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Linear Thermoelastic Model

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Synonyms

Anisotropic materials; Initial flux; Initial stresses; Isotropic materials; Linear theory

Overview

The linear theory of thermoelasticity considers the interaction between the deformation of elastic materials and the thermal field for infinitesimal deformations and small variations of the temperature. The linear thermoelasticity gives the tools to investigate the stresses produced by the temperature field and to calculate the distribution of the temperature due to the action of internal forces. The rapid development of this theory was stimulated by various engineering science. We have to point out that, for the isotropic materials, the equations of linear thermoelasticity were postulated by Duhamel [1, 2] and Neumann [3]. The history of thermoelasticity is fully discussed by Truesdell [4] and Hetnarski and Eslami [5] (see also the reference list of the work by Carlson [6]).

Our analysis is based on the works by Truesdell and Noll [7], Truesdell [4], Nowacki [8], Carlson [6], Eringen [9], and Hetnarski and Eslami [5]. More details regarding the nonlinear theory can be found in the article of the present encyclopedia devoted to this subject [10].

The main purpose of this work is to present the equations of linear theory of thermoelasticity. First, we present the equations of the nonlinear theory. Then, we consider the case of infinitesimal deformations and of small variations of the temperature, and we give the forms of the basic equations for anisotropic and for isotropic materials. The equations are formulated in Cartesian coordinates and also in cylindrical and spherical coordinates. In the last section, we give a short description of the linear theory of thermoelastic materials with initial stresses and initial heat flux. To this aim, we use the results from the paper [11] (see also [12]).

The Equations of Nonlinear Theory

We consider a body made by a thermoelastic material, which at the time t_0 occupies the region B of the three-dimensional Euclidian space E^3, whose boundary is the smooth surface ∂B. In the following, the configuration of the body at the initial time t_0 is considered as the reference configuration.

We refer the initial configuration of the body to a fixed rectangular Cartesian system of axes OX_K $(K = 1, 2, 3)$. We denote by X_K $(K = 1, 2, 3)$ the coordinates of a generic material point M_0, of the domain B. We suppose that after the deformation process, at the time t, the body occupies a new domain $\mathcal{B}$ which has the boundary $\partial\mathcal{B}$, the material point M_0 arriving in the position M. We will refer the configuration $\mathcal{B}$ of the body at the time t to a new fixed rectangular Cartesian system of axes ox_i $(i = 1, 2, 3)$. The coordinates of the position M are denoted by x_i $(i = 1, 2, 3)$. In the rest of this entry, $\mathbf{X}$ denotes the position vector with the components (X_1, X_2, X_3), and $\mathbf{x}$ denotes the position vector (x_1, x_2, x_3).

In the following, Latin subscripts take the values 1, 2, 3, and summation is carried out over repeated indices. Typical conventions for differential operations are implied such as a superposed dot or comma followed by a subscript to denote the partial derivative with respect to time or to the corresponding Cartesian coordinate, respectively.

We assume that the body should not penetrate itself and hence there is a one-to-one application between B and $\mathcal{B}$. Let us consider a fixed time interval $[t_0, t_1)$, where $t_0 \geq 0$, while $t_1 > 0$ can be infinite.

The deformation of the body will be described by the relation [9]

$$\mathbf{x} = \mathbf{x}(\mathbf{X},\ t),\ (\mathbf{X},\ t) \in \overline{B} \times [t_0, t_1) \tag{1}$$

This application is of class C^2 on $\overline{B} \times [t_0, t_1)$, and we have

$$J = \det\left(\frac{\partial x_i}{\partial X_K}\right) > 0 \tag{2}$$

The coordinates X_K are called material coordinates, while the coordinates x_i are called spatial coordinates.

Considered two positive integers M and N, we say that a function f defined on $B \times (t_0, t_1)$ is of class $C^{M,N}$, if the functions

$$\partial^m f^{(n)} \equiv \frac{\partial^m}{\partial X_P \partial X_Q \ldots \partial X_K}\left(\frac{\partial^n f}{\partial t^n}\right)$$
$$m \in \{0, 1, \ldots M\}, n \in \{0, 1, \ldots, N\}$$
$$m + n \leq \max\{M, N\}$$

exist and are continuous on $B \times (t_0, t_1)$.

