ON SOME DIFFERENCE EQUATIONS

BY

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Abstract. Asymptotic properties of some non-linear difference equations are considered.

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Volterra difference equations arise in the mathematical modelling of some real phenomena and also in various procedures of numerical solutions of some differential and integral equations. This motivates an essential interest in investigating the asymptotic properties of the solutions and in developing appropriate methods of the analysis.

The purpose of this paper is to prove theorems concerning asymptotic properties (convergent to a constant vector as \( n \to \infty \)) of some linear and nonlinear difference equations.

Some results concerning stability and asymptotic behaviour (convergent to a constant vector and approximation of solution for some system) of solutions of Volterra difference equations have been established for example in papers [1, 2, 3, 4].

We adopt the following notations in this paper:
\( Z \) is the set of all non-negative integers, \( N(n_0) = \{n_0, n_0 + 1, \ldots \} \), \( n_0 \in Z \), \( R_k \)-the \( k \)-dimensional real Euclidean space with norm \( |x| = \sum_{i=1}^{k} |x_i| \), \( x = (x_1, \ldots, x_k) \), \( M^k \)-the space of all \( k \times k \) matrices \( A = (a_{ij}) \) with \( \bullet \) given by \( |A| = \sum_{i=1}^{k} \sum_{j=1}^{k} |a_{ij}| \). The identity matrix is denoted by \( E \).
We start with a Lemma which connected asymptotic properties of solutions of Volterra difference equations in two cases.

**Lemma 1.** Assume \( k \geq 1 \) and for every \( v(1 \leq v \leq k) \) the functions \( \varphi_v, g_v, \psi_v: N(n_0) \rightarrow R \) satisfy the conditions:

\[
\text{(1a)} \quad \lim_{n \to \infty} |g_v(n)| = \infty \quad \text{and} \quad \sum_{s=n_0}^{n} |\psi_v(s)| \leq K|g_v(n)| \\
\text{or} \]

\[
\text{(1b)} \quad \lim_{n \to \infty} |g_v(n)| = 0, \quad g_v(n) \neq 0 \quad \text{and} \quad \sum_{s=n}^{\infty} |\psi_v(s)| \leq K|g_v(n)| 
\]

for \( n \in N(n_0) \) with some \( K \geq 1 \),  

\[
\text{(1c)} \quad \lim_{n \to \infty} \varphi_v(n) = 0.
\]

Let

\[
J_v^{(0)}(z) = \frac{\lambda}{|g_v(n)|} \sum_{s=n_0}^{n} |\psi_v(s)|z(s) \quad \text{in the case (1a)}
\]

and

\[
J_v^{(0)}(z) = \frac{\lambda}{|g_v(n)|} \sum_{s=n}^{\infty} |\psi_v(s)|z(s) \quad \text{in the case (1b)},
\]

where \( \lambda \)-satisfies the inequality \( 0 < \lambda < \frac{1}{kK} \).

Then the system (or equation in the case \( k = 1 \))

\[
(1) \quad z_v(n) = \varphi_v(n) + J_v^{(0)} \left( \sum_{j=1}^{k} z_j(n) \right), \quad v = 1, \ldots, k,
\]

has, for \( n \geq n_0 \), a solution \( \{\bar{z}_n, \ldots, \bar{z}_k\} \) defined by

\[
(2) \quad \bar{z}_v(n) = \varphi_v(n) + \sum_{m=0}^{\infty} \sum_{j=1}^{k} J_v^{(m)}(\varphi_j(n)), \quad v = 1, \ldots, k,
\]

where

\[
(3) \quad J_v^{(m+1)}(z) = \sum_{j=1}^{k} J_v^{(0)}(J_j^{(m)}(z)), \quad v = 1, \ldots, k, \quad m = 0, 1, 2, \ldots
\]
such that \( \lim_{n \to \infty} \bar{z}_v(n) = 0, \quad v = 1, \ldots, k \).

