

Generalized orthogonality between rays and wavefronts in anisotropic inhomogeneous media

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Abstract

We prove that in elastic anisotropic inhomogeneous media, rays and wavefronts are orthogonal to each other with respect to the metric induced by the phase-velocity function. The standard orthogonality of rays and wavefronts in elastic isotropic inhomogeneous media is a special case of this formulation.

1 Introduction

It has been shown in several papers, e.g., Antonelli et al (2003), Bona and Slawinski (2002) and Antonelli et al (2002) that Finsler geometry provides a fruitful platform for the study of seismic ray theory in anisotropic inhomogeneous media. In this paper, we show that the Finsler metric, discussed in the above papers, provides a natural context for the study of rays and wavefronts in such media.

In general, in anisotropic media, rays and wavefronts are not orthogonal to each other in the sense of Euclidean geometry. However, in this paper we prove that rays and wavefronts are orthogonal to each other in the sense of the geometry imposed by the properties of the medium, which are stated in

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the context of a given phase-velocity function. Since this imposed geometry is different from the Euclidean geometry we need to introduce its fundamental concepts in the context of differential geometry.

We begin this paper by introducing the differential-geometry concepts that are pertinent to ray theory in anisotropic inhomogeneous media. Therein, we state definitions and equations that we need to investigate rays and wavefronts. Subsequently, we use our ray-theory Hamiltonian to obtain a generalized metric. Then, we show the generalized orthogonality between rays and wavefronts in anisotropic inhomogeneous media. We conclude with a discussion of several related ray-theory properties. Orthogonality of geodesics (rays) and geodesic balls (wavefronts) in the context of Finsler geometry is known as Gauss Lemma, see Bao et al (2000). The proof we give here is different and uses Hamilton equations of rays. Moreover we prefer to work on the cotangent space of a manifold rather than on the tangent space because the eikonal equations are defined there and the Hamiltonian metric may not be positive definite everywhere. This implies that the Legendre transformation may not be well defined somewhere and therefore we cannot use Euler Lagrange equations to give the usual proof for the orthogonality.

2 Geometrical background

2.1 Geometric space

To obtain the results of this paper, in this section we define several necessary entities and the spaces to which they belong.

The geometry of an elastic medium is the geometry of a triple (M, ρ, \mathbf{c}) , where M is a three-dimensional space, ρ is the mass-density function, and \mathbf{c} is a fourth-rank tensor with particular properties. We note that — in a mathematical context — M is a manifold.

To study rays and wavefronts in the context of differential geometry, we identify the physical space, M , of the elastic medium with an open subset of the Euclidean three-dimensional space, where the coordinates are given by functions x^1, x^2, x^3 . At every point \mathbf{x} of M , we consider the differential of functions ψ at this point, namely,

$$d\psi|_{\mathbf{x}} = \sum_{i=1}^3 \frac{\partial \psi}{\partial x^i} dx^i. \quad (1)$$

The set of the differentials at \mathbf{x} of all functions on M is a three-dimensional vector space. We denote this space by $T_{\mathbf{x}}^*M$, while we denote the collection of $T_{\mathbf{x}}^*M$ at all points \mathbf{x} by T^*M . We note that T^*M is often called the momentum space in classical mechanics, the phase space in Hamiltonian mechanics and corresponds to the cotangent space in differential geometry.

Our geometry must be associated with T^*M since the key entities that are discussed below — such as the phase slowness, the Hamiltonian, Hamilton’s equations and the Hamiltonian metric — are defined in this space.

Each element \mathbf{p} of $T_{\mathbf{x}}^*M$ can be expressed as $\mathbf{p} = p_1 dx^1 + p_2 dx^2 + p_3 dx^3$, where — in view of expression (1) — we have

$$p_i := \frac{\partial \psi}{\partial x^i}, \quad i \in \{1, 2, 3\}. \quad (2)$$

Then, the coordinates of T^*M are denoted by (x^i, p_i) , where $i = 1, 2, 3$. Symbol $:=$ emphasizes that expression (2) is a definition not an equation.

Since the entities that we are about to define are, in general, not differentiable at $\underline{\mathbf{p}} = 0$, we remove $(\mathbf{x}, \mathbf{0})$ from T^*M . We denote the remaining space by $\widehat{T^*M}$. Furthermore, as these entities are homogeneous in \mathbf{p} , if we do not remove $(\mathbf{x}, \mathbf{0})$, these entities would be reduced to special cases that would limit the generality of our subsequent formulation.

