Subseries of a monotone $P$-divergent double series

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Abstract This paper presents the following general characterization of $P$-divergency of double sub-series. Let $(I_m, J_n)$ be a Pringsheim increasing double sequence of distance positive order pair of integers, and let $\sum d_{m,n}$ be a monotone $P$-divergent double series. A necessary and sufficient condition for the $P$-divergence of the subseries $\sum d_{I_m,J_n}$ is that $I_m$ and $J_n$ are bounded.

Keywords RH-regular · Double sequence · Pringsheim limit point · $P$-convergent

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1 Introduction

With divergent ordinary series it is natural to ask how many terms can one remove and still assure the divergence of such series. If a series is merely divergent that at most a finite number may be neglected. The same is also true for double sequences. However, for single dimensional series if the series is also monotone then more terms may be neglected as proved by Hamming in [1]. The goal of this paper is to extend Hamming’s results to double dimensional series. This will be accomplished with the presentation of the following theorem.

Let $(I_m, J_n)$ be a Pringsheim increasing double sequence of distance positive order pair of integers, and let $\sum d_{m,n}$ be a monotone $P$-divergent double series. A necessary and sufficient condition for the $P$-divergence of the subseries $\sum d_{I_m,J_n}$ is that $I_m$ and $J_n$ are bounded.
2 Definitions, notations, and preliminary results

Before continuing with this paper we present some definitions and preliminaries.

**Definition 2.1 (Pringsheim, 1900)** A double sequence \( x = [x_{k,l}] \) has Pringsheim limit \( L \) (denoted by \( P\text{-}\lim x = L \)) provided that given \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_{k,l} - L| < \epsilon \) whenever \( k, l > N \). Such an \( x \) is describe more briefly as “\( P\text{-}\)convergent”.

**Definition 2.2 (Patterson, 2000)** The double sequence \( y \) is a double subsequence of \( x \) provided that there exist increasing index sequences \( \{n_j\} \) and \( \{k_j\} \) such that if \( x_j = x_{n_j,k_j} \), then \( y \) is formed by

\[
\begin{align*}
x_1 & \quad x_2 \quad x_5 \quad x_{10} \\
x_4 & \quad x_3 \quad x_6 \\
x_9 & \quad x_8 \quad x_7 \\
& \quad \cdots \quad \cdots \\
& \quad \cdots \quad \cdots 
\end{align*}
\]

A number \( \beta \) is called a Pringsheim limit point of the double sequence \( x = [x_{n,k}] \) provided that there exists a subsequence \( y = [y_{n,k}] \) of \([x_{n,k}]\) that has Pringsheim limit \( \beta : P\text{-}\lim y_{n,k} = \beta \). With this definition we can finally presented a complementary notions of \( P\)-convergence double, that is, a double sequence \( x \) is divergent in the Pringsheim sense (\( P\)-divergent) provided that \( x \) does not converge in the Pringsheim sense (\( P\)-convergent). For more recent developments on double sequences one can consult the papers (see, [2]-[7], [8], [9]) where more references can be found.

3 Main results

**Theorem 3.1** Let \((I_n, J_n)\) be a Pringsheim increasing double sequence of distance positive order pair of integers, and let \( \sum d_{m,n} \) be a monotone \( P\)-divergent double series. A necessary and sufficient condition for the \( P\)-divergence of the subseries \( \sum d_{I_m,J_n} \) is that \( \frac{I_n}{m} \) and \( \frac{J_n}{n} \) are bounded.