**Proof.** For \( n \in N(n_0), \quad v = 1, \ldots, k \) and \( m = 0, 1, 2, \ldots \) from (3) we will prove by induction the inequality

(4) \[ 0 \leq J_v^{(m)}(1) \leq (\lambda kK)^{m+1}, \quad m = 0, 1, 2, \ldots \]

By the assumption (1a) or (1b) we have \( 0 \leq J_v^{(0)}(1) \leq \lambda K \) and inequality (4) holds for \( m = 0 \). Assume that the formula holds for \( m > 0 \). We will prove it for \( m + 1 \). From (3) it follows that

(4) \[ J_v^{(m+1)}(1) = \sum_{j=1}^{k} J_v^{(0)}(J_j^{(m)}(1)) \leq \sum_{j=1}^{k} J_v^{(0)}((\lambda K)^{m+1}k^m) \leq (\lambda kK)^{m+2}. \]

Because \( \lim_{n \to \infty} \varphi_v(n) = 0 \), then we can choose \( M > 0 \) so that \( |\varphi_v(n)| \leq M \) for \( n \in N(n_0) \) and \( v = 1, 2, \ldots, k \). Let us put

(5) \[ \bar{z}_v(n) = \varphi_v(n) + \sum_{m=0}^\infty \sum_{j=1}^{k} J_v^{(m)}(\varphi_j(n)), \quad v = 1, \ldots, k, \]

the series being uniformly convergent for \( n \in N(n_0) \) by (4). Moreover from (5) we have

\[ |\bar{z}_v(n)| \leq M + M\lambda k^2K(1 - \lambda kK)^{-1} \]

for \( v = 1, \ldots, k, \quad n \in N(n_0) \).

We show that the functions \( \bar{z}_v \) satisfy (1). Namely we have

\[
\varphi_v(n) + J_v^{(0)} \left( \sum_{j=1}^{k} \bar{z}_j(n) \right) = \varphi_v(n) + \sum_{j=1}^{k} J_v^{(0)}(\varphi_j) + \]

\[ + \sum_{j=1}^{k} J_v^{(0)} \left( \sum_{m=0}^{\infty} \sum_{l=1}^{k} J_j^{(m)}(\varphi_l) \right) = \]

\[ = \varphi_v(n) + \sum_{j=1}^{k} J_v^{(0)}(\varphi_j) + \sum_{m=0}^{\infty} \sum_{l=1}^{k} \sum_{j=1}^{k} J_v^{(0)}(J_j^{(m)}(\varphi_l)) = \]
\[ = \varphi_v(n) + \sum_{j=1}^{k} J_v^{(0)}(\varphi_j) + \sum_{m=0}^{\infty} \sum_{l=1}^{k} J_v^{(m+1)}(\varphi_l) = \]

\[ = \varphi_v(n) + \sum_{m=0}^{\infty} \sum_{l=1}^{k} J_v^{(m)}(\varphi_l) = \tilde{z}_v(n). \]

We next prove that \( \lim_{n \to \infty} \tilde{z}_v(n) = 0 \). Let \( L_v = \lim_{n \to \infty} |\tilde{z}_v(n)| \), then from (1) we obtain \( L_v \leq \lambda K \sum_{j=1}^{k} L_j \).

Hence

\[ \sum_{v=1}^{k} L_v \leq \lambda k K \sum_{j=1}^{k} L_j \text{ and } (1 - \lambda k K) \sum_{v=1}^{k} L_v \leq 0. \]

We obtain from this \( L_v = 0 \) for \( v = 1, \ldots, k \) and Lemma follows.

Let us consider the nonlinear system (or equation in the case \( k = 1 \))

\[ (6) \quad g_v(n) = \frac{\gamma_v}{g_v(n)} + r_v P_v(f_v(n, y_1(n), \ldots, y_k(n))), \quad v = 1, \ldots, k \]

where

\[ (7) \quad P_v(z) = \frac{1}{g_v(n)} \sum_{s=n_0}^{n} \psi_v(s)z(s), \quad r_v = 1 \]

and function \( g_v(n) \) satisfies condition

\[ (1a') \quad \lim_{n \to \infty} |g_v(n)| = \infty \text{ and } \sum_{s=n_0}^{n} |\psi_v(s)| \leq K|g_v(n)| \]

or

\[ (8) \quad P_v(z) = \frac{1}{g_v(n)} \sum_{s=n}^{\infty} \psi_v(s)z(s), \quad r_v = -1 \]

and function \( g_v(n) \) satisfies condition

\[ (1b') \quad \lim_{n \to \infty} |g_v(n)| = 0, g_v(n) \neq 0 \text{ and } \sum_{s=n}^{\infty} |\psi_v(s)| \leq K|g_v(n)|, \]

\[ v = 1, \ldots, k, \quad K \geq 1. \]
We assume \( \gamma_v = 1 \) in the case (1a') and \( \gamma_v = 0 \) in the case (1b').