We denote by $\mathbf{E}_k$ the directional vector of the axis OX_K, and by $\mathbf{e}_i$ the directional vector of the axis ox_i. It is easy to remark that

$$\mathbf{E}_K \cdot \mathbf{E}_L = \delta_{KL},\quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{3}$$

where δ_{KL} and δ_{ij} are Kronecker's symbols.

Hence, we have

$$\mathbf{X} = X_K \mathbf{E}_K,\ \mathbf{x} = x_i \mathbf{e}_i \tag{4}$$

Let us introduce the vectors $\mathbf{c} = \overrightarrow{Oo}$ and $\mathbf{u} = \overrightarrow{M_0 M}$, and let us remark that we have

$$\begin{aligned} \mathbf{u} &= u_i \mathbf{e}_i = U_K \mathbf{E}_K \\ \mathbf{c} &= c_i \mathbf{e}_i = C_K \mathbf{E}_K \\ \mathbf{u} &= \mathbf{x} - \mathbf{X} + \mathbf{c} \end{aligned} \tag{5}$$

We call the vector $\mathbf{u}$ the displacement vector.

In the nonlinear theory, the following measures are used to describe the deformations of the continuum:

$$\begin{aligned} 2E_{KL} &= C_{KL} - \delta_{KL} \\ 2e_{ij} &= \delta_{ij} - c_{ij} \end{aligned} \tag{6}$$

where

$$\begin{aligned} C_{KL} &= x_{i,K} x_{i,L} \\ c_{ij} &= X_{K,i} X_{K,j} \end{aligned} \tag{7}$$

The tensors c_{ij} and C_{KL} are called the Cauchy deformation tensor and the Green deformation tensor, respectively, while E_{KL} and e_{ij} are the Lagrangian strain tensor and the Eulerian strain tensor, respectively.

The strain tensors E_{KL} and e_{ij} may be expressed in terms of the components of the displacement vector as

$$2E_{KL} = U_{K,L} + U_{L,K} + U_{M,K} U_{M,L} \tag{8}$$

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{s,i}u_{s,j} \tag{9}$$

The equations of nonlinear thermoelasticity consist of (see [6])

$$\begin{aligned} &\rho_0 = \rho J \\ &S_{Ki,K} + \rho_0 b_i = \rho_0\, \ddot{u}_i \\ &\rho_0 \dot{i} = S_{KL}\, \dot{E}_{KL} - Q_{K,K} + \rho_0 r \\ &S_{KL} = S_{LK} \\ &\rho_0 (T\dot{S} - r) + Q_{K,K} - \left(\frac{Q_K}{T}\right)_{,K} \geq 0 \end{aligned} \tag{10}$$

where

$$S_{Ki} = x_{i,L} S_{KL} \tag{11}$$

The quantities from the above relations have the following physical significations [5]:

1. ρ_0 is the mass density of the continuum at the initial time, while ρ is the mass density in the actual configuration.
2. S_{Ki} and S_{KL} are the Piola-Kirchhoff stress tensors.
3. b_i is the body force per unit mass.
4. i is the internal energy per unit mass.
5. Q_K is the heat flux vector.
6. S is the entropy per unit mass.
7. T is the absolute temperature.
8. r is the heat supply per unit mass.

We consider that T is a positive function of class $C^{2,1}$ and S is of class $C^{0,1}$ on $B \times (t_0, t_1)$.

We say that a media is a thermoelastic material if the following constitutive equations hold

$$\begin{aligned} \psi &= \tilde{\psi}(E_{KL}, T, T_{,B};\, X_N) \\ S_{KL} &= \tilde{S}_{KL}\,(E_{KL}, T, T_{,B};\, X_N) \\ S &= \tilde{S}(E_{KL}, T, T_{,B};\, X_N) \\ Q_K &= \tilde{Q}_K\,(E_{KL}, T, T_{,B};\, X_N) \end{aligned} \tag{12}$$

where

$$\psi = i - TS \tag{13}$$

is the Helmholtz free energy. We assume that the response functions are of class C^2 on their domain.

