

Loss of ellipticity for non-coaxial plastic deformations in additive logarithmic finite strain plasticity

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Abstract

In this paper we consider the additive logarithmic finite strain plasticity formulation from the view point of loss of ellipticity in elastic unloading. We prove that even if an elastic energy $F \mapsto W(F) = \widehat{W}(\log U)$ defined in terms of logarithmic strain $\log U$, where $U = \sqrt{F^T F}$, happens to be everywhere rank-one convex as a function of F , the new function $F \mapsto \widetilde{W}(F) = \widehat{W}(\log U - \log U_p)$ need not remain rank-one convex at some given plastic stretch U_p (viz. $E_p^{\log} := \log U_p$). This is in complete contrast to multiplicative plasticity (and infinitesimal plasticity) in which $F \mapsto W(F F_p^{-1})$ remains rank-one convex at every plastic distortion F_p if $F \mapsto W(F)$ is rank-one convex ($\nabla u \mapsto \|\text{sym} \nabla u - \varepsilon_p\|^2$ remains convex). We show this disturbing feature of the additive logarithmic plasticity model with the help of a recently introduced family of exponentiated Hencky energies.

Key words: Hencky strain, logarithmic strain, natural strain, true strain, Hencky energy, multiplicative decomposition, elasto-plasticity, ellipticity domain, isotropic formulation, additive plasticity.

1 Introduction

Geometrically nonlinear plasticity is a field of intensive ongoing research. There exist several fundamentally different approaches which reduce, more or less, to the infinitesimal model based on the additive split of the infinitesimal strain tensor $\varepsilon = \text{sym} \nabla u$ into symmetric elastic and plastic strains [53, 62, 50, 52] $\varepsilon = \varepsilon_e + \varepsilon_p$. We assume the reader to be familiar with the general framework of finite strain plasticity models.

The involved nonlinearities make it difficult, both from an analysis and algorithmic point of view to obtain definitive results. There exists, however, one well known and much used methodology to reduce the algorithmic complexity dramatically. It is based on the matrix-logarithm and the introduction of a so called plastic metric $C_p \in \text{PSym}(3)$ together with an additive ansatz for elastic strains $E_e = \log C - \log C_p$. We refer to these models as additive logarithmic¹. Within these, basically the small strain linearized framework is simply lifted to the geometrically nonlinear setting through the properties of the logarithm. Thus, frame-indifference, thermodynamical admissibility, plastic volume constraint, associative flow rule, principle of maximal dissipation etc. are all easily satisfied.

In this paper, however, we want to exhibit a major drawback of this approach which makes the additive logarithmic ansatz in our view inadmissible in those cases where large plastic deformations need to be considered, as are encountered e.g. in elastic spring back processes in the automobile industry. Without loss of generality, we will concentrate our exposition to the completely isotropic setting, in which most prominently the quadratic Hencky-logarithmic strain energy appears.

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¹We refrain completely from discussing hypoplastic models based on the so-called logarithmic rate, see e.g. [6, 83] which also involve the Hencky strain.

1.1 Notation

For $a, b \in \mathbb{R}^n$ we let $\langle a, b \rangle_{\mathbb{R}^n}$ denote the scalar product on \mathbb{R}^n with associated vector norm $\|a\|_{\mathbb{R}^n}^2 = \langle a, a \rangle_{\mathbb{R}^n}$. We denote by $\mathbb{R}^{n \times n}$ the set of real $n \times n$ second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{R}^{n \times n}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{n \times n}} = \text{tr}(XY^T)$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{n \times n}}$. In the following we do not adopt any summation convention and we omit the subscript $\mathbb{R}^{n \times n}$ in writing the Frobenius tensor norm. The identity tensor on $\mathbb{R}^{n \times n}$ will be denoted by $\mathbb{1}$, so that $\text{tr}(X) = \langle X, \mathbb{1} \rangle$, while $\text{dev}_n X = X - \frac{1}{n} \text{tr}(X) \cdot \mathbb{1}$ is the n -dimensional deviatoric part of a second order tensor $X \in \mathbb{R}^{n \times n}$. We let $\text{Sym}(n)$ and $\text{PSym}(n)$ denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\text{GL}(n) := \{X \in \mathbb{R}^{n \times n} \mid \det X \neq 0\}$ denotes the general linear group, $\text{SL}(n) := \{X \in \text{GL}(n) \mid \det X = 1\}$, $\text{O}(n) := \{X \in \text{GL}(n) \mid X^T X = \mathbb{1}\}$, $\text{SO}(n) := \{X \in \text{GL}(n, \mathbb{R}) \mid X^T X = \mathbb{1}, \det X = 1\}$, $\text{GL}^+(n) := \{X \in \mathbb{R}^{n \times n} \mid \det X > 0\}$ is the group of invertible matrices with positive determinant, $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid X^T = -X\}$ is the Lie-algebra of skew symmetric tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid \text{tr}(X) = 0\}$ is the Lie-algebra of traceless tensors. For $X \in \text{PSym}(n)$, $X = \sum_{i=1}^n \lambda_i N_i \otimes N_i$, where λ_i are the eigenvalues and N_i are the eigenvectors of X , we consider $\log X = \sum_{i=1}^n \log \lambda_i (N_i \otimes N_i)$. Here and in the following the superscript T is used to denote transposition. For all vectors $\xi, \eta \in \mathbb{R}^3$ we have the (dyadic) tensor product $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$. The set of positive real numbers is denoted by $\mathbb{R}_+ := (0, \infty)$, while $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$.