**Theorem 2.** Suppose that:

1° for every \( 1 \leq v \leq k \) the function \( g_v(n) \) satisfies condition (1a') or (1b')

2° there exists \( \eta > 0 \) and values \( c_v \) such that \( f_v(n, y_1, \ldots, y_k) \), \( v = 1, \ldots, k \) are continuous functions of the variables \( y_1, \ldots, y_k \) and satisfy the inequalities

\[
|f_v(n, u_1, \ldots, u_k) - f_v(n, v_1, \ldots, v_k)| \leq \lambda \sum_{j=1}^{k} |u_j - v_j|
\]

for \( n \in N(n_0) \), \( |y_j - c_j| \leq \eta \), \( |u_j - c_j| \leq \eta \), \( |v_j - c_j| \leq \eta \) (\( j = 1, \ldots, k \)) with some \( \lambda \) satisfying the inequality \( 0 < \lambda < (kK)^{-1} \),

3° \( \lim_{n \to \infty} f_v(n, c_1, \ldots, c_k) = c_v \), \( v = 1, \ldots, k \)

4° \( \Delta |g_v(n)| > 0 \) (\( \Delta |g_v(n)| < 0 \)) for all large \( n \in N(n_0) \)

5° for each \( n \in N(n_0) \) there exists \( K_1 > 0 \) such that \( \frac{|\psi_v(n + 1)|}{\Delta |g_v(n)|} \leq K_1 \), \( K_1 \leq K \)

6° \( \lim_{n \to \infty} \frac{1}{g_v(n)} \sum_{s=n_0}^{n} \psi_v(s) = 1 \) for \( v = 1, \ldots, k \).

Then the system (6) has, for sufficiently large \( n \), a solution \( y^* \) such that \( \lim_{n \to \infty} y^*_v(n) = c_v \) for \( v = 1, \ldots, k \).

**Proof.** We first suppose that \( c_v = 0 \) for \( v = 1, \ldots, k \) and choose \( n_1 \in N(n_0) \) such that for \( n \geq n_1 \) we have

\[
|f_v(n, 0, \ldots, 0)| \leq \eta \frac{(1 - k\lambda K)}{4K}
\]

and

\[
\frac{1}{|g_v(n)|} \leq \eta \frac{(1 - k\lambda K)}{K}
\]

if (1a') holds, for \( v = 1, \ldots, k \).
We set, for \( n \geq n_1 \) and \( v = 1, \ldots, k \),

\[
y_{v0}(n) = \frac{\gamma_k}{g_v(n)},
\]

\[
y_{vm+1}(n) = y_{v0}(n) + r_v P_v(f_v(n, y_{1m}(n), \ldots, y_{km}(n))), \quad m = 0, 1, \ldots
\]

By assumptions 1°, (9) and 3° we obtain

\[
|f_v(n, y_{10}, \ldots, y_{ko})| \leq |f_v(n, 0, \ldots, 0)| + 
\]

\[
+ \lambda \sum_{v=1}^k |y_v^{(n)}| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad v = 1, \ldots, k.
\]

Then there exists \( P_v(f_v(n, y_{10}, \ldots, y_{ko})) \). Applying l’Hospital’s theorem, we have

\[
\lim_{n \to \infty} |P_v(f_v(n, y_{10}(n), \ldots, y_{ko}(n)))| \leq 
\]

\[
\leq \lim_{n \to \infty} \frac{1}{|g_v(n)|} \sum_{s=n_0}^n |\psi_v(s)||f_v(s, y_{10}(s), \ldots, y_{ko}(s))| \leq 
\]

\[
\leq K \lim_{n \to \infty} |f_v(n+1, y_{10}(n+1), \ldots, y_{ko}(n+1))| = 0
\]

We obtain from this that \( \lim_{n \to \infty} y_{v1}(n) = 0 \). Analogues we prove by induction for \( m = 1, 2, \ldots \) the existence of \( P_v(f_v(n, y_{1m}, \ldots, y_{km})) \).

We will show by induction for \( m = 0, 1 \ldots \) that

\[
(10) \quad |y_{vm} - y_{v0}| \leq \frac{n}{2} \quad \text{and} \quad |y_{vm}| \leq \eta
\]

for \( n \geq n_1 \) and \( v = 1, \ldots, k \).