To give a seismological motivation, we note that expression (2) plays an important role in ray theory. If we consider a particular function $\psi(\mathbf{x})$, we can write a moving wavefront as $\psi(\mathbf{x}) = t$, where t denotes time. At a given instant, wavefronts are level sets of this particular function. Hence, in view of expression (2) — in this case — \mathbf{p} is normal to the wavefront. Since — in this case — units of the p_i are the units of slowness, we may refer to \mathbf{p} as the phase-slowness vector. We note that in this context, the removal of $\mathbf{p} = 0$ has an immediate physical meaning since we remove the case that would correspond to phase slowness that is equal to zero.

At this point, we also wish to emphasize the distinction between the formulation of various entities in T^*M and their meaning along particular curves in this space that correspond to rays and wavefronts in ray theory. Rigorous treatment of ray theory cannot be performed without global definitions. In other words, while we are interested only in the properties of rays and wavefronts, to study these properties, we require global definitions for our Hamiltonian and its related entities.

Let us now return to triple (M, ρ, \mathbf{c}) . At each point \mathbf{x} of M , the elasticity tensor is a fourth-rank tensor whose components in our coordinate system

are $c^{ijkl}(\mathbf{x})$, where $i, j, k, l \in \{1, 2, 3\}$. Using standard methods for solving equations of motion in anisotropic inhomogeneous media, which employ an asymptotic trial solution (e.g., Bóna and Slawinski, 2003), we obtain

$$[\Gamma^{ij}(\mathbf{x}, \mathbf{p}) - \delta^{ij}] A_j(\mathbf{x}) = \mathbf{0}, \quad i, j \in \{1, 2, 3\}, \quad (3)$$

where $\mathbf{A}(\mathbf{x})$ is the wavefront amplitude and $\Gamma(\mathbf{x}, \mathbf{p})$ is the matrix whose entries are

$$\tilde{\Gamma}^{ij}(\mathbf{x}, \mathbf{p}) = \sum_{k=1}^3 \sum_{l=1}^3 \frac{c^{ijkl}(\mathbf{x})}{\rho(\mathbf{x})} p_k p_l, \quad i, j \in \{1, 2, 3\}, \quad (4)$$

with $\rho(\mathbf{x})$ being mass density. Matrix $\Gamma^{ij}(\mathbf{x}, \mathbf{p})$ is called Christoffel's matrix. Due to the properties of the elasticity tensor, $c^{ijkl}(\mathbf{x})$, Christoffel's matrix is symmetric and positive-definite. Consequently, the three eigenvalues of this matrix are real and positive. We denote them by $G_\alpha(\mathbf{x}, \mathbf{p})$, where $\alpha \in \{1, 2, 3\}$. It can be shown that the eigenvalues, G_α , are homogeneous of degree 2 in \mathbf{p} (e.g., Červený, 2001, p. 22). In other words, $G_\alpha(\mathbf{x}, \lambda\mathbf{p}) = \lambda^2 G_\alpha(\mathbf{x}, \mathbf{p})$, for any real number λ . Also, functions G_α are differentiable on $\widetilde{T^*M}$. We note that — in view of the homogeneity of G_α — including $\mathbf{p} = 0$ in the domain of differentiability would have limited G_α to a quadratic function in \mathbf{p} . Consequently, our subsequent formulation would have been limited to elliptical anisotropy.

We wish to consider a given eigenvalue, G_α . Let us refer to it as G and, for convenience, let

$$H(\mathbf{x}, \mathbf{p}) := \frac{1}{2} G(\mathbf{x}, \mathbf{p}), \quad (5)$$

to which we shall refer as the ray-theory Hamiltonian. In view of the properties of G , we see that H is also homogeneous of degree 2 in \mathbf{p} . Since H is homogeneous of degree 2 in \mathbf{p} , by factoring $p^2 = \mathbf{p} \cdot \mathbf{p}$, we rewrite Hamiltonian (5) as

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2} p^2 v^2(\mathbf{x}, \mathbf{p}), \quad (6)$$

where, as we immediately see,

$$v^2(\mathbf{x}, \mathbf{p}) := \frac{2H(\mathbf{x}, \mathbf{p})}{p^2}. \quad (7)$$

Function v is homogeneous of degree 0 in \mathbf{p} . In other words, $v(x^i, \lambda p_i) = v(x^i, p_i)$. This property means that the value of v depends on direction of \mathbf{p} but not on the magnitude of \mathbf{p} .