Let us consider $W(F)$ to be the strain energy function of an elastic material in which $F = \nabla \varphi$ is the gradient of a deformation from a reference configuration to a configuration in the Euclidean 3-space; $W(F)$ is measured per unit volume of the reference configuration. The domain of $W(\cdot)$ is $\text{GL}^+(n)$. We denote by $C = F^T F$ the right Cauchy-Green strain tensor, by $B = F F^T$ the left Cauchy-Green (or Finger) strain tensor, by $U = \sqrt{F^T F}$ the right stretch tensor, i.e., the unique element of $\text{PSym}(n)$ for which $U^2 = C$ and by V the left stretch tensor, i.e., the unique element of $\text{PSym}(n)$ for which $V^2 = B$. Here, we are only concerned with rotationally symmetric functions (objective and isotropic), i.e., $W(F) = W(Q_1^T F Q_2) \forall F = R U = V R \in \text{GL}^+(n)$, $Q_1, Q_2, R \in \text{SO}(n)$. We denote by $S_1 = D_F[W(F)]$ the first Piola-Kirchhoff stress tensor and by $\sigma = \frac{1}{J} S_1 F^T$ the Cauchy stress tensor. In this paper, the scalar product will be always denoted by $\langle \cdot, \cdot \rangle$, while “ \cdot ” will denote the multiplication with scalars or the multiplication of matrices. We will also use “ \cdot ” to denote by $D_F[W(F)] \cdot H$ the Fréchet derivative of the function $W(F)$ of a tensor F applied to the tensor-valued increment H .

Further $\mu > 0$ is the infinitesimal shear modulus, $\kappa = \frac{2\mu + 3\lambda}{3} > 0$ is the infinitesimal bulk modulus with λ the first Lamé constant, $C_e := F_e^T F_e$ is the elastic strain tensor, U_e is the right elastic stretch tensor, i.e. the unique element of $\text{PSym}(n)$ for which $U_e^2 = C_e$ and

$$F = F_e \cdot F_p \quad (1.1)$$

is the multiplicative decomposition of the deformation gradient [32, 53, 8, 44, 43, 56], while $C_p := F_p^T F_p$ is the plastic metric and $U_p \in \text{PSym}(n)$ is the plastic stretch, $U_p^2 = C_p$.

1.2 The Hencky energy and elasto-plasticity

As hinted at above, the logarithmic strain space description is arguably the simplest algorithmic approach to finite plasticity, suitable for the phenomenological description of isotropic polycrystalline metals if the structure of geometrically linear theories is used with respect to the Lagrangian logarithmic strain $\log C$. As deduced by Itskov [30], the logarithmic strain yields the most appropriate results in case of rigid-plastic material subjected to simple shear.

In isotropic finite strain computational hyperelasto-plasticity [3, 74, 75, 68, 15, 69] the mostly used elastic energy is the quadratic Hencky logarithmic energy [79, 51, 19, 83, 6, 27] (see also [25, 70, 66, 63, 20, 46]). Among the works which use the Hencky strain in elasto-plasticity we may also mention [76, 40, 63, 7, 66, 65, 16, 21]. The Hencky energy W_H is the energy considered by the late J.C. Simo (see Eq. (3.4), page 147, from [77] and also [1]) because

$$\begin{aligned} W_H(F_e) &:= \mu \|\text{dev}_n \log \sqrt{F_e^T F_e}\|^2 + \frac{\kappa}{2} [\text{tr}(\log \sqrt{F_e^T F_e})]^2 = \frac{\mu}{4} \|\text{dev}_n \log F_e^T F_e\|^2 + \frac{\kappa}{8} [\text{tr}(\log F_e^T F_e)]^2 \\ &= \frac{\mu}{4} \|\text{dev}_n \log F_e^T F_e\|^2 + \frac{\kappa}{2} [\log(\det F_e)]^2. \end{aligned} \quad (1.2)$$

The Hencky energy W_{H} has the correct behaviour for extreme strains in the sense that $W(F_e) \rightarrow \infty$ as $\det F_e \rightarrow 0$ and, likewise $W(F_e) \rightarrow \infty$ as $\det F_e \rightarrow \infty$ [78, page 392], but it cannot be a polyconvex function of the deformation gradient. However, the model provides a good approximation for moderately large elastic strains, which is superior to the usual Saint-Venant-Kirchhoff model of finite elasticity. Several models of such a type have been considered in [62, 31]. The advantage of using the energy W_{H} compared to other elastic energies stems from the fact that computational implementations of elasto-plasticity [20] based on the additive decomposition $\varepsilon = \varepsilon_e + \varepsilon_p$ in infinitesimal models² [17, 61, 28], can be used with nearly no changes also in isotropic finite strain problems [78, page 392]. The computation of the elastic equilibrium at given plastic distortion F_p suffers, however, under the well-known non-ellipticity of W_{H} [4, 2, 48, 29]. We know that W_{H} is Legendre-Hadamard elliptic in a neighbourhood of the identity if $\lambda, \mu > 0$, $\lambda_i \in [0.21162\dots, 1.39561\dots]$ (see [4, 5]), therefore $F \mapsto W_{\text{H}}(F)$ is Legendre-Hadamard elliptic for moderately large strains in the previous sense.