It follows, using the entropy inequality, that (see [6])

$$\begin{aligned} U &= \hat{U}(E_{KL}, T;\, X_N) \\ S_{KL} &= \frac{\partial U}{\partial E_{KL}} \\ \rho_0 S &= -\frac{\partial U}{\partial T} \\ Q_K &= \hat{Q}_K\,(E_{KL}, T, T_{,M}; X_N) \end{aligned} \tag{14}$$

where $U = \rho_0 \psi$.

Moreover, the heat flux vector must verify the following inequality:

$$Q_K T_{,K} \leq 0 \tag{15}$$

A consequence of the inequality (15) is the fact that the heat flux vanishes when the gradient of the temperature vanishes, that is,

$$Q_K(E_{KL}, T, 0;\, X_N) = 0 \tag{16}$$

In conclusion, the basic equations of nonlinear thermoelasticity consist of (10), the constitutive (14), and the geometrical (8), on $B \times (t_0, t_1)$.

More details about the above equations can be found in the books by Truesdell and Noll [7] and Eringen [9] and in the paper of the present encyclopedia by Galeş [10].

Linear Theory

In this section, we present the equations of linear theory of thermoelasticity. We use the following notations $X_i = \delta_{iA} X_A$, where δ_{iA} is Kronecker's symbol, and $f_{,i} = \frac{\partial f}{\partial X_i}$.

We denote by T_0 the absolute temperature in the reference configuration, and we suppose that T_0 is a prescribed positive constant. We also suppose that in the natural state, the body is free from stresses and has zero entropy.

The variation of temperature is given by

$$\theta = T - T_0 \tag{17}$$

In the linear theory, we will suppose that $u_i = \varepsilon u_i'$ and $\theta = \varepsilon \theta'$ where ε is a constant small enough to have $\varepsilon^n \simeq 0$, for $n \geq 2$, while u_i' and θ' are independent of ε. In fact, we assume that all quantities $u_i,\ \theta, S_{Ki}, Q_K, S$ are of the form $\varepsilon\phi$ with ε a very small constant and ϕ independent of ε. Moreover, in the linear theory, we consider only one fixed system of rectangular axes, and so we have

$$x_i = X_i + u_i \tag{18}$$

Let us remark that, in the framework of the linear theory, the partial derivatives of a function $\hat{f} = \varepsilon f$ with respect to the spatial coordinates, where $\varepsilon^n \simeq 0$ for $n \geq 2$, can be approximated by the partial derivatives with respect to the material coordinates, that is,

$$\begin{aligned}\frac{\partial \hat{f}}{\partial X_1} &= \frac{\partial \hat{f}}{\partial x_j}\frac{\partial x_j}{\partial X_1}\\ &= \frac{\partial \hat{f}}{\partial x_j}\left(\delta_{1j} + \frac{\partial u_j}{\partial X_1}\right) = \frac{\partial \hat{f}}{\partial x_1} + O(\varepsilon^2)\end{aligned}$$

Hence, in the linear theory, the Lagrangian and the Eulerian strain tensors coincide. We denote the components of the strain tensor by e_{ij}, so we have

$$2e_{ij} = u_{i,j} + u_{j,i} \tag{19}$$

This tensor is called the infinitesimal strain tensor. In the linear theory, the Piola-Kirchhoff stress tensors coincide. In the following, we denote the stress tensor and the heat flux by σ_{ij} and q_i, respectively.

In the linear theory, the free energy is considered to be a second-order polynomial in terms of the strain tensor and the variation of temperature. Thus, it has the following form:

$$\begin{aligned}U = \hat{U}(e_{ij}, \theta) &= c_0 - c_1\theta + c_{ij}e_{ij}\\ &+ \frac{1}{2}C_{ijkl}e_{ij}e_{kl} - M_{ij}e_{ij}\theta - \frac{1}{2}a\theta^2\end{aligned} \tag{20}$$

where $c_0, c_1, c_{ij}, C_{ijkl}, M_{ij}$ and a are constitutive coefficients characterizing the type of material. They have the following properties of symmetry:

$$\begin{aligned}c_{ij} = c_{ji}, M_{ij} = M_{ji}\\ C_{ijkl} = C_{klij} = C_{jikl}\end{aligned} \tag{21}$$

The quantity

$$c_\varepsilon = \frac{1}{\rho_0}T_0 a \tag{22}$$

is the specific heat per unit mass corresponding to the natural state ($e_{ij} = 0, \theta = 0$), while

$$c = \rho_0 c_\varepsilon \tag{23}$$

is the specific heat per unit volume.