From definition of \( y_{v0} \) we get \( |y_{v0}| \leq \eta \). \( \frac{1-k\lambda K}{4} < \frac{\eta}{2} \) and (10) hold for \( m = 0 \). Suppose that (10) is true for some \( m (m \geq 0) \). Then by (9) we have successively

\[
|f_v(n, y_{1m}, \ldots, y_{km})| \leq |f_v(n, 0, \ldots, 0)| + \lambda \sum_{j=1}^k |y_{j0}| + \lambda \sum_{j=1}^k |y_{jm} - y_{j0}| \leq 
\]
\[ \leq \eta \left( \frac{1 - k\lambda K}{4K} + \lambda k\eta \frac{(1 - k\lambda K)}{4} + k\lambda \frac{\eta}{2} \right) \leq \frac{\eta}{2K}. \]

\[ |y_{v+1} - y_0| = |P_v(f_v(n, y_{1m}, \ldots, y_{km})| \leq \]

\[ \leq \frac{1}{|g_v(n)|} \sum_{s=n_0}^{n} |\psi_v(s)| |f_v(s, y_{1m}, \ldots, y_{km})| \leq \]

\[ \leq \frac{\eta}{2K} \frac{1}{|g_v(n)|} \sum_{s=n_0}^{n} |\psi_v(s)| \leq \frac{\eta}{2}. \]

From this we obtain \(|y_{v+1}| < \eta|, since \(|y_0| < \frac{\eta}{2}|. Let \(J_v^{(m)}(z) be defined as in Lemma with \(n_1 instead of \(n_0. We set

\[ \varphi_j(n) = \sum_{v=1}^{k} |f_v(n, 0, \ldots, 0)| (k\lambda)^{-1} + |y_{j0}|. \]

We will prove by induction for \(m = 0, 1, \ldots ; n \geq n_1; v = 1, \ldots, k \) the inequality.

\[ (11) \quad |y_{v+1} - y_v| \leq J_v^{(m)} \left( \sum_{j=1}^{k} \varphi_j \right). \]

We obtain for \(m = 0\)

\[ |y_1 - y_0| = |P_v(f_v(n, y_{10}, \ldots, y_{k0}))| \leq \]

\[ \leq \frac{\lambda}{|g_v(n)|} \sum_{s=n_0}^{n} |\psi_v(s)| \left| \frac{f_v(s, y_{10}(s), \ldots, y_{k0}(s))}{\lambda} \right| = \]

\[ = J_v^{(0)} \left( \frac{|f_v(n, y_{10}, \ldots, y_{k0})|}{\lambda} \right) \leq \]

\[ \leq J_v^{(0)} \left( \frac{|f_v(n, 0, \ldots, 0)|}{\lambda} + \sum_{j=1}^{k} |y_{j0}| \right) = J_v^{(0)} \left( \sum_{j=1}^{k} \varphi_j \right). \]
Supposing that (11) is verified for some \( m (m \geq 0) \), it is

\[
|y_{vm+1} - y_{vm}| \leq J_v^{(m)} (\sum_{j=1}^{k} \varphi_j)
\]

let us prove it for \( m+1 \). Then

\[
|y_{vm+2} - y_{vm+1}| = |P_v(f_v(n, y_{1m+1}, \ldots, y_{km+1})) - P_v(f_v(n, y_{1m}, \ldots, y_{km}))| \leq
\]

\[
\leq \frac{\lambda}{|g_v(n)|} \sum_{s=n_0}^{n} |\psi_v(s)| \sum_{j=1}^{k} |y_{jm+1} - y_{jm}| = J_v^{(0)} \left( \sum_{j=1}^{k} |y_{jm+1} - y_{jm}| \right) \leq
\]

\[
\leq J_v^{(0)} \left( \sum_{j=1}^{k} J_v^{(m)} \left( \sum_{s=1}^{k} \varphi_s \right) \right) = J_v^{(m+1)} \left( \sum_{s=1}^{k} \varphi_s \right).
\]