Moreover, the elastic Hencky energy has been shown [37, 54] to have a fundamental differential geometric meaning, not shared by any other elastic energy, i.e.

$$\begin{aligned} \text{dist}_{\text{geod}}^2 \left(\frac{F}{(\det F)^{1/n}}, \text{SO}(n) \right) &= \text{dist}_{\text{geod}, \text{SL}(n)}^2 \left(\frac{F}{(\det F)^{1/n}}, \text{SO}(n) \right) = \|\text{dev}_n \log U\|^2, \\ \text{dist}_{\text{geod}}^2 \left((\det F)^{1/n} \cdot \mathbf{1}, \text{SO}(n) \right) &= \text{dist}_{\text{geod}, \mathbb{R}_+ \cdot \mathbf{1}}^2 \left((\det F)^{1/n} \cdot \mathbf{1}, \mathbf{1} \right) = |\log \det F|^2, \end{aligned} \quad (1.3)$$

where $\text{dist}_{\text{geod}, \mathbb{R}_+ \cdot \mathbf{1}}^2$ and $\text{dist}_{\text{geod}, \text{SL}(n)}^2$ are the canonical left invariant geodesic distances on the Lie-group $\text{SL}(n)$ and on the group $\mathbb{R}_+ \cdot \mathbf{1}$, respectively (see [55]). For these investigations new mathematical tools had to be discovered [60] also having consequences for the classical polar decomposition.

1.3 Additive metric plasticity

The Green-Naghdi additive plasticity models [23, 24] are well known (see e.g. [10, 11, 79]). They are based on the additive split of the total Green-Saint-Venant strain $E = \frac{1}{2}(C - \mathbf{1})$ into

$$E = E_e + E_p, \quad (1.4)$$

where $E_e = \frac{1}{2}(C_e - \mathbf{1})$ is the elastic Green-Saint-Venant strain and $E_p = \frac{1}{2}(C_p - \mathbf{1})$ is the plastic strain. While $E_e = E - E_p$ was identified with elastic strain in the original work of Green and Naghdi [23], in later works [24] this identification was dropped (see also [36]). This decomposition is justified as an approximation that is valid [36] when (i) small plastic deformations are accompanied by moderate elastic strains, (ii) small elastic strains are accompanied by moderate plastic deformations, or (iii) small strains are accompanied by moderate rotations. A formulation based on elastic strain-measures like

$$W(E - E_p) := \frac{\mu}{4} \|E - E_p\|^2 + \frac{\lambda}{8} [\text{tr}(E - E_p)]^2 \quad (1.5)$$

is not even rank-one convex for zero plastic strain (this is the well-known deficiency of the Saint-Venant-Kirchhoff model [48, 67]).

Another finite plasticity model using logarithmic strains is taking the additive elastic Hencky energy [42, 41, 64, 35, 34, 73, 72] in the format

$$\widehat{W}_{\text{H}}(\log U - \log U_p) = \mu \|\text{dev}_3[\log U - \log U_p]\|^2 + \frac{\kappa}{2} [\text{tr}(\log U)]^2,$$

as a starting point, in which plastic incompressibility $\text{tr}(\log U_p) = 0$ ($\det U_p = 1$) is already included. This approach is based on the ad hoc introduction of the symmetric elastic strain tensor

$$E_e^{\log} := E^{\log} - E_p^{\log} = \log U - \log U_p = \frac{1}{2} \log C - \frac{1}{2} \log C_p,$$

²We need to be a little more specific. For the additive model in the format $\|\log C - \log C_p\|^2$ the complete systems of equations of the plastic flow rule are **identical** to the infinitesimal additive model, while for the truly multiplicative model the return mapping algorithm is **similar** to the infinitesimal case.

in the spirit of [23]. An additive decomposition of logarithmic strain has been used in [71] to construct a viscoplasticity theory. Note that the additive decomposition of the total strain is also considered in thermo-mechanics, see e.g. [9, 33]. Eisenberg et al. [18] employ an additive decomposition of strain for thermo-plastic materials and provide experimental correlation to theory. For numerical results see also [82, 81].

Despite appearance, we need to remark that this additive model has not much in common with models based on the multiplicative decomposition, which will be shortly discussed in Section 2. Indeed, the independent plastic variable in the additive model is, in fact, $E_p^{\text{log}} = \frac{1}{2} \log C_p = \log U_p$, which is determined to be trace free. Moreover, the definition of E_p^{log} avoids any ambiguity of plastic rotation. Here, $C_p = U_p^2 \in \text{PSym}(3)$ takes only formally the role of a local plastic metric. The flow rule will be an evolution equation in terms of the independent variable $E_p^{\text{log}} = \log U_p \in \mathfrak{sl}(3) \cap \text{Sym}(3)$

$$\frac{d}{dt}[\log U_p] \in \partial\mathcal{X}(\text{dev}_3 \Sigma) \quad \Leftrightarrow \quad \frac{d}{dt}[\log U_p] = \lambda_p^+ \frac{\text{dev}_3 \Sigma}{\|\text{dev}_3 \Sigma\|}, \quad (1.6)$$

