where $\Lambda^{(j)}$ are defined in (23) and solutions can be represented by the single-layer potential

$$U(x) = V(h)(x), \quad x \in \Omega^+$$

where the density vector $h \in C^{1,b}(S)$ solves the integral equation

$$\left[-2^{-1}I_6 + \mathcal{K}\right] h(x) = F(x), \qquad x \in S$$

A solution vector U is defined modulo a rigid displacement, while the generalized stress vector TU is determined uniquely.

Similar existence results hold also true for weak solutions in smooth and Lipschitz domains (see [9–11], and [12]).

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Boundary Variational Functional

▶ Variational Formulation and Nonsmooth Optimization Algorithms in Elastostatic Contact Problems for Cracked Body

Boundary–Initial Value Problems

▶ Boundary–Initial Value Problems of Thermoelastodynamics

Boundary–Initial Value Problems of Thermoelastodynamics

Ionel-Dumitrel Ghiba
Department of Mathematics, "Alexandru Ioan Cuza" University of Iaşi, Iaşi, Romania
"Octav Mayer" Institute of Mathematics, Romanian Academy, Iaşi, Romania

Synonyms

Boundary–initial value problems; Linear theory; Thermoelasticity

Overview

In this work, we formulate boundary–initial value problems of the linear thermoelasticity. We formulate the forward in time-coupled problem and the backward in time-coupled problem.

Moreover, we consider the uncoupled problem of thermoelasticity, the problem of the quasi-static theory, and the problem of the equilibrium theory.

Our analysis is based on the works by Truesdell and Noll [9], Truesdell [10], Nowacki [8], Carlson [4], Eringen [5], and Hetnarski and Eslami [7]. The history of the thermoelasticity is fully discussed by Truesdell [10] and Hetnarski and Eslami [7] (see also the reference list of the work by Carlson [4]).

Boundary–Initial Value Problems

In this work, we consider a thermoelastic material which at time $t_0 = 0$ occupies the region B of the three-dimensional Euclidian space E^3, whose boundary is the smooth surface ∂B. In the following, the configuration of the body at the initial time $t_0 = 0$ is considered as the reference configuration.

Throughout this chapter, Latin subscripts take the values $1, 2, 3$, and summation is carried out over repeated indices. Typical conventions for differential operations are implied such as a superposed dot or comma, followed by a subscript to denote the partial derivative with respect to time or to the corresponding Cartesian coordinate, respectively.

We refer the configurations of the body to a fixed system of rectangular axes. In the rest of this chapter, $\mathbf{x}$ denotes the position vector with the components (x_1, x_2, x_3) of a generic point P of the domain B.

Let us consider a fixed time interval $[0, t_1)$, where $t_1 > 0$ can be infinite. Considered two positive integers M and N, we say that a function f defined on $B \times (0, t_1)$ is of class $C^{M,N}$ if the functions

$$\partial^m f^{(n)} \equiv \frac{\partial^m}{\partial x_p \partial x_q \ldots \partial x_k}\left(\frac{\partial^n f}{\partial t^n}\right)$$
$$m \in \{0, 1, \ldots M\}, n \in \{0, 1, \ldots, N\}$$
$$m + n \leq \max\{M, N\}$$

exist and are continuous on $B \times (0, t_1)$.

We denote by T_0 the absolute temperature in the reference configuration, and we suppose that T_0 is a prescribed positive constant. We also suppose that in the natural state, the body is free of initial stresses and entropy.

We assume that the components u_i of the displacement vector are of class C^2 on $B \times (0, t_1)$, while we assume that the variation of temperature θ is of class $C^{2,1}$ on $B \times (0, t_1)$ and continuous together with $\dot{\theta}$ and $\theta_{,i}$ on $\overline{B} \times [0, t_1)$.

