text{and} \quad \lim_{n \to \infty} q_n = q \quad \text{weakly in} \quad L^2(0, T; H).
\]

Set \( u_n = u_{p_n} \) for each \( n \in \mathbb{N} \). Then there exist \( (g_n)_{n \in \mathbb{N}} \) and \( (f_n)_{n \in \mathbb{N}} \) such that

\[
\begin{aligned}
&u_n'(t) + g_n(t) = p_n(t) \\
g_n \in SL^2(\partial \varphi(u_n(\cdot))) \\
f_n \in SL^2(F(\cdot, u_n(\cdot))) \\
u_n(0) = -u(T),
\end{aligned}
\]

and \( q_n = f_n + h \) for each \( n \in \mathbb{N} \). Since \( (p_n)_{n \in \mathbb{N}} \) is bounded in \( L^2(0, T; H) \) and \( \varphi \) is of compact type (cf. \((H_1)\)), Lemma 3.3 in [1] in conjunction with (ii) in \((H_2)\) implies, upon passage to the limit in (2.9) as \( n \to \infty \), that \( (p, q) \) belongs to the graph of \( Q \). Now the multivalued version of Theorem 1 in [3] comes into play and shows that \( Q \) has at least one fixed point in \( K \) and this completes the proof.

3. Examples. We consider first the Navier–Stokes system in dimension 3 with anti–periodic conditions

\[
\begin{aligned}
&u_t - \Delta u + u \cdot \nabla u = h - \nabla p \\
div u = 0 \\
u = 0 \\
u(0, x) = -u(T, x)
\end{aligned}
\]

in \([0, T] \times \Omega\) and \((0, T] \times \Omega\) respectively the pressure of an incompressible fluid moving inside \( \Omega \). Denoting by \( H(\Omega) \) the space \([L^2(\Omega)]^3\) and by \( H_\sigma(\Omega) \) the completion of all \( C_0^\infty \) divergence free vector fields on \( \Omega \) with respect to the \( H(\Omega) \) norm, (3.1) can be rewritten as a problem of the form (1.1) in the space \( H_\sigma(\Omega) \) where \( \partial \varphi(u) = -P_\Omega \Delta u \) and \( F(t, u) = P_\Omega (u \nabla u) \), \( P_\Omega \) being the orthogonal projection of \( H(\Omega) \) on \( H_\sigma(\Omega) \). It is well–known that \( \varphi \) satisfies \((H_1)\). See for instance [9] Section 10 in Chapter 4. Furthermore, by Lemma 4.10.2 and Lemma 4.10.5 in [9] it readily follows that \( F \) satisfies \((H_3)\). So, from Theorem 1, we get
Theorem 3.1. For each $T > 0$ there exists $\rho > 0$ such that, for each $h \in L^2(0, T; H(\Omega))$ with $\|P_Ih\|_{L^2(0, T; H(\Omega))} \leq \rho$, the problem (3.1) has at least one solution $u \in W^{1, 2}(0, T; H(\Omega))$ satisfying:

$$u(t) \in H^2(\Omega) \text{ a.e. on } [0, T]$$
$$t \mapsto u_t, \text{ and } t \mapsto P_Iu(t) \text{ belong to } L^2(0, T; H(\Omega))$$

$$t \mapsto \varphi(u(t)) \text{ is absolutely continuous on } [0, T].$$

Next, let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma$, let $\beta : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ be a maximal monotone operator with $\beta = \partial j$ and consider the following anti–periodic problem with dynamic boundary conditions

$$\begin{cases}
  u_t - \Delta u = f_\Omega(x, u, \nabla u) + h_\Gamma(t) & \text{in } [0, T] \times \Omega \\
  u_t + u_\nu + \beta(u) \ni f_\Gamma(x, u, \nabla u) + h_\Gamma(t) & \text{on } [0, T] \times \Gamma \\
  u(0, x) = -u(T, x) & \text{in } \Omega \\
  u(0, x) = -u(T, x) & \text{on } \Gamma,
\end{cases}$$

Here and thereafter, $u_\nu$ stands for the outward normal derivative of $u$, $f_\Omega : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $f_\Gamma : \Gamma \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous functions and $h_\Omega \in L^2(\Omega)$, $h_\Gamma \in L^2(\Gamma)$.

The problem (3.2) can be equivalently rewritten in the abstract form (1.1) as follows. First take $H = L^2(\Omega) \times L^2(\Gamma)$ endowed with the usual inner product $\langle u, v \rangle = \langle u_\Omega, v_\Omega \rangle_{L^2(\Omega)} + \langle u_\Gamma, v_\Gamma \rangle_{L^2(\Gamma)}$, where $u_\Omega, v_\Omega$ are the $L^2(\Omega)$ components of $u, v$ and $u_\Gamma, v_\Gamma$ are the $L^2(\Gamma)$ components of $u, v$.

Next let us define the function $\varphi : H \rightarrow \mathbb{R}_+$ by

$$\varphi(u) = \begin{cases}
  \frac{1}{2} \int_\Omega \|\nabla u\|^2 dx + \int_\Gamma j(u_\Gamma) d\sigma & \text{if } u \in D(\varphi) \\
  +\infty & \text{otherwise}
\end{cases}$$

where

$$D(\varphi) = \{ u \in H; \ u_\Omega \in H^1(\Omega), \ j(u_\Gamma) \in L^1(\Gamma) \text{ and } u_\Omega|_\Gamma = u_\Gamma \}.$$ 

Finally define $F : [0, T] \times H^2(\Omega) \times H^{3/2}(\Gamma) \rightarrow H$ by

$$F(t, u) = (f_\Omega(x, u_\Omega(x)), f_\Gamma(y, u_\Gamma(y)))$$

for a.a. $x \in \Omega$ and a.a. $y \in \Gamma$, and $h \in L^2(0, T; H)$ by $h = (h_\Omega, h_\Gamma)$.

The hypotheses we need in what follows are listed below.
(H₃) \( j : \mathbb{R} \to \mathbb{R}_+ \) is a proper, convex, even and l.s.c. function with \( j(0) = 0 \) which satisfies
\[
    cv^2 \leq j(v)
\]
for each \( v \in D(j) \), where \( c > 0 \).

(H₄) \( f_\Omega \) and \( f_\Gamma \) are continuous and there exist \( c_1 > 0, c_2 > 0 \) and \( d_1, d_2 \in \mathbb{R} \) such that
\[
    |f_\Omega(x, u, w)| \leq c_1|u||w|_{\mathbb{R}^n} + d_1
\]
and
\[
    |f_\Gamma(y, u, w)| \leq c_2|u||w|_{\mathbb{R}^n} + d_2
\]
for each \( u \in \mathbb{R}, w \in \mathbb{R}^n, x \in \Omega \) and \( y \in \Gamma \).

An application of Theorem 2.1 yields

**Theorem 3.2.** If (H₃) and (H₄) are satisfied, then for each \( T > 0 \) there exists \( \rho > 0 \) such that for each \( h \in L^2(0, T; H) \) with \( \|h\|_{L^2(0, T; H)} \leq \rho \), the problem (3.2) has at least one solution satisfying
\[
    u \in W^{1, 2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)).
\]

**Proof.** From [4] we know that in the hypotheses (H₃) \( \varphi \) is of compact type, while from (H₄) it readily follows that \( F \) satisfies (H₂).

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