where $\Sigma = D[\widehat{W}_H(\log U - \log U_p)]$, $\partial\mathcal{X}(\text{dev}_3 \Sigma)$ is the subdifferential of the indicator function $\mathcal{X}(\text{dev}_3 \Sigma)$ of the convex elastic domain $\mathcal{E}_e(\Sigma, \frac{2}{3} \sigma_y^2) := \left\{ \Sigma \in \text{Sym}(3) \mid \|\text{dev}_3 \Sigma\|^2 \leq \frac{2}{3} \sigma_y^2 \right\}$, where σ_y is the yield limit and the plastic multiplier λ_p^+ satisfies the KarushKuhnTucker (KKT)-optimality constraints

$$\lambda_p^+ \geq 0, \quad \mathcal{X}(\text{dev}_3 \Sigma) \leq 0, \quad \lambda_p^+ \mathcal{X}(\text{dev}_3 \Sigma) = 0. \quad (1.7)$$

Note that since U and U_p do not commute, $\log U - \log U_p \neq \log(U U_p^{-1})$ in general. Having $\text{tr}(\log U_p) = 0$ consistent with the flow rule, it follows $\det U_p = 1$. Moreover, $\log U_p \in \text{Sym}(3)$ implies $U_p \in \text{PSym}(3)$. The advantage of this model is that its structure w.r.t. to plasticity is identical to the infinitesimal model, while all proper invariances (objectivity and isotropy) of the geometrically nonlinear theory are retained. In contrast to multiplicative approaches the thermodynamic driving force is not the Eshelby tensor Σ_E and this model gives decreasing shear stress in plastic simple shear which is physically unacceptable [30].

Now, it could be argued that the reason for the former deficiency is the dependence on the elastic strain $E_e^{\text{log}} = \log U - \log U_p$, together with using the (already) non-elliptic elastic Hencky energy giving rise to a non-elliptic elastic formulation. However, the additive logarithmic metric ansatz has another serious shortcoming which is the focus of this contribution and treated next.

2 Rank one convexity and plastic flow

Next, our goal is to understand the response of a plasticity formulation at given plastic distortion or plastic strain. Let us first consider small strain plasticity based on the additive decomposition of elastic strain

$$\text{sym} \nabla u = \varepsilon = \varepsilon_e + \varepsilon_p, \quad \varepsilon_p \in \text{Sym}(3). \quad (2.1)$$

The governing equations for isotropic perfect plasticity are

$$\text{Div} \sigma = f, \quad \sigma = 2\mu(\text{sym} \nabla u - \varepsilon_p) + \lambda(\text{sym} \nabla u - \varepsilon_p) \mathbf{1}, \quad \frac{d}{dt}[\varepsilon_p] \in \partial\mathcal{X}(\text{dev}_3 \Sigma_{\text{lin}}), \quad (2.2)$$

where σ is the Cauchy stress tensor and $\Sigma_{\text{lin}} = D_{\varepsilon_p} W_{\text{lin}}(\varepsilon - \varepsilon_p)$, f is the body force, $\text{Div} \sigma$ is a vector having as components the divergence of the rows of the Cauchy stress tensor σ and $\partial\mathcal{X}$ is the subdifferential of the indicator function \mathcal{X} of the convex elastic domain

$$\mathcal{E}_e(\Sigma_{\text{lin}}, \frac{2}{3} \sigma_y^2) = \left\{ \Sigma_{\text{lin}} \in \text{Sym}(3) \mid \|\text{dev}_3 \Sigma_{\text{lin}}\|^2 \leq \frac{2}{3} \sigma_y^2 \right\} \subset \text{Sym}(3). \quad (2.3)$$

At frozen plastic straining ε_p , the elastically stored energy is given by

$$W_{\text{lin}}(\nabla u, \varepsilon_p) = \mu \|\text{sym} \nabla u - \varepsilon_p\|^2 + \frac{\lambda}{2} [\text{tr}(\text{sym} \nabla u - \varepsilon_p)]^2. \quad (2.4)$$

A simple observation is that the convexity of the function W_{lin} w.r.t. ∇u is not influenced by the plastic flow at all. Indeed, even the second derivative of W_{lin} (i.e. the elastic moduli) is unchanged. In other words, by plastic flow alone, the elastic unloading response is not touched upon.