Expression (6) is the form of our Hamiltonian to be used in the remainder of this paper. We note that our Hamiltonian is a product of two functions, namely, p^2 and $v^2(\mathbf{x}, \mathbf{p})$. The former function accounts for the homogeneity of H , while the latter one, which is homogeneous of degree 0 in \mathbf{p} , contains all the information about the medium. For instance, if v does not depend on \mathbf{p} , the medium is isotropic.

Having defined the phase-slowness vector, \mathbf{p} , function v , and the ray-theory Hamiltonian, H , which are all associated with T^*M , we are now ready to formulate the equations that will allow us to discuss wavefronts and rays. At this point, we will focus on level sets on T^*M that correspond to the wavefronts.

2.2 Wavefront and ray equations

To discuss the wavefronts, we invoke equation (3) and study the eigenvalues that are associated with this equation. For nontrivial solvability of equation (3), we require

$$\det [\Gamma^{ij}(\mathbf{x}, \mathbf{p}) - \delta^{ij}] = 0, \quad (8)$$

which is an eigenvalue equation. Hence — in the context of equation (3) — each of the three corresponding eigenvalues of matrix (4) gives us an equation, which can be written as

$$G(\mathbf{x}, \mathbf{p}) = 1. \quad (9)$$

This is an eikonal equation allowing us to describe the wavefronts. Following expressions (5) and (6), we can restate eikonal equation (9) as

$$p^2 = \frac{1}{v^2(\mathbf{x}, \mathbf{p})}. \quad (10)$$

In view of equation (10) and since wavefronts are loci of constant phase, we refer to v , defined in expression (7), as the phase-velocity function. Equations (9) and (10) are valid for anisotropic inhomogeneous media since, considering wavefronts and their normals, we see that \mathbf{x} refers to the dependence on position and \mathbf{p} refers to the dependence on direction.

We wish to explicitly state the eikonal equation as a differential equation. In view of expression (2), we can write equation (10) as

$$\sum_{i=1}^3 \left(\frac{\partial \psi}{\partial x^i} \right)^2 = \frac{1}{v^2 \left(x^i, \frac{\partial \psi}{\partial x^i} \right)}. \quad (11)$$

This is a first-order nonlinear partial differential equation for function ψ . To state equation (11) in terms of our Hamiltonian, H , in view of expression (6), we can write

$$2H(\mathbf{x}, \mathbf{p}) \equiv 2H \left(x^i, \frac{\partial \psi}{\partial x^i} \right) = 1, \quad (12)$$

which is akin to equation (9) and is a standard form of the Hamilton-Jacobi equation. The solution of equation (12) is function ψ , whose level sets are the wavefronts.

Solving the eikonal equation by the method of characteristics (e.g., Courant and Hilbert, 1989), we obtain Hamilton's ray equations, namely,

$$\begin{cases} \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \end{cases}, \quad i \in \{1, 2, 3\}. \quad (13)$$

Rays are curves whose components, $[x^1(t), x^2(t), x^3(t)]$, are solutions of system (13).

Having formulated the equations describing wavefronts and rays, we are now ready to study their geometrical relation, which is a function of a metric that characterizes a given geometry. Thus, we begin by formulating pertinent metrics.

2.3 Hamiltonian metric

In general, a given elastic medium exhibits two metric structures that are of interest in our study. They are the Euclidean metric δ_{ij} , where δ_{ij} is Kronecker's delta, and metric $g_{ij}(\mathbf{x}, \mathbf{p})$, to which we refer to as the Hamiltonian metric. The Euclidean metric is intrinsically associated with the physical

space, while the Hamiltonian metric is induced by the phase-velocity function that corresponds to a given type of waves that propagate in the medium. Since, in general, there are three distinct phase-velocity functions, which correspond to the three types of waves that propagate in an anisotropic inhomogeneous medium, at every point of the medium there are three Hamiltonian metrics.

Now, we wish to obtain the Hamiltonian metric that corresponds to the properties of a given medium.