Because we have assumed that in the natural state the body is free from stresses and has zero entropy, it follows that $c_0 = 0, c_1 = 0$ and $c_{ij} = 0$. So, the free energy has the form

$$U = \frac{1}{2}C_{ijkl}e_{ij}e_{kl} - M_{ij}e_{ij}\theta - \frac{1}{2}a\theta^2 \tag{24}$$

It follows from (14) that

$$\begin{aligned}\sigma_{ij} &= \frac{1}{2}\left(\frac{\partial U}{\partial e_{ij}} + \frac{\partial U}{\partial e_{ji}}\right)\\ &= C_{ijkl}e_{kl} - M_{ij}\theta\\ \rho_0 S &= -\frac{\partial U}{\partial \theta} = M_{ij}e_{ij} + a\theta\end{aligned} \tag{25}$$

Moreover, in view of the relations (15) and (16), the heat flux vector is given by

$$q_i = -k_{ij}\theta_{,j} \tag{26}$$

with

$$k_{ij}\theta_{,i}\theta_{,j} \geq 0 \tag{27}$$

The tensor k_{ij} is called the conductivity tensor.

In view of the above analysis, we remark that the energy equation can be written in the following form:

$$\rho_0 T_0 \dot{S} + q_{i,i} = \rho_0 r \tag{28}$$

Concluding, the equations of linear theory of thermoelasticity are:

– The equations of motion

$$\sigma_{ji,j} + \rho_0 b_i = \rho_0 \ddot{u}_i \tag{29}$$

– The energy equation

$$\rho_0 T_0 \dot{S} + q_{i,i} = \rho_0 r \tag{30}$$

– The constitutive equations

$$\begin{aligned} \sigma_{ij} &= C_{ijkl} e_{kl} - M_{ij}\theta \\ \rho_0 S &= M_{ij} e_{ij} + a\theta \\ q_i &= -k_{ij}\theta_{,j} \end{aligned} \tag{31}$$

– The geometrical equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{32}$$

To these equations, we must adjoin boundary conditions and initial conditions. The boundary conditions can be of Dirichlet type and of Neumann type, or we can have mixed boundary conditions.

Let us consider now, in the three-dimensional Euclidian space, a symmetric tensor **a** of second order. As it is known, the coefficients of the following polynomial expression (in ξ)

$$\begin{aligned} \det(\mathrm{a_{ij}} - \xi\delta_{\mathrm{ij}}) = -\xi^3 + \mathrm{I}_1(\mathbf{a})\xi^2 \\ -I_2(\mathbf{a})\xi + I_3(\mathbf{a}) \end{aligned} \tag{33}$$

are the principal invariants of the tensor **a** under the full group of orthogonal transformations of the rectangular frame of reference. For more details regarding the invariants of vectors and tensors, the reader is referred to the book by Eringen [9].

Moreover, we recall that any other invariant of the tensor **a** can be written as function of the invariants $I_j(\mathbf{a})$ which are given by

$$\begin{aligned} I_1(\mathbf{a}) &= a_{ii} \\ I_2(\mathbf{a}) &= \frac{1}{2}\left(a_{ii}a_{jj} - a_{rs}a_{rs}\right) \\ I_3(\mathbf{a}) &= \det(\mathrm{a_{ij}}) \end{aligned} \tag{34}$$

For isotropic materials, the function U must be form invariant under the full orthogonal group of transformations of the material frame. This means that U shall be a function of the invariants of e_{ij}, that is,

$$U = \tilde{U}(I_1(\mathbf{e}), I_2(\mathbf{e}), I_3(\mathbf{e}), \theta; X_i) \tag{35}$$

The effect of material symmetry on the form of the elasticity tensor C_{ijkl} is discussed in details by Gurtin [13].

Often, the invariants $I_i(\mathbf{e})$ are replaced by the invariants defined by

$$\begin{aligned} J_1 &= e_{ii} \\ J_2 &= e_{ij}e_{ij} \\ J_3 &= \det(\delta_{\mathrm{ij}} + 2e_{\mathrm{ij}}) \end{aligned} \tag{36}$$

For isotropic materials, the Helmholtz free energy is given by

$$U = \frac{1}{2}\lambda J_1^2 + \mu J_2 - mJ_1\theta - \frac{1}{2}a\theta \tag{37}$$

where λ, μ, m and a are constitutive coefficients. Moreover, we have

$$q_i = -k\theta_{,i} \tag{38}$$

with $k \geq 0$.