In the linear theory, we suppose that $\mathbf{u} = \varepsilon \mathbf{u}'$ and $\theta = \varepsilon \theta'$ where ε is a constant small enough to have $\varepsilon^n \simeq 0$, for $n \geq 2$, and $\mathbf{u}'$ and θ' are independent of ε. We consider the components of the infinitesimal strain tensor e_{ij} given by

$$2e_{ij} = u_{i,j} + u_{j,i} \tag{1}$$

In the following, we denote the stress tensor and the heat flux by σ_{ij} and q_i, respectively. Moreover, we will use the following notations:

i) ρ_0 is the mass density of the continuum at the initial time.
ii) S is the entropy per unit mass.
iii) $\mathbf{b}$ is the body force per unit mass.
iv) r is the heat supply per unit mass.

The equations of the linear theory of thermoelasticity consist of (see [4])

– The equations of motion

$$\sigma_{ji,j} + \rho_0 b_i = \rho_0 \ddot{u}_i \tag{2}$$

– The energy equation

$$\rho_0 T_0 \dot{S} + q_{i,i} = \rho_0 r \tag{3}$$

– The constitutive equations

$$\begin{aligned} \sigma_{ij} &= C_{ijkl} e_{kl} - M_{ij}\theta \\ \rho_0 S &= M_{ij} e_{ij} + a\theta \\ q_i &= -k_{ij}\theta_{,j} \end{aligned} \tag{4}$$

– The geometrical equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

The constitutive coefficients $C_{ijkl}, M_{ij}, k_{ij}, \rho_0$ and a are depending on the spatial variables x_1, x_2, x_3, and they have the following properties of symmetry:

$$M_{ij} = M_{ji} \quad C_{ijkl} = C_{klij} = C_{jikl} \tag{6}$$

and

$$k_{ij}\theta_{,i}\theta_{,j} \geq 0 \tag{7}$$

To these equations, we must adjoin boundary conditions and initial conditions. The boundary conditions can be of Dirichlet type or of Neumann type, or we can have mixed boundary conditions.

The initial conditions have the following form:

$$\begin{aligned} u_i(\mathbf{x},0) &= u_{0i}(\mathbf{x}) \\ \dot{u}_i(\mathbf{x},0) &= \dot{u}_{0i}(\mathbf{x}) \\ S(\mathbf{x},0) &= S_0(\mathbf{x}), \quad \mathbf{x} \in \overline{B} \end{aligned} \tag{8}$$

and, in the case of the mixed problem, the boundary conditions are

$$u_i(\mathbf{x},t) = \hat{u}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_1 \times [0,t_1) \tag{9}$$

$$\begin{aligned} s_i(\mathbf{x},t) &= \sigma_{ji}(\mathbf{x},t)n_j \\ &= \hat{s}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_2 \times [0,t_1) \end{aligned} \tag{10}$$

$$\theta(\mathbf{x},t) = \hat{\theta}(\mathbf{x},t), \quad \text{on} \quad \Sigma_3 \times [0,t_1) \tag{11}$$

$$\begin{aligned} q(\mathbf{x},t) &= q_i(\mathbf{x},t)n_i \\ &= \hat{q}(\mathbf{x},t), \quad \text{on} \quad \Sigma_4 \times [0,t_1) \end{aligned} \tag{12}$$

where Σ_s $(s = 1, \ldots, 4)$ are subsets of the boundary ∂B so that $\Sigma_1 \cup \overline{\Sigma}_2 = \Sigma_3 \cup \overline{\Sigma}_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$, $\mathbf{n}$ is the unit outward normal to the boundary, and $u_{0i}, \dot{u}_{0i}, S_0, \hat{u}_i, \hat{s}_i, \hat{\theta}$, and $\hat{q}$ are prescribed fields.

In the linear theory of isotropic materials, we have only five constitutive coefficients λ, μ, m, k, and a so that the constitutive equations (4) become

$$q_i = -k\theta_{,i} \tag{13}$$

$$\begin{aligned} \sigma_{ij} &= \lambda e_{rr}\delta_{ij} + \mu e_{ij} - m\theta\delta_{ij} \\ \rho_0 S &= m e_{rr} + a\theta \end{aligned} \tag{14}$$

Here, the scalars λ and μ are called Lamé moduli, μ is the shear modulus, m is the stress–temperature modulus, and k is the conductivity coefficient assumed to be positive.