Next, we turn to finite strain isotropic multiplicative plasticity. The system of equations can be written

$$\text{Div } S_1(F F_p^{-1}) = f, \quad S_1(F F_p^{-1}) = D_F[W(F F_p^{-1})] = D_F[W(F_e)], \quad -F_p \frac{d}{dt}[F_p^{-1}] \in \partial\mathcal{X}(\text{dev}_3 \Sigma_E), \quad (2.5)$$

where $\partial\mathcal{X}$ is the subdifferential of the indicator function \mathcal{X} of the convex elastic domain

$$\mathcal{E}_e(\Sigma_E, \frac{2}{3} \sigma_y^2) = \left\{ \Sigma_e \in \text{Sym}(3) \mid \|\text{dev}_3 \Sigma_e\|^2 \leq \frac{2}{3} \sigma_y^2 \right\} \subset \text{Sym}(3). \quad (2.6)$$

Here, Σ_E is the elastic Eshelby tensor

$$\Sigma_E := F_e^T D_{F_e}[W(F_e)] - W(F_e) \cdot \mathbf{1},$$

driving the plastic evolution (see e.g. [53, 39, 14, 12, 13]) and f is the body force. As in the small strain case, rank-one convexity is preserved. However, the elastic moduli may decrease with evolving plastic flow. To substantiate this claim let us recall the following simple, yet fundamental observation:

Lemma 2.1. [51, 49] (rank-one convexity and multiplicative decomposition) *If the elastic energy $F \mapsto W(F)$ is rank-one convex, it follows that the elasto-plastic formulation*

$$F \mapsto W(F, F_p) := W(F F_p^{-1}) = W(F_e) \quad (2.7)$$

remains rank-one convex w.r.t F for all given plastic distortions F_p . Hence, in the multiplicative setting, the elastic rank-one convexity is independent of the plastic flow. In multiplicative plasticity rank-one convexity is thus configuration independent.

The same constitutive invariance property is true for convexity, polyconvexity and quasiconvexity [51, 31]. Therefore, the multiplicative approach is ideally suited as far as preservation of ellipticity properties for elastic unloading is concerned and marks a sharp contrast to additive logarithmic modelling frameworks, as will be seen.

These observations, together with the experimental evidence that elastic unloading is always seen to be a stable process motivates us to postulate:

Postulate 2.2. [Stability at frozen plastic flow] *At frozen plastic flow the purely elastic response (elasticity with eigenstresses) should define a well-posed nonlinear elasticity problem in the sense that rank-one convexity is preserved.*

3 Additive logarithmic plasticity does not preserve rank-one convexity during plastic flow

Let us start our argument by returning to a simplified one-dimensional situation for the sake of clarity. Firstly, we observe that the classical quadratic Hencky strain energy

$$t \mapsto (\log t)^2 \quad (3.1)$$

is not convex (not rank-one convex) and therefore nothing can be gained by using the additive logarithmic ansatz

$$t \mapsto (\log t - \log s)^2, \quad t \in (0, \infty) \quad (3.2)$$

where $s \in (0, \infty)$ is representing the plastic straining. However, the exponentiated Hencky energy