Using the relations (14), we deduce that

$$\begin{aligned} \sigma_{ij} = \lambda e_{rr}\delta_{ij} + \mu e_{ij} - m\theta\delta_{ij} \\ \rho_0 S = m e_{rr} + a\theta \end{aligned} \tag{39}$$

The scalars λ and μ are called Lamé moduli, μ is the shear modulus, m is the stress-temperature modulus, and k is the conductivity coefficient.

If $\mu \neq 0$ and $3\lambda + 2\mu \neq 0$, then the relations $(39)_1$ can be inverted to yield

$$\begin{aligned} e_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_{rr}\delta_{ij} \\ + \alpha\theta\delta_{ij} \end{aligned} \tag{40}$$

The quantity

$$\alpha = \frac{m}{3\lambda + 2\mu} \tag{41}$$

is called the coefficient of thermal expansion.

Let us define Young's modulus E and Poisson's ratio v by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, v = \frac{\lambda}{2(\lambda + \mu)} \tag{42}$$

The relations (40) can be written in the following form:

$$e_{ij} = \frac{1 + v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{rr}\delta_{ij} + \alpha\theta\delta_{ij} \tag{43}$$

Cylindrical and Spherical Coordinates

In the study of many problems of the mechanics of continuous media, it is useful to employ curvilinear coordinates. In this section, we present the equations of the linear theory of thermoelasticity in cylindrical and spherical coordinates, respectively. The equations of the theory of thermoelasticity in arbitrary curvilinear coordinates are presented in [14] (see also [15] and [16]).

Let us consider the cylindrical coordinates (ϱ, ϕ, z). Thus, we have

$$x_1 = \varrho cos\phi,\ x_2 = \varrho sin\phi,\ x_3 = z \tag{44}$$

$$\varrho \in [0, \infty), \phi \in [0, 2\pi), z \in \mathbb{R}$$

The physical components of the displacement vector are denoted by u_ϱ, u_ϕ, and u_z, while the physical components of the strain tensor are denoted by $e_{\varrho\varrho}$, $e_{\phi\phi}$, e_{zz} $e_{\varrho\phi}$, $e_{\varrho z}$, and $e_{\phi z}$.

We have the following relations between the physical components of the strain tensor and the physical components of the displacement vector:

$$\begin{aligned}
e_{\varrho\varrho} &= \frac{\partial u_\varrho}{\partial \varrho}, e_{\phi\phi} = \frac{1}{\varrho}\left(\frac{\partial u_\phi}{\partial \phi} + u_\varrho\right), e_{zz} = \frac{\partial u_z}{\partial z} \\
e_{\varrho\phi} &= \frac{1}{2}\left(\frac{1}{\varrho}\frac{\partial u_\varrho}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{1}{\varrho}u_\phi\right) \\
e_{\phi z} &= \frac{1}{2}\left(\frac{\partial u_\phi}{\partial z} + \frac{1}{\varrho}\frac{\partial u_z}{\partial \phi}\right) \\
e_{\varrho z} &= \frac{1}{2}\left(\frac{\partial u_\varrho}{\partial z} + \frac{\partial u_z}{\partial \varrho}\right)
\end{aligned} \tag{45}$$

Throughout this section, we consider the equations of the linear theory of thermoelasticity for homogeneous and isotropic media.

If we denote by b_ϱ, b_ϕ and b_z the physical components of the mass forces, then the equations of motions are

$$\begin{aligned}
\frac{\partial \sigma_{\varrho\varrho}}{\partial \varrho} + \frac{1}{\varrho}\frac{\partial \sigma_{\varrho\phi}}{\partial \phi} + \frac{\partial \sigma_{\varrho z}}{\partial z} + \frac{1}{\varrho}(\sigma_{\varrho\varrho} - \sigma_{\phi\phi}) + \rho_0 b_\varrho &= \rho_0\frac{\partial^2 u_\varrho}{\partial t^2} \\
\frac{\partial \sigma_{\varrho\phi}}{\partial \varrho} + \frac{1}{\varrho}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\partial \sigma_{\phi z}}{\partial z} + \frac{2}{\varrho}\sigma_{r\phi} + \rho_0 b_\phi &= \rho_0\frac{\partial^2 u_\phi}{\partial t^2} \\
\frac{\partial \sigma_{\varrho z}}{\partial \varrho} + \frac{1}{\varrho}\frac{\partial \sigma_{\phi z}}{\partial \phi} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{\varrho}\sigma_{rz} + \rho_0 b_z &= \rho_0\frac{\partial^2 u_z}{\partial t^2}
\end{aligned} \tag{46}$$