Since \( \lim_{n \to \infty} f_v(n, 0, \ldots, 0) = 0 \), \( v = 1, \ldots, k \), it immediately follows that \( \lim_{n \to \infty} \varphi_j(n) = 0 \) for \( j = 1, \ldots, k \). On the other hand \( \varphi_j(n) \) are defined for \( n \geq n_1 \), they are bounded for these \( n \). By (11) and (4) we infer that the series \( \sum_{m=0}^{\infty} |y_{vm+1} - y_{vm}| \) are uniformly convergent for \( n \geq n_1 \), \( v = 1, \ldots, k \)

and there exists the limits \( \bar{y}_v = \lim_{m \to \infty} y_{vm} = y_v + \sum_{m=0}^{\infty} (y_{vm+1} - y_{vm}) \). Since \( y_{vm} \) are defined for \( n \geq n_1 \), then \( \bar{y}_v \) are also defined for these \( n \), and \( |\bar{y}_v| \leq \eta \) for \( n \geq n_1 \). Hence, by lemma

\[
|\bar{y}_v| \leq |y_v| + \sum_{m=0}^{\infty} |y_{vm+1} - y_{vm}| \leq \varphi_v + \sum_{m=0}^{\infty} J_v^{(m)} \left( \sum_{j=1}^{k} \varphi_j \right) = \bar{z}_v \to 0
\]

as \( n \to \infty \). Finally, we will prove that the equalities

\[
\lim_{m \to \infty} P_v(f_v(n, y_{1m}, \ldots, y_{km})) = P_v(f_v(n, \bar{y}_1, \ldots, \bar{y}_k))
\]

hold uniformly for \( n \geq n_1 \) and \( v = 1, \ldots, k \). For a given \( \varepsilon > 0 \) we choose an index \( l \) such that \( |y_{vm} - \bar{y}_v| < \varepsilon \) for \( m \geq l \), \( v = 1, \ldots, k \) and \( n \geq n_1 \). Then

\[
|P_v(F_v(n, y_{1m}, \ldots, y_{km})) - P_v(F_v(n, \bar{y}_1, \ldots, \bar{y}_k))| \leq
\]

\[
\leq
\]

\[
\leq
\]
\[ \leq \frac{\lambda}{g_v(n)} \sum_{s=n_0}^{n} |\psi_v(s)| \sum_{j=1}^{k} |y_{jm} - \bar{y}_j| \leq \lambda kK \varepsilon < \varepsilon. \]

We obtain from this that $\bar{y}_v$ is the solution of system (6). In the case $c_v \neq 0$, we substitute into (6) $y_v(n) = u_v(n) + c_v$,

\[ f_v(n, y_1, \ldots, y_k) = f_v^*(n, u_1, \ldots, u_k) + c_v \]

\[ h_v(n) = \frac{1}{g_v(n)} + \left( \frac{1}{g_v(n)} \sum_{s=n_0}^{n} \psi_v(s) - 1 \right) c_v \]

and we obtain the system

\[ (12) \quad u_v(n) = h_v(n) + \frac{1}{g_v(n)} \sum_{s=n_0}^{n} \psi_v(s) f_v^*(s, u_1(s), \ldots, u_k(s)), \quad v = 1, \ldots, k. \]

where the functions $f_v^*$ (instead of $f_v$) satisfy hypotheses of Theorem for $c_v = 0$. Since $\lim_{n \to \infty} h_v(n) = 0$ for $v = 1, \ldots, k$, we obtain that there exists, for $n \geq n_1$, solution $\{u_1, \ldots, u_k\}$ of the system (12) such that $\lim_{n \to \infty} u_v(n) = 0$ for $v = 1, \ldots, k$. This gives

\[ \lim_{n \to \infty} \bar{y}_v(n) = c_v \]

when substituted in $\bar{y}_v(n) = \bar{u}_v(n) + c_v$.

**Lemma 3.** Assume $k \geq 1$ and for every $v(1 \leq v \leq k)$ the functions $\varphi_v : N(n_0) \to R$, $N_v : N(n_0) \times N(n_0) \to R$ satisfies the conditions:

(i) $\lim_{n \to \infty} \varphi_v(n) = 0$,

(ii) $\lim_{n \to \infty} \sum_{s=n_0}^{n} |N_v(n, s)| \leq K < \infty$.