$$t \mapsto e^{(\log t)^2} \quad (3.3)$$

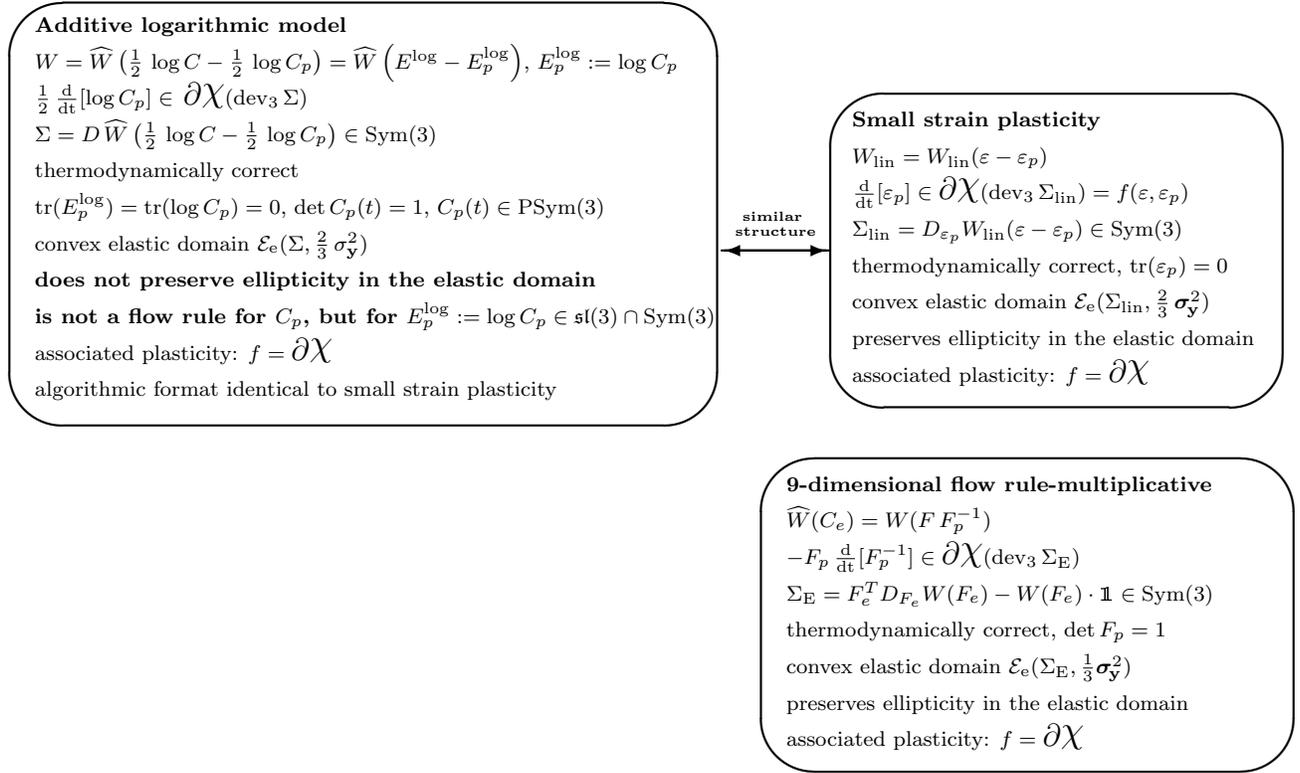


Figure 1: An additive logarithmic model, additive small strain plasticity. All these models are associative, since all flow rules are in the format $\frac{d}{dt}[P]P^{-1} \in -\partial \mathcal{X}$ or $\frac{d}{dt}[\varepsilon_p] \in \partial \mathcal{X}$ and for comparison the 9-dimensional flow rule based on the multiplicative decomposition.

is in fact a convex function of t and the function

$$t \mapsto e^{(\log t - \log s)^2}, \quad t \in (0, \infty) \quad (3.4)$$

remains convex in t for all $s \in (0, \infty)$ since $e^{(\log t - \log s)^2} = e^{(\log \frac{t}{s})^2}$ is only a linear scaling in the argument of a convex function. Therefore in the one-dimensional setting the additive logarithmic framework preserves ellipticity (convexity).

The last observation in (3.4) motivated us in a previous work [58] to consider the following family of energies

$$W_{\text{eH}}(F) = W_{\text{eH}}^{\text{iso}} \left(\frac{F}{\det F^{\frac{1}{n}}} \right) + W_{\text{eH}}^{\text{vol}}(\det F^{\frac{1}{n}} \cdot \mathbf{1}) = \begin{cases} \frac{\mu}{k} e^{k \|\text{dev}_n \log U\|^2} + \frac{\kappa}{2\widehat{k}} e^{\widehat{k} [(\log \det U)]^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0. \end{cases} \quad (3.5)$$

We have called this the family of **exponentiated Hencky energies**. For the two-dimensional situation $n = 2$ and for $\mu > 0, \kappa > 0$, we have established that the functions $W_{\text{eH}} : \mathbb{R}^{2 \times 2} \rightarrow \overline{\mathbb{R}}_+$ from the family of exponentiated Hencky type energies are **rank-one convex** [58] for additional dimensionless material parameters $k \geq \frac{1}{4}$ and $\widehat{k} \geq \frac{1}{8}$. Moreover, these energies are **polyconvex** [59, 22] and the corresponding minimization problem admits at least one solution.

However, in the two-dimensional setting the function

$$F \mapsto W(F, U_p) = e^{\|\text{dev}_2 \log U - \text{dev}_2 \log U_p\|^2} = e^{\frac{1}{4} \|\text{dev}_2 \log C - \text{dev}_2 \log C_p\|^2} \quad (3.6)$$

loses ellipticity for essentially non-coaxial plastic deformations, i.e., $C C_p^{-1} \neq C_p^{-1} C$. Otherwise, if we have the commutation relation $C C_p^{-1} = C_p^{-1} C$, then

$$e^{\frac{1}{4} \|\text{dev}_2 \log C - \text{dev}_2 \log C_p\|^2} = e^{\frac{1}{4} \|\text{dev}_2 \log(C C_p^{-1})\|^2} \quad (3.7)$$

by the properties of the matrix logarithm and the function

$$F \mapsto W(F, C_p) = e^{\frac{1}{4} \|\text{dev}_2 \log(C C_p^{-1})\|^2} \quad (3.8)$$

is always Legendre-Hadamard elliptic w.r.t. F for given plastic metric $C_p \in \text{PSym}(2)$. Here, we have used that

$$e^{\frac{1}{4} \|\text{dev}_2 \log(C C_p^{-1})\|^2} = e^{\frac{1}{4} \|\text{dev}_2 \log C_e\|^2} \quad (3.9)$$

since the principal invariants of $C C_p^{-1}$ and C_e are equal, and therefore the eigenvalues of $C C_p^{-1}$ are equal to the eigenvalues of C_e , see [57]. Then we may invoke Lemma 2.1 applied to $W(F_e) = e^{\frac{1}{4} \|\text{dev}_2 \log F_e^T F_e\|^2}$.

Therefore, considering the family W_{eH} of exponentiated Hencky energies, the main result of this paper shows the following unacceptable feature: even if the elastic energy is everywhere rank-one convex as a function of F , i.e. $F \mapsto \widehat{W}(\log U)$ is rank-one convex, the new function $F \mapsto \widehat{W}(\log U - \log U_p)$ need not remain rank-one convex at some given plastic stretch U_p (viz. E_p^{\log}). This type of loss of ellipticity is relevant in elastic unloading problems at given plastic deformation. It naturally appears in computation of the elastic spring back. In other words we demonstrate that, relative to an initial natural configuration prior to the occurrence of further yielding, the elastic rank-one convexity property may be lost with development of plastic flow alone. We note again that this type of degenerate response cannot occur in elasto plasticity based on the multiplicative decomposition of the deformation gradient into incompatible elastic and plastic distortion $F = F_e \cdot F_p$, thus giving more credit to these latter type of models.