In cylindrical coordinates, the physical components of the stress tensor are

$$\begin{aligned}
\sigma_{\varrho\varrho} &= \lambda(e_{\varrho\varrho} + e_{\phi\phi} + e_{zz}) + 2\mu e_{\varrho\varrho} - \beta\theta \\
\sigma_{\phi\phi} &= \lambda(e_{\varrho\varrho} + e_{\phi\phi} + e_{zz}) + 2\mu e_{\phi\phi} - \beta\theta \\
\sigma_{zz} &= \lambda(e_{\varrho\varrho} + e_{\phi\phi} + e_{zz}) + 2\mu e_{zz} - \beta\theta \\
\sigma_{\varrho\phi} &= 2\mu e_{\varrho\phi} \\
\sigma_{\varrho z} &= 2\mu e_{\varrho z} \\
\sigma_{\phi z} &= 2\mu e_{\phi z}
\end{aligned} \tag{47}$$

The entropy is given by

$$\rho_0 S = \beta(e_{\varrho\varrho} + e_{\phi\phi} + e_{zz}) + a\theta \quad (48)$$

while the heat flux vector has the following physical components:

$$q_\varrho = -k\frac{\partial\theta}{\partial\varrho}, q_\phi = -k\frac{1}{\varrho}\frac{\partial\theta}{\partial\phi}, q_z = -k\frac{\partial\theta}{\partial z} \quad (49)$$

Hence, the energy equation becomes

$$\begin{aligned}&\beta T_0\frac{\partial}{\partial t}(e_{\varrho\varrho} + e_{\phi\phi} + e_{zz}) + T_0 a\frac{\partial\theta}{\partial t}\\ &= k\left[\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\frac{\partial\theta}{\partial\varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2\theta}{\partial\phi^2} + \frac{\partial^2\theta}{\partial z^2}\right] + \rho_0 r\end{aligned} \quad (50)$$

Let us now introduce the spherical coordinate system by means of the mapping

$$\begin{aligned}&x_1 = R\sin\phi\cos\psi, \mathrm{x}_2 = \mathrm{R}\sin\phi\sin\psi,\\ &x_3 = R\cos\phi\end{aligned} \quad (51)$$

$$R \in [0, \infty), \phi \in [0, \pi), \psi \in [0, 2\pi)$$

In this case, the physical components of the strain tensor are

$$\begin{aligned}e_{RR} &= \frac{\partial u_R}{\partial R}, e_{\phi\phi} = \frac{1}{R}\left(\frac{\partial u_\phi}{\partial\phi} + u_R\right)\\ e_{\psi\psi} &= \frac{1}{R\sin\phi}\frac{\partial u_\psi}{\partial\psi} + \frac{1}{R}u_R + \frac{\cot\phi}{R}u_\phi\\ e_{\phi\psi} &= \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_\psi}{\partial\phi} - \frac{\cot\phi}{R}u_\psi + \frac{1}{R\sin\phi}\frac{\partial u_\phi}{\partial\psi}\right)\\ e_{\psi R} &= \frac{1}{2}\left(\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial\psi} - \frac{1}{R}u_\psi + \frac{\partial u_\psi}{\partial R}\right)\\ e_{\phi R} &= \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial\phi} - \frac{1}{R}u_\phi + \frac{\partial u_\phi}{\partial R}\right)\end{aligned} \quad (52)$$

where u_R, u_ϕ and u_ψ are the physical components of the displacement vector.