Let

\[ J_v^{(0)} = \lambda \sum_{s=n_0}^{\infty} |N_v(n, s)| \varphi(s), \quad \lambda \text{satisfies the inequality } 0 < \lambda < \frac{1}{kK_1}, \quad K_1 \geq K. \]

Then the system (or equation in the case $k = 1$)

\[ z_v(n) = \varphi_v(n) + J_v^{(0)} \left( \sum_{j=1}^{k} z_j(n) \right), \quad v = 1, \ldots, k \]
has for \( n \geq n_0 \) a solution \( \{ \bar{z}_1, \ldots, \bar{z}_k \} \) defined by

\[
\bar{z}_v(n) = \varphi_v(n) + \sum_{m=0}^{\infty} \sum_{j=1}^{k} J_v^{(m)}(\varphi_j(n))
\]

where

\[
J_v^{(m+1)}(z) = \sum_{j=1}^{k} J_v^{(0)}(J_j^{(m)}(z)), \quad v = 1, \ldots, k, \quad m = 0, 1, 2, \ldots,
\]

such that \( \lim_{n \to \infty} \bar{z}_v(n) = 0, \quad v = 1, \ldots, k. \)

Proof [see Lemma 1].

The system (6) can be generalized in the form

\[
y_v(n) = \varphi_v(n) + \sum_{s=n_0}^{\infty} N_v(n, s)f_v(s, y_1(s), \ldots, y_k(s)), \quad v = 1, \ldots, n
\]

where \( \lim_{n \to \infty} \varphi_v(n) = 0 \) and the functions \( N_v(n, s) \) satisfy conditions:

(i) \( \lim_{n \to \infty} \sum_{s=n_0}^{\infty} |N_v(n, s)| \leq K < \infty, \)

(ii) \( \lim_{n \to \infty} N_v(n, s) = 0 \) uniformly for \( s \geq n_0 \)

(iii) \( \lim_{n \to \infty} \sum_{s=n_0}^{\infty} N_v(n, s) = 1. \)

As in the proof of Theorem 2 one can prove the existence, for large \( n \), of a solution \( \{ \bar{y}_1, \ldots, \bar{y}_k \} \) of system (13) such that \( \lim_{n \to \infty} \bar{y}_v(n) = c_v. \)

We consider asymptotic properties (for \( n \to \infty \)) of solutions of the difference equation

\[
a(n)y(n + 1) = y(n) + f(n, y(n)).
\]

Theorem 4. Suppose that

1° \( a : (0, \infty) \to (1, \infty), \)
2° \( \lim_{n \to \infty} g(n) = \infty \) where \( g(n) = \prod_{l=0}^{n-1} a(l), \sum_{l=1}^{\infty} \frac{1}{a(l)} \leq K < \infty \),

3° the function \( f \) (for \( k = 1 \)) satisfies conditions 2°, 3° of Theorem 2,

4° the conditions 5°, 6° of Theorem 2 are satisfied with

\[
\psi(n) = \frac{\Delta g(n)}{(a(n) - 1)a(n)}.
\]

Then the difference equation (14) has for sufficiently large \( n \) solution \( \{y(n)\} \) such that.

\[
\lim_{n \to \infty} y(n) = c.
\]

**Proof.** We set \( g(n) = \prod_{l=0}^{n-1} a(l), \psi(n) = \frac{\Delta g(n)}{(a(n) - 1)a(n)} \). Then we obtain formally from (14) the equation

\[
y(n) = \frac{x_0}{g(n)} + \frac{1}{g(n)} \sum_{s=0}^{n-1} \psi(s)f(s, y(s)).
\]

We complete the proof by applying Theorem 2.

Let us now consider the difference equation given by

\[
\Delta y(n) = F(n, y(n)),
\]

where \( F_y(n, y) \neq 0 \) exists and is a continuous function with respect \( y \) for \( n \geq n_0 \).

Sitting \( y(n) = r(n)u(n) \) into (16) we get

\[
\frac{1}{F_y(n, r(n))} u(n + 1) = u(n) + F^*(n, u(n)),
\]

where

\[
F^*(n, u(n)) = \frac{F(n, r(n)u(n))}{r(n + 1)F_y(n, r(n))} + \left[ \frac{1}{F_y(n, r(n))} \left( \frac{\Delta r(n)}{r(n + 1)} - 1 \right) \right] u(n).
\]

For this equation we can formulate analogues results as for equation (14).

**Remark.** Let \( F^*(n, 1) \to \infty \) for \( n \to \infty \), then we obtain that for sufficiently large \( n \) an integral \( \tilde{y}(n) \) of (16) has property \( \tilde{y}(n) \sim r(n) \) for \( n \to \infty \).
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