We assume that the constitutive coefficients are of class C^1 on $\overline{B}$ while ρ_0 and a are assumed to be continuous on $\overline{B}$ and $\rho_0 > 0$.

We also suppose that the prescribed data are given so that [4]:

i) b_i and r are continuous on $\overline{B} \times [0,t_1)$.
ii) $u_{0i}, \dot{u}_{0i}$, and S_0 are continuous on $\overline{B}$.
iii) $\hat{u}_i$ are continuous on $\Sigma_1 \times [0,t_1)$.
iv) $\hat{s}_i$ are smooth functions on $\Sigma_2 \times [0,t_1)$ and continuous as functions of time.
v) $\hat{\theta}$ is continuous on $\Sigma_3 \times [0,t_1)$.
vi) $\hat{q}$ is smooth on $\Sigma_4 \times [0,t_1)$ and continuous as function of time.

By an admissible thermoelastic process, in the linear theory of thermoelasticity, we mean an ordered array $[u_i, e_{ij}, \sigma_{ij}, \theta, S, q_i]$ with the properties [4]:

i) u_i are of class C^2 on $B \times (0,t_1)$.
ii) $u_i, \dot{u}_i, \ddot{u}_i, u_{i,j}, \dot{u}_{i,j}$ are continuous on $\overline{B} \times [0,t_1)$.
iii) e_{ij} are the components of a symmetric tensor, continuous on $\overline{B} \times [0,t_1)$.
iv) σ_{ij} are the components of a symmetric tensor, of class $C^{1,0}$ on $B \times (0,t_1)$.
v) σ_{ij} and $\sigma_{ji,j}$ are continuous on $\overline{B} \times [0,t_1)$.
vi) θ is of class $C^{2,1}$ on $B \times (0,t_1)$.
vii) $\theta, \theta_{,i}, \dot{\theta}$ are continuous on $\overline{B} \times [0,t_1)$.
viii) S is of class $C^{0,1}$ on $B \times (0,t_1)$.
ix) $S, \dot{S}$ are continuous on $\overline{B} \times [0,t_1)$.
x) q_i are of class $C^{1,0}$ on $B \times (0,t_1)$.
xi) q_i and $q_{i,i}$ are continuous on $\overline{B} \times [0,t_1)$.

The following remark is an immediate consequence of the above definitions and linearity of field equations.

Remark 1. Carlson [4] Let $[u_i, e_{ij}, \sigma_{ij}, \theta, S, q_i]$ and $[\tilde{u}_i, \tilde{e}_{ij}, \tilde{\sigma}_{ij}, \tilde{\theta}, \tilde{S}, \tilde{q}_i]$ be thermoelastic processes corresponding to the external force systems $[\mathbf{s}, \mathbf{f}]$ and $[\tilde{\mathbf{s}}, \tilde{\mathbf{f}}]$, respectively, and to the external thermal systems $[q, r]$ and $[\tilde{q}, \tilde{r}]$, respectively.

If α and $\tilde{\alpha}$ are scalars, then $[\alpha u_i + \tilde{\alpha}\tilde{u}_i, \alpha e_{ij} + \tilde{\alpha}\tilde{e}_{ij}, \alpha\sigma_{ij} + \tilde{\alpha}\tilde{\sigma}_{ij}, \alpha\theta + \tilde{\alpha}\tilde{\theta}, \alpha S + \tilde{\alpha}\tilde{S}, \alpha q_i + \tilde{\alpha}\tilde{q}_i]$ is a thermoelastic process corresponding to the external force system $[\alpha\mathbf{s} + \tilde{\alpha}\tilde{\mathbf{s}}, \alpha\mathbf{f} + \tilde{\alpha}\tilde{\mathbf{f}}]$ and to the external thermal system $[\alpha q + \tilde{\alpha}\tilde{q}, \alpha r + \tilde{\alpha}\tilde{r}]$.

The above remark proves that the set of all admissible thermoelastic processes may be organized as a linear space endowed with natural addition and scalar multiplication.