We consider now the isotropic exponentiated energy $F \mapsto W_{\text{eH}}^{\text{iso}}(F) = e^{\|\text{dev}_2 \log U\|^2}$ corresponding to the material parameter $k = 1$ and we provide the proof that the new function $F \mapsto \widehat{W}_{\text{eH}}^{\text{iso}}(\log U - \log U_p)$ need not remain rank-one convex at some given plastic stretch U_p .

Lemma 3.1. *The function $F \mapsto W(F) = e^{\|\text{dev}_2 \log U - \text{dev}_2 \log U_p\|^2}$ is not rank-one convex for some given plastic stretch $U_p \in \text{PSym}(2)$, while $F \mapsto W(F) = e^{\|\text{dev}_2 \log U\|^2}$ is rank-one convex.*

Proof. The rank-one convexity of the isotropic exponentiated energy $F \mapsto W_{\text{eH}}^{\text{iso}}(F) = e^{\|\text{dev}_2 \log U\|^2}$ follows as a particular case of the result established in [58].

In order to prove that the function $F \mapsto W(F) = e^{\|\text{dev}_2 \log U - \text{dev}_2 \log U_p\|^2}$ is not rank-one convex for some given $U_p \in \text{PSym}(2)$, it suffices to consider the elastic simple shear case. We choose the vectors $\eta, \xi \in \mathbb{R}^3$ so that $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\eta \otimes \xi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and we will be able to find a plastic stretch $U_p \in \text{PSym}(2)$ such that the scalar function $h : \mathbb{R} \rightarrow \mathbb{R}$,

$$h(t) = W(\mathbb{1} + t(\eta \otimes \xi)) = e^{\|\text{dev}_2 \log\{[\mathbb{1} + t(\eta \otimes \xi)]^T [\mathbb{1} + t(\eta \otimes \xi)]\} - \text{dev}_2 \log U_p\|^2}$$

is not convex as function of t . This is sufficient for loss of rank-one convexity. Thus we consider $F = \mathbb{1} + t(\eta \otimes \xi) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

For this total deformation we have that the polar decomposition $F = R \cdot U = V \cdot R$ into the right Biot stretch tensor $U = \sqrt{F^T F}$ of the deformation and the orthogonal polar factor R is given by

$$U = \frac{1}{\sqrt{t^2 + 4}} \begin{pmatrix} 2 & t \\ t & t^2 + 2 \end{pmatrix}, \quad R = \frac{1}{\sqrt{t^2 + 4}} \begin{pmatrix} 2 & t \\ -t & 2 \end{pmatrix}. \quad (3.10)$$

Let us rewrite the right Biot stretch tensor U in the following form

$$U = \frac{\lambda_1}{\lambda_1^2 + 1} \begin{pmatrix} 2 & \lambda_1 - \frac{1}{\lambda_1} \\ \lambda_1 - \frac{1}{\lambda_1} & \lambda_1^2 + \frac{1}{\lambda_1^2} \end{pmatrix}, \quad (3.11)$$

where $\lambda_1 = \frac{1}{2}(\sqrt{t^2 + 4} + t)$ denotes the first eigenvalue of U . Further, U can be orthogonally diagonalized to

$$U = Q \cdot \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \lambda_1 \end{pmatrix} \cdot Q^T, \quad (3.12)$$

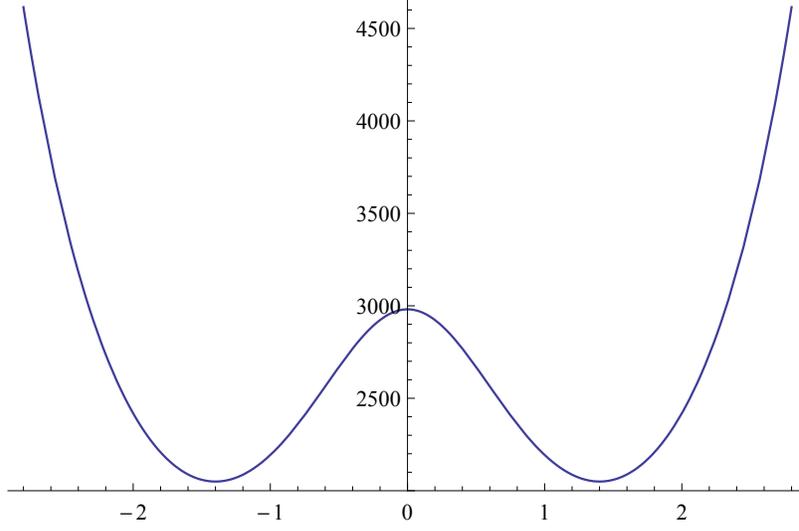


Figure 2: The graphical representation of the function $t \mapsto h_{-2,0}(t) = e^{2 \log^2 [\frac{1}{2}(\sqrt{t^2+4}+t)] - 8t \frac{\log [\frac{1}{2}(\sqrt{t^2+4}+t)]}{t^2+4} + 8}$.

where

$$Q = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1^2 + 1}} \begin{pmatrix} -\sqrt{\lambda_1} & \frac{1}{\sqrt{\lambda_1}} \\ \frac{1}{\sqrt{\lambda_1}} & \sqrt{\lambda_1} \end{pmatrix}. \quad (3.13)$$