**Proof.** If \( \frac{I_n}{m} \) and \( \frac{J_n}{n} \) the result is clear. Now let us establish the necessary part. Let us assume without loss of generality that \( \limsup \frac{I_n}{m} = \infty \) and \( \limsup \frac{J_n}{n} = \infty \) and exhibit a monotone \( P\)-divergent double series \( \sum d_{m,n} \) for which the subseries \( \sum d_{I_m,J_n} \) is \( P\)-convergent. Let \( m_0 = I_{m_0} = 0 \) and \( n_0 = J_{n_0} = 0 \) and choose \( m_1 \) and \( n_1 \) such that \( \frac{I_{m_k+1} - I_{m_k}}{m_k+1 + m_k} > 2^k \) and \( \frac{J_{n_1+1} - J_{n_1}}{n_1 + n_1} > 2^l \) hold for \( k = l = 0 \). Then choose the following ordered integer sequences \( m_2, m_3, m_4, m_5, \ldots \), and \( n_2, n_3, n_4, n_5, \ldots \) such that \( m_{k+1} - m_k > m_k - m_{k-1} \) and \( n_{l+1} - n_l > n_l - n_{l-1} \) with \( \frac{I_{m_k+1} - I_{m_k}}{m_k+1 + m_k} > 2^k \) and \( \frac{J_{n_l+1} - J_{n_l}}{n_l + n_l} > 2^l \), for \( k, l = 0, 1, 2, \ldots \). We then choose constants \( B(U) \) and \( B(V) \) such that \( 2 \geq B[I_{m_k+1}] \geq B[J_{m_k+2}] \geq \cdots \geq B[I_{m_k+1}] \geq 1 \) and \( 2 \geq B[\{J_{m_k+1}\}] \geq 2 \geq B[J_{m_k+2}] \geq \cdots \geq B[J_{m_k+1}] \geq 1 \). For \( k, l = 0, 1, 2, \ldots \) let \( d_{U,V} = \frac{B(U)B(V)}{2^k + \frac{|m_{k+1} - m_k|}{|n_{l+1} - n_l|} I_{m_k}} \), \( I_{m_k} < U \leq I_{m_k+1}J_{n_l} < V \leq J_{n_l+1} \). Therefore \( \sum d_{m,n} \) is a monotone Pringsheim decreasing with
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$P - \lim_{m,n} d_{m,n} = 0$. But for each $(k, l)$ with $k, l = 1, 2, 3, \ldots$, 

$$
\sum_{\{(U, V) : I_{m_k} < U \leq I_{m_{k+1}}, J_{n_l} < V \leq J_{n_{l+1}}\}} B(U) \bar{B}(V) \frac{1}{2^{k+l}[m_{k+1} - m_k][n_{l+1} - n_l]} > \sum_{\{(U, V) : I_{m_k} < U \leq I_{m_{k+1}}, J_{n_l} < V \leq J_{n_{l+1}}\}} \frac{1}{2^{k+l}[m_{k+1} - m_k][n_{l+1} - n_l]} \left[ I_{m_{k+1}} - I_{m_k} \right] \left[ J_{n_{l+1}} - J_{n_l} \right] \frac{1}{2^{k+l}[m_{k+1} - m_k][n_{l+1} - n_l]} > 1
$$

and thus $\sum d_{m,n}$ is $P$-divergent. Finally, for each $k, l = 1, 2, 3, \ldots$ we have 

$$
\sum_{\{(U, V) : I_{m_k} < U \leq I_{m_{k+1}}, n_l < V \leq n_{l+1}\}} d_{I_{m_k}, J_{n_l}} = \sum_{\{(U, V) : I_{m_k} < U \leq I_{m_{k+1}}, n_l < V \leq n_{l+1}\}} \frac{B(I_{U}) \bar{B}(J_{V})}{2^{k+l}[m_{k+1} - m_k][n_{l+1} - n_l]} \leq \sum_{\{(U, V) : I_{m_k} < U \leq I_{m_{k+1}}, n_l < V \leq n_{l+1}\}} \frac{2}{2^{k+l}[m_{k+1} - m_k][n_{l+1} - n_l]} = \frac{1}{2^{k+l-2}}
$$

and thus $\sum d_{m,n}$ $P$-converges. Therefore the series $\sum d_{m,n}$ satisfies the conditions. 

$\square$

References

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