By a solution of the mixed boundary–initial value problem, we mean an admissible thermoelastic process which satisfies the (2)–(5) and the conditions (8)–(12).

Let us introduce the relations (4) and (5) into (2) and (3). Then, we can formulate the boundary–initial value problem in terms of displacement u_i and temperature variation θ only. Thus, we have the differential system

$$\begin{aligned} (C_{ijkl}u_{k,l})_{,j} - (M_{ij}\theta)_{,j} - \rho_0\ddot{u}_i &= -\rho_0 b_i \\ (k_{ij}\theta_{,j})_{,i} - T_0 M_{ij}\dot{u}_{i,j} - c\dot{\theta} &= -\rho_0 r \end{aligned} \tag{15}$$

with the initial conditions

$$\begin{aligned} u_i(\mathbf{x},0) &= u_{0i}(\mathbf{x}) \\ \dot{u}_i(\mathbf{x},0) &= \dot{u}_{0i}(\mathbf{x}) \\ \theta(\mathbf{x},0) &= \theta_0(\mathbf{x}), \quad \mathbf{x} \in \overline{B} \end{aligned} \tag{16}$$

and the boundary conditions

$$u_i(\mathbf{x},t) = \hat{u}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_1 \times [0,t_1) \tag{17}$$

$$\begin{aligned} &(C_{ijkl}u_{k,l} - M_{ij}\theta)(\mathbf{x},t)n_j \\ &\quad = \hat{s}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_2 \times [0,t_1) \end{aligned} \tag{18}$$

$$\theta(\mathbf{x},t) = \hat{\theta}(\mathbf{x},t), \quad \text{on} \quad \Sigma_3 \times [0,t_1) \tag{19}$$

$$k_{ij}\theta_{,j}(\mathbf{x},t)n_i = \hat{q}(\mathbf{x},t), \quad \text{on} \quad \Sigma_4 \times [0,t_1) \tag{20}$$

In the above relations, we have used the specific heat $c = T_0 a$, and we have introduced the notation:

$$a\theta_0 = \rho_0 S_0 - M_{ij}u_{0i,j} \tag{21}$$

If the body is homogeneous, then the system of partial differential equations (15) reduces to

$$\begin{aligned} C_{ijkl}u_{k,lj} - M_{ij}\theta_{,j} - \rho_0\ddot{u}_i &= -\rho_0 b_i \\ k_{ij}\theta_{,ji} - T_0 M_{ij}\dot{u}_{i,j} - c\dot{\theta} &= -\rho_0 r \end{aligned} \tag{22}$$

Moreover, if the body is homogeneous and isotropic, then the equations are

$$\begin{aligned} \mu u_{i,jj} + (\lambda+\mu)u_{j,ji} - m\theta_{,i} - \rho_0\ddot{u}_i &= -\rho_0 b_i \\ k\theta_{,ii} - T_0 m\dot{u}_{j,j} - c\dot{\theta} &= -\rho_0 r \end{aligned} \tag{23}$$

Concerning the solution of the mixed boundary value problem, we have the following uniqueness result [4]:

Theorem 1. *Suppose the elasticity tensor C_{ijkl} is positive semi-definite and the specific heat c is strictly positive. Then the mixed problem has at most one solution.*

Sometimes in applications, the boundary condition for the heat flux is considered in the form of the following convection condition:

$$\mathbf{q} \cdot \mathbf{n} = h(T - T_e) \quad \text{on} \quad \partial B \tag{24}$$

where T is the temperature of the solid's boundary, T_e is the ambient temperature and h is the convection coefficient. The last two quantities, h and T_e, are determined by experiments. Other types of boundary conditions for the heat flux can be found in the books [3, 7, 8].

Up to now, we have formulated forward in time problems. In the last part of this section, we formulate the boundary–final value problem known as the backward in time problem. We consider the boundary–final value problem of the linear theory of thermoelasticity on the interval $(-t_1, 0]$, where $t_1 > 0$ may be infinite. All the quantities have the same significations as in the formulation of the forward in time problem defined above.