Hence, the principal logarithm of U is

$$\log U = Q \cdot \begin{pmatrix} -\log \lambda_1 & 0 \\ 0 & \log \lambda_1 \end{pmatrix} \cdot Q^T = \frac{1}{\sqrt{t^2+4}} \cdot \begin{pmatrix} -t \log \lambda_1 & 2 \log \lambda_1 \\ 2 \log \lambda_1 & t \log \lambda_1 \end{pmatrix}. \quad (3.14)$$

We still need to determine a suitable plastic stretch $U_p \in \text{SL}(2)$. The general form of a matrix $U_p \in \text{PSym}(2)$ such that $\text{tr}(\log U_p) = 0$ is

$$\log U_p = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \Rightarrow U_p = \exp \begin{pmatrix} a & b \\ b & -a \end{pmatrix}. \quad (3.15)$$

We obtain that

$$\begin{aligned} \|\text{dev}_2 \log U - \text{dev}_2 \log U_p\|^2 &= \|\text{dev}_2 \log U\|^2 - 2 \langle \text{dev}_2 \log U, \text{dev}_2 \log U_p \rangle + \|\text{dev}_2 \log U_p\|^2 \\ &= 2 \log^2 \lambda_1 - 2 \frac{\log \lambda_1}{t^2+4} (-2at + 4b) + 2a^2 + 2b^2. \end{aligned} \quad (3.16)$$

With this representation we are able to disprove rank-one convexity of the function $F \mapsto e^{\|\text{dev}_2 \log U - \text{dev}_2 \log U_p\|^2}$. We have to check the convexity of the function

$$h_{a,b}(t) = W(\mathbb{1} + t(\eta \otimes \xi)) = e^{2 \log^2 [\frac{1}{2}(\sqrt{t^2+4}+t)] - 2 \frac{\log [\frac{1}{2}(\sqrt{t^2+4}+t)]}{t^2+4} (-2at+4b) + 2a^2 + 2b^2}, \quad t \in \mathbb{R}.$$

For our purpose, it is enough to find a matrix U_p , i.e. some values for $a, b \in \mathbb{R}$, such that the function $t \mapsto h_{a,b}(t)$ is not convex. For simplicity, let us choose $b = 0$. We remark from Figure 2 that for $b = 0$ and $a = -2$, i.e. for $U_p = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^2 \end{pmatrix} \in \text{SL}(2)$, the function

$$t \mapsto h_{-2,0}(t) = e^{2 \log^2 [\frac{1}{2}(\sqrt{t^2+4}+t)] - 8t \frac{\log [\frac{1}{2}(\sqrt{t^2+4}+t)]}{t^2+4} + 8}, \quad t \in \mathbb{R}.$$

is not convex. In the following we prove this observation analytically.

Let us suppose that the function $h_{-2,0}$ is convex. On one hand, this assumption implies that the first derivative $t \mapsto h'_{-2,0}(t)$ is a monotone increasing function on $[0, \infty)$ and we would have

$$h'_{-2,0}(t) \geq h'_{-2,0}(0) \quad \forall t > 0. \quad (3.17)$$

On the other hand, since $h_{-2,0}(t) = h_{-2,0}(-t) \forall t \in \mathbb{R}$, we deduce that $h'_{-2,0}(t) = -h'_{-2,0}(-t)$ for all $t \in \mathbb{R}$, which implies that $h'_{-2,0}(0) = 0$. Hence, we have obtained that

$$h'_{-2,0}(t) \geq 0 \quad \forall t > 0, \quad (3.18)$$

which means that $h_{-2,0}$ is a monotone increasing function on $[0, \infty)$. However, this is not true, since

$$h_{-2,0}(0) = e^8 \approx 2980.96 \quad \text{and} \quad h_{-2,0}(1) \approx 2193.36. \quad (3.19)$$

Therefore, the assumption that the function $h_{-2,0}$ is convex leads to a contradiction.

In conclusion, the function $h_{-2,0}$ is not convex, which implies that for U_p chosen as above the energy $F \mapsto e^{\|\text{dev}_2 \log U - \text{dev}_2 \log U_p\|^2}$ is not rank-one convex, while $F \mapsto e^{\|\text{dev}_2 \log U\|^2}$ is rank-one convex [58]. This final remark completes the proof. \square

4 Final remarks

We have shown that the multiplicative plasticity models preserve ellipticity in purely elastic processes at frozen plastic variable provided that the initial elastic response is elliptic, see [57]. Preservation of Legendre-Hadamard ellipticity is, in our view, a property which should be satisfied by any hyperelastic-plastic model since the elastically unloaded material specimen should respond reasonably under further purely elastic loading. However, the much used additive logarithmic model does not preserve Legendre-Hadamard ellipticity in general. In Figure 1 we summarize some properties of the additive logarithmic model and the multiplicative decomposition and we compare it with the small strain plasticity model. In contrast, the formulation based on $e^{\|\text{dev}_2 \log C_e\|^2}$ and the multiplicative decomposition remains always rank-one convex. Moreover, the change of a given FEM-implementation of W_H into W_{eH} is nearly free of costs [80, 38, 47, 26]. For more constitutive issues regarding the interesting properties of W_{eH} we refer to [58, 45].

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