Consider the ray-theory Hamiltonian given by expression (6). Using this Hamiltonian, we write a convenient metric (e.g., Rund, 1959) whose components are given by

$$g^{ij}(\mathbf{x}, \mathbf{p}) := \frac{\partial^2 H}{\partial p_i \partial p_j}(\mathbf{x}, \mathbf{p}), \quad i, j \in \{1, 2, 3\}. \quad (14)$$

Inserting expression (6) into expression (14), we can obtain an explicit expression for the components of this metric, namely,

$$g^{ij} = v^2 \delta^{ij} + 2v \left(p^i \frac{\partial v}{\partial p_j} + p^j \frac{\partial v}{\partial p_i} \right) + p^2 \left(v \frac{\partial^2 v}{\partial p_i \partial p_j} + \frac{\partial v}{\partial p_i} \frac{\partial v}{\partial p_j} \right), \quad i, j \in \{1, 2, 3\}. \quad (15)$$

Thus, our Hamiltonian metric naturally emerges from the properties of a given medium, which are contained in the phase-velocity function, v .

We require that H be regular, which means that the matrix with entries given in expression (14) be nondegenerate on T^*M . In other words, $\det [g^{ij}] \neq 0$. We note that, physically, $\det [g^{ij}] = 0$ corresponds to the inflection points of a phase-slowness surface. In this paper, however, we do not study these singular points. In view of expression (6), the assumption of differentiability of H is equivalent to the assumption of differentiability of v .

Examining expression (14), we observe the following properties of the Hamiltonian metric (e.g., Miron, et al., 2001). Since H is homogeneous of degree 2 in the p_i , it follows that the components of the Hamiltonian metric are homogeneous of degree 0 in the p_i . Furthermore, since $\det [g^{ij}] \neq 0$, we also have the inverse of the Hamiltonian metric, which we denote by $g_{ij}(\mathbf{x}, \mathbf{p})$; in other words,

$$\sum_{j=1}^3 g_{ij}(\mathbf{x}, \mathbf{p}) g^{jk}(\mathbf{x}, \mathbf{p}) = \delta_i^k, \quad i, k \in \{1, 2, 3\}. \quad (16)$$

Examining expression (15), we recognize that we can also view this expression as the relation between the Hamiltonian metric, g^{ij} , and the Euclidean metric, δ^{ij} . We note that for an isotropic medium, where v is independent of \mathbf{p} , equation (15) reduces to $g^{ij} = v^2\delta^{ij}$. This means that in an isotropic medium the two metrics differ by a multiplicative scalar factor; in other words, they are conformal to one another. This also justifies our choice of metric (14).

Having formulated the Hamiltonian metric, we are now ready to complete our study of a geometrical relation between rays and wavefronts, namely, their orthogonality.

3 Orthogonality between rays and wavefronts

3.1 Euclidean and Hamiltonian gradients

To discuss orthogonality, we use the fact that a gradient of a function with respect to a given metric is a vector that is orthogonal to the level sets of this function with respect to this metric. For function ψ on M , we can define its gradient as being either the vector whose components are

$$(\nabla\psi)^i = \sum_{j=1}^3 \delta^{ij} \frac{\partial\psi}{\partial x^j}, \quad i \in \{1, 2, 3\}, \quad (17)$$

with respect to the Euclidean metric, or as the vector whose components are

$$(\nabla_g\psi)^i = \sum_{j=1}^3 g^{ij} \frac{\partial\psi}{\partial x^j}, \quad i \in \{1, 2, 3\}, \quad (18)$$

with respect to the Hamiltonian metric (e.g., Anastasiei, 1998). In other words, expression (17) defines the Euclidean gradient, while expression (18) defines the Hamiltonian gradient. Consequently, $\nabla\psi$ is orthogonal to the level sets of ψ with respect to the Euclidean metric, while $\nabla_g\psi$ is orthogonal to the level sets of ψ with respect to the Hamiltonian metric. For conciseness of terminology, when dealing with the latter case, we will refer to it as the Hamilton-orthogonality. In the next section, we show that $\nabla_g\psi$ is Hamilton-orthogonal to the level sets of ψ . Before showing this orthogonality, we wish to show that gradients $\nabla\psi$ and $\nabla_g\psi$ are distinct from one another.

In expression (15), we obtained the relation between the Euclidean and Hamiltonian metrics. Using this relation we can also derive an analytical expression relating the components of the two corresponding gradients, as follows. In view of expressions (15), (17) and (18), we can write

$$(\nabla_g \psi)^i = v^2 (\nabla \psi)^i + vp^2 \frac{\partial v}{\partial p_i}, \quad i \in \{1, 2, 3\}. \quad (19)$$

Thus, in general, the two gradients are distinct from one another.