In terms of displacement u_i and temperature variation θ, the boundary–final value problem is defined by the equations

$$\left(C_{ijkl}u_{k,l}\right)_{,j} - \left(M_{ij}\theta\right)_{,j} - \rho_0\ddot{u}_i = -\rho_0 b_i$$
$$\left(k_{ij}\theta_{,j}\right)_{,i} - T_0 M_{ij}\dot{u}_{i,j} - c\dot{\theta} = -\rho_0 r, \quad \text{in} \quad B\times(-t_1,0) \tag{25}$$

the final conditions

$$\begin{aligned} u_i(\mathbf{x},0) &= u_{0i}(\mathbf{x}) \\ \dot{u}_i(\mathbf{x},0) &= \dot{u}_{0i}(\mathbf{x}) \\ \theta(\mathbf{x},0) &= \theta_0(\mathbf{x}), \quad \mathbf{x}\in\overline{B} \end{aligned} \tag{26}$$

and the boundary conditions

$$u_i(\mathbf{x},t) = \hat{u}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_1\times(-t_1,0] \tag{27}$$

$$\begin{aligned}&\left(C_{ijkl}u_{k,l} - M_{ij}\theta\right)(\mathbf{x},t)n_j \\ &\quad = \hat{s}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_2\times(-t_1,0]\end{aligned} \tag{28}$$

$$\theta(\mathbf{x},t) = \hat{\theta}(\mathbf{x},t), \quad \text{on} \quad \Sigma_3\times(-t_1,0] \tag{29}$$

$$k_{ij}\theta_{,j}(\mathbf{x},t)n_i = \hat{q}(\mathbf{x},t), \quad \text{on} \quad \Sigma_4\times(-t_1,0] \tag{30}$$

where Σ_s $(s = 1,\ldots,4)$ are subsets of the boundary ∂B so that $\Sigma_1\cup\overline{\Sigma}_2 = \Sigma_3\cup\overline{\Sigma}_4 = \partial B$, $\Sigma_1\cap\Sigma_2 = \Sigma_3\cap\Sigma_4 = \emptyset$, $\mathbf{n}$ is the unit outward normal to the boundary, and $u_{0i}, \dot{u}_{0i}$, S_0, $\hat{u}_i$, $\hat{s}_i$, $\hat{\theta}$, and $\hat{q}$ are prescribed fields.

The backward in time problems lead to ill-posed problems. By means of the change $t \rightsquigarrow -t$ we can transform the above boundary–final value problem into a boundary–initial problem defined by the equations

$$\left(C_{ijkl}u_{k,l}\right)_{,j} - \left(M_{ij}\theta\right)_{,j} - \rho_0\ddot{u}_i = -\rho_0 b_i$$
$$\left(k_{ij}\theta_{,j}\right)_{,i} + T_0 M_{ij}\dot{u}_{i,j} + c\dot{\theta} = -\rho_0 r, \quad \text{in} \quad B\times(0,t_1) \tag{31}$$

the initial conditions

$$\begin{aligned} u_i(\mathbf{x},0) &= u_{0i}(\mathbf{x}) \\ \dot{u}_i(\mathbf{x},0) &= \dot{u}_{0i}(\mathbf{x}) \\ \theta(\mathbf{x},0) &= \theta_0(\mathbf{x}), \quad \mathbf{x}\in\overline{B} \end{aligned} \tag{32}$$

and the boundary conditions

$$u_i(\mathbf{x},t) = \hat{u}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_1\times[0,t_1) \tag{33}$$

$$\begin{aligned}&\left(C_{ijkl}u_{k,l} - M_{ij}\theta\right)(\mathbf{x},t)n_j \\ &\quad = \hat{s}_i(\mathbf{x},t), \quad \text{on} \quad \Sigma_2\times[0,t_1)\end{aligned} \tag{34}$$

$$\theta = \hat{\theta}, \quad \text{on} \quad \Sigma_3\times[0,t_1) \tag{35}$$

$$k_{ij}\theta_{,j}(\mathbf{x},t)n_i = \hat{q}(\mathbf{x},t), \quad \text{on} \quad \Sigma_4\times[0,t_1) \tag{36}$$

We remark the energy equation is changed by this transformation, while the first three equations have the same form as in the case of final value problem. This class of problems was first considered by Ames and Payne [1] (see also the book by Ames and Straughan [2]).

Uncoupled Problems

In the study of certain materials, it has been observed that in the energy equation $(15)_2$, the term $T_0 M_{ij}\dot{u}_{i,j}$ can be neglected, and the corresponding predicted results are in concordance with the experiments. In such a case, the energy equation $(15)_2$ is replaced by

$$\left(k_{ij}\theta_{,j}\right)_{,i} - c\dot{\theta} = -\rho_0 r \tag{37}$$

and the mixed problem becomes high simplified. In fact, the mixed problem leads to the separate study of two boundary–initial value problems. The first is concerned with the above equation together with the initial and boundary conditions for the temperature variation θ, a problem relating only the temperature variation. Assuming solved this problem for the temperature variation θ, the second boundary–initial problem consists of the differential system $(15)_1$ with the initial and boundary conditions in terms of the displacement u_i in which the temperature variation is assumed prescribed. This last boundary–initial value problem represents a boundary–initial value problem of the linear elastodynamics in which the components of the body force vector are

$$b_i - \frac{1}{\rho_0}\left(M_{ij}\theta\right)_{,j} \tag{38}$$

and the stress boundary condition is

$$s_i(\mathbf{x}, t) = \tilde{s}_i(\mathbf{x}, t) + M_{ij}\theta(\mathbf{x}, t)n_j \tag{39}$$

This last case is called the uncoupled theory of the thermoelasticity. The coupled theory concerns the studies of the interaction between the deformation of elastic materials and the thermal field. Thermoelasticity gives the tools to investigate the stresses produced by the temperature field and to calculate the distribution of temperature due to the action of internal forces.

In the uncoupled theory, the function θ vanishes when $r, \theta_0, \hat{\theta}$, and $\hat{q}$ are zero. In the coupled theory, this is not true: there is a variation of the temperature due to the mechanical deformation. This variation also produces a mechanical deformation.

Sometimes the inertial terms are not taken into account. Then, the (2) is replaced by

$$\sigma_{ji,j} + \rho_0 b_i = 0 \tag{40}$$

In such a case, we obtain the so-called quasi-static theory of thermoelasticity. Then the basic equations of the quasi-static theory are (40) and (3)–(5).

Let us consider now the equilibrium theory. Then the fundamental system of field equations consist of

– The equations of equilibrium

$$\sigma_{ji,j} + \rho_0 b_i = 0 \tag{41}$$

– The energy equation

$$q_{i,i} = \rho_0 r \tag{42}$$

– The constitutive equations

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}e_{kl} - M_{ij}\theta \\ \rho_0 S &= M_{ij}e_{ij} + a\theta \\ q_i &= -k_{ij}\theta_{,j} \end{aligned} \tag{43}$$

– The geometrical equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{44}$$

To these equations, we must adjoin boundary conditions which can be of Dirichlet type or of Neumann type, or we can have mixed boundary conditions.

In the case of the mixed problem, the boundary conditions are

$$u_i(\mathbf{x}) = \hat{u}_i(\mathbf{x}), \quad \text{on} \quad \Sigma_1 \tag{45}$$

$$s_i(\mathbf{x}) = \sigma_{ji}(\mathbf{x})n_j = \hat{s}_i(\mathbf{x}), \quad \text{on} \quad \Sigma_2 \tag{46}$$

$$\theta(\mathbf{x}) = \hat{\theta}(\mathbf{x}), \quad \text{on} \quad \Sigma_3 \tag{47}$$

$$q(\mathbf{x}) = q_i(\mathbf{x})n_i = \hat{q}(\mathbf{x}), \quad \text{on} \quad \Sigma_4 \tag{48}$$

where Σ_s $(s = 1, \ldots, 4)$ are subsets of the boundary ∂B such that $\Sigma_1 \cup \overline{\Sigma}_2 = \Sigma_3 \cup \overline{\Sigma}_4 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \emptyset$, and $\hat{u}_i$, $\hat{s}_i$, $\hat{\theta}$, and $\hat{q}$ are prescribed fields.