3.2 Hamilton-orthogonality of $\nabla_g \psi$ and wavefronts

Herein, we rigorously show that vector $\nabla_g \psi$ is Hamilton-orthogonal to the wavefronts. This is equivalent to saying that $\nabla_g \psi$ is Hamilton-orthogonal to any curve that belongs to a level set of ψ .

Consider such a curve described by our coordinate functions as

$$\psi(\mathbf{x}(t)) = C, \quad (20)$$

where C denotes a constant. Taking the derivative of both sides of equation (20) with respect to t , we obtain

$$\sum_{i=1}^3 \frac{\partial \psi}{\partial x^i} \frac{dx^i}{dt} = 0. \quad (21)$$

To state expression (21) in terms of the Hamiltonian gradient, using expression (16), we can rewrite expression (21) as

$$\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 g_{ji} g^{jk} \frac{\partial \psi}{\partial x^k} \frac{dx^i}{dt} = 0. \quad (22)$$

Recognizing that $\sum_{k=1}^3 g^{jk} \partial \psi / \partial x^k$ is expression (18), we can write expression (22) as

$$\sum_{i=1}^3 \sum_{j=1}^3 g_{ji} (\nabla_g \psi)^j \frac{dx^i}{dt} = 0. \quad (23)$$

Equation (23) states that the scalar product — with respect to the Hamiltonian metric — of the Hamiltonian gradient and the vector tangent to the wavefront vanishes. Thus, the Hamiltonian gradient is Hamilton-orthogonal to the wavefront.

3.3 Hamilton-orthogonality of rays and wavefronts

To show the Hamilton-orthogonality of rays and wavefronts, it now suffices to show that the vector tangent to the ray coincides with the Hamiltonian-gradient vector $\nabla_g \psi$.

Consider a ray described in our coordinates as a curve given by $\mathbf{x}(t)$. The vector tangent to this ray can be written as $d\mathbf{x}/dt$. In view of the first equation of system (13), we can write the components of $d\mathbf{x}/dt$ as

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad i \in \{1, 2, 3\}. \quad (24)$$

We want to show that $\partial H/\partial p_i$ are the components of the Hamiltonian gradient of ψ .

Since H is homogeneous of degree 2 in the p_i , by Euler's homogeneous-function theorem, we can write

$$\sum_{i=1}^3 \frac{\partial H}{\partial p_i} p_i = 2H. \quad (25)$$

Furthermore, since H is homogeneous of degree 2 in the p_i , it follows that $\partial H/\partial p_i$ is homogeneous of degree 1 in the p_i . Following Euler's homogeneous-function theorem, we obtain

$$\sum_{j=1}^3 \frac{\partial}{\partial p_j} \left(\frac{\partial H}{\partial p_i} \right) p_j = \frac{\partial H}{\partial p_i}, \quad i \in \{1, 2, 3\}. \quad (26)$$

Multiplying both sides of equation (26) by p_i and summing over i , we get

$$\sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial^2 H}{\partial p_i \partial p_j} p_i p_j = \sum_{i=1}^3 \frac{\partial H}{\partial p_i} p_i. \quad (27)$$

Following expressions (14) and (25), we can rewrite equation (27) as

$$2H(\mathbf{x}, \mathbf{p}) = \sum_{i=1}^3 \sum_{j=1}^3 g^{ij}(\mathbf{x}, \mathbf{p}) p_i p_j. \quad (28)$$

Taking partial derivatives of equation (28) with respect to p_i , we obtain

$$2 \frac{\partial H}{\partial p_i} = 2 \sum_{j=1}^3 g^{ij} p_j + \sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial g^{kj}}{\partial p_i} p_k p_j, \quad i \in \{1, 2, 3\}. \quad (29)$$

Now, we will show that the double summation is identically zero. Recalling expression (14), we can write

$$\frac{\partial g^{kj}}{\partial p_i} = \frac{1}{2} \frac{\partial^3 H}{\partial p_i \partial p_k \partial p_j} = \frac{\partial g^{ij}}{\partial p_k}, \quad i, j, k \in \{1, 2, 3\}, \quad (30)$$

where the second equality results from the equality of the mixed partial derivatives. Using expression (30), we can rewrite the double summation as

$$\sum_{j=1}^3 \sum_{k=1}^3 \frac{\partial g^{kj}}{\partial p_i} p_k p_j = \sum_{j=1}^3 \left(\sum_{k=1}^3 \frac{\partial g^{ij}}{\partial p_k} p_k \right) p_j, \quad i \in \{1, 2, 3\}. \quad (31)$$