We suppose that the prescribed data are given so that [4]:

i) b_i and r are continuous on $\overline{B}$.
ii) $\hat{u}_i$ are continuous on Σ_1.
iii) $\hat{s}_i$ are smooth functions on Σ_2.
iv) $\hat{\theta}$ is continuous on Σ_3.
v) $\hat{q}$ is smooth on Σ_4.

Let us remark that the above system is uncoupled in the sense that the temperature can be found by solving the heat flow problem given by

$$\begin{aligned} (k_{ij}\theta_{,j})_{,i} &= -\rho r, \quad \text{in} \quad B \\ \theta(\mathbf{x}) &= \hat{\theta}(\mathbf{x}), \quad \text{on} \quad \Sigma_3 \\ q_i(\mathbf{x})n_i &= \hat{q}(\mathbf{x}), \quad \text{on} \quad \Sigma_4 \end{aligned} \tag{49}$$

From the above problem, we can remark that the mechanical deformation does not influence the variation of the temperature. In the following, we can suppose that the temperature field is already determined.

By an admissible state, in the linear equilibrium theory of thermoelasticity, we mean an ordered array $[u_i, e_{ij}, \sigma_{ij}]$ with the properties [4]:

i) u_i are of class C^2 on B.
ii) u_i and $u_{i,j}$ are continuous on $\overline{B}$.
iii) e_{ij} are the components of a symmetric tensor, continuous on $\overline{B}$.

iv) σ_{ij} are the components of a symmetric tensor, of class C^1 on B.
v) σ_{ij} and $\sigma_{ji,j}$ are continuous on $\overline{B}$.

Thus, the equilibrium problem consists in finding an admissible state which satisfies the problem defined by the (41), (43), and (44) and the boundary conditions (45) and (46), where θ is a known function. We also have that the set of all admissible states may be organized as a linear space endowed with natural addition and scalar multiplication.

Regarding the equilibrium problem, we have the following uniqueness theorem [4]:

Theorem 2. *Let the elasticity tensor C_{ijkl} be positive definite. Then any two solutions of the mixed equilibrium problem are equal modulo a rigid displacement. Moreover, if Σ_1 is nonempty, the mixed problem has at most one solution.*

In the equilibrium problem, let us consider that $\Sigma_2 = \partial B$. So, we know the surface traction on entire boundary of the body. For this problem, we intend to give a formulation of the problem only in terms of the stress tensor.

We assume that the tensor C_{ijkl} is invertible. From the constitutive equations $(43)_1$, we have that there exists a tensor A_{ijkl} so that

$$e_{ij} = A_{ijkl}\sigma_{kl} + \alpha_{ij}\theta \tag{50}$$

where

$$\alpha_{ij} = A_{ijkl}M_{kl} \tag{51}$$

We assume that A_{ijkl} and α_{ij} are of class C^2 on $\overline{B}$ and that the domain B is simply connected. In view of the compatibility conditions (see [6]), it follows that the stress tensor σ_{ij}, of class C^2 on $\overline{B}$, corresponds to a solution of the equilibrium problem if and only if it is solution of the problem defined by the equations

$$\begin{gathered} \sigma_{ji,j} + \rho_0 b_i = 0 \\ \varepsilon_{pim}\varepsilon_{qjn}(A_{ijkl}\sigma_{kl} + \alpha_{ij}\theta)_{,mn} = 0, \quad \text{in } B \end{gathered} \tag{52}$$

and the boundary condition

$$\sigma_{ij}n_j = \hat{s}_i, \quad \text{on } \partial B \tag{53}$$

where ε_{ijk} is the alternating symbol.

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Boundary-Value Problems Resulting in Thermoelastic Shock Wave Propagation

Yury A. Rossikhin and Marina V. Shitikova
Research Center for Wave Dynamics in Solids and Structures, Voronezh State University of Architecture and Civil Engineering, Voronezh, Russia

Synonyms

Boundary value problems; Conditions of compatibility; Ray series; Shock waves