Since g^{ij} is homogeneous of degree 0 in the p_i , by Euler's homogeneous-function theorem, the term in parentheses vanishes. Consequently, the double summation is identically zero and expression (29) is reduced to

$$\frac{\partial H}{\partial p_i} = \sum_{j=1}^3 g^{ij} p_j, \quad i \in \{1, 2, 3\}. \quad (32)$$

Using expression (32), we can now write equation (24) as

$$\frac{dx^i}{dt} = \sum_{j=1}^3 g^{ij} p_j, \quad i \in \{1, 2, 3\}. \quad (33)$$

In view of expressions (2) and (18), we can rewrite equation (33) as

$$\frac{dx^i}{dt} = (\nabla_g \psi)^i, \quad i \in \{1, 2, 3\}. \quad (34)$$

Equation (34) states that the vector tangent to the ray coincides with the Hamiltonian gradient of ψ , whose level sets are the wavefronts. Hence, the proof of the statement that rays and wavefronts are Hamilton-orthogonal to each other is complete.

4 Discussion and conclusions

This paper proves the Hamilton-orthogonality between rays and wavefronts in anisotropic inhomogeneous media. In other words, the Euclidean orthogonality associated with isotropic media is extended to the anisotropic ones.

This can be explained in the following way. The Hamiltonian metric, which is derived from the angle-dependent phase velocity, contains this angular dependence. This means that the anisotropy has been accounted for by the metric itself.

In addition, this study allows us to illustrate the following ray-theory properties.

We recall that the Hamiltonian metric is expressed in terms of the phase-velocity function, as shown in expression (15). It is also possible to obtain the phase-velocity function from the Hamiltonian metric alone. To see this, consider expression (28). Dividing both sides of equation (28) by p^2 and in view of expression (6), we obtain

$$v^2(\mathbf{x}, \mathbf{p}) = \sum_{i=1}^3 \sum_{j=1}^3 g^{ij}(\mathbf{x}, \mathbf{p}) \frac{p_i p_j}{p p} = \sum_{i=1}^3 \sum_{j=1}^3 g^{ij}(\mathbf{x}, \mathbf{n}) n_i n_j. \quad (35)$$

In expression (35), $p = \sqrt{\mathbf{p} \cdot \mathbf{p}}$ and $n_i = p_i/p$.

We note that the fact that rays and wavefronts are Hamilton-orthogonal to each other means that each elementary wavefront generated by a point source is a unit ball of the geometry induced by the phase-velocity function. In other words — in the context of the Hamiltonian metric — each elementary wavefront can be viewed as a sphere with unit radius. To see this, we can use expression (28) to rewrite Hamilton-Jacobi equation (12) as

$$\sum_{i=1}^3 \sum_{j=1}^3 g^{ij} \left(x^k, \frac{\partial \psi}{\partial x^k} \right) \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} = 1. \quad (36)$$

Equation (36) can be expressed in terms of the Hamiltonian gradient, as follows. The squared magnitude of the Hamiltonian gradient $\nabla_g \psi$ with respect to the Hamiltonian metric is

$$(\nabla_g \psi)^2 = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 g_{kl} g^{ki} \frac{\partial \psi}{\partial x^i} g^{lj} \frac{\partial \psi}{\partial x^j} = \sum_{i=1}^3 \sum_{j=1}^3 g^{ij} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j}. \quad (37)$$

In view of expression (37), we can concisely write equation (36) as

$$(\nabla_g \psi)^2 = 1, \quad (38)$$

which is a unit ball. Notably, this is the indicatrix of the Finsler geometry (e.g., Bao et al., 2000), and we see that the geodesics are orthogonal to indicatrices.

The results we have shown include, in particular, the case of isotropy. If a medium is isotropic, then the phase-velocity function depends on position only, namely, $v = v(\mathbf{x})$. In such a case, expression (19) can be written as

$$\nabla_g \psi = v^2 \nabla \psi. \quad (39)$$

Also, in such a case, expression (15) reduces to

$$g^{ij} = v^2 \delta^{ij}, \quad i, j \in \{1, 2, 3\}, \quad (40)$$

which means that, for isotropy, the Hamiltonian and Euclidean metrics are conformal to one another. From equations (39) and (40), we see that, in isotropic media, the Hamilton-orthogonality reduces to the standard Euclidean orthogonality.

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