The equations of motion become

$$\begin{aligned}&\frac{\partial\sigma_{RR}}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial\sigma_{R\psi}}{\partial\psi} + \frac{1}{R}\frac{\partial\sigma_{R\phi}}{\partial\phi} + \frac{1}{R}(2\sigma_{RR} - \sigma_{\phi\phi} - \sigma_{\psi\psi} + \sigma_{R\phi}\cot\phi) + \rho_0 b_\mathrm{R} = \rho_0\frac{\partial^2\mathrm{u_R}}{\partial \mathrm{t}^2}\\ &\frac{\partial\sigma_{R\phi}}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial\sigma_{\phi\psi}}{\partial\psi} + \frac{1}{R}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + \frac{1}{R}[3\sigma_{R\phi} + (\sigma_{\phi\phi} - \sigma_{\psi\psi})\cot\phi] + \rho_0 b_\phi = \rho_0\frac{\partial^2\mathrm{u}_\phi}{\partial \mathrm{t}^2}\\ &\frac{\partial\sigma_{R\psi}}{\partial R} + \frac{1}{R\sin\phi}\frac{\partial\sigma_{\psi\psi}}{\partial\psi} + \frac{1}{R}\frac{\partial\sigma_{\phi\psi}}{\partial\phi} + \frac{1}{R}(3\sigma_{R\psi} + 2\sigma_{\phi\psi}\cot\phi) + \rho_0 b_\psi = \rho_0\frac{\partial^2\mathrm{u}_\psi}{\partial \mathrm{t}^2}\end{aligned} \quad (53)$$

In spherical coordinates, the physical components of the stress tensor are

$$\begin{aligned}\sigma_{RR} &= \lambda(e_{RR} + e_{\phi\phi} + e_{\psi\psi}) + 2\mu e_{RR} - \beta\theta\\ \sigma_{\phi\phi} &= \lambda(e_{RR} + e_{\phi\phi} + e_{\psi\psi}) + 2\mu e_{\phi\phi} - \beta\theta\\ \sigma_{\psi\psi} &= \lambda(e_{RR} + e_{\phi\phi} + e_{\psi\psi}) + 2\mu e_{\psi\psi} - \beta\theta\\ \sigma_{R\phi} &= 2\mu e_{\phi R}\\ \sigma_{R\psi} &= 2\mu e_{\psi R}\\ \sigma_{\phi\psi} &= 2\mu e_{\phi\psi}\end{aligned} \quad (54)$$

The entropy and the physical components of the heat flux vector are given by

$$\rho_0 S = \beta(e_{RR} + e_{\phi\phi} + e_{\psi\psi}) + a\theta \quad (55)$$

and

$$\begin{aligned}&q_R = -k\frac{\partial\theta}{\partial R}, q_\phi = -k\frac{1}{R}\frac{\partial\theta}{\partial\phi},\\ &q_\psi = -k\frac{1}{R\sin\phi}\frac{\partial\theta}{\partial\psi}\end{aligned} \quad (56)$$

respectively.

In consequence, the energy equation has the following form:

$$\beta T_0 \frac{\partial}{\partial t}(e_{RR} + e_{\phi\phi} + e_{\psi\psi}) + T_0 a \frac{\partial \theta}{\partial t} = k\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \theta}{\partial R}\right) + \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial \theta}{\partial \phi}\right) + \frac{1}{R^2 \sin^2\phi}\frac{\partial^2 \theta}{\partial \psi^2}\right] + \rho_0 r \quad (57)$$

Linearized Theory of Materials with Initial Stresses and Initial Flux

In the first part of this section, we consider three states of the body: the reference configuration B (at the time t_0) and other two configurations, $\mathcal{B}$ and $\mathcal{B}^*$, at some intermediate moments. We call $\mathcal{B}$ the primary state and $\mathcal{B}^*$ the secondary state. Moreover, we shall designate as incremental those quantities associated with the difference of motion between the secondary and primary states.

Let us consider that the material point having the coordinates X_K in the reference configuration has the coordinates x_i in the primary state and y_i in the secondary state. The quantities associated with the secondary state will be denoted with an asterisk. We define the incremental displacement by

$$u_i = y_i - x_i \quad (58)$$

We have

$$y_i = y_i(X_1, X_2, X_3, t) \quad (59)$$

We define the incremental quantity

$$\theta = T^* - T \quad (60)$$

Because we consider the linear theory, we assume that $u_i = \varepsilon u_i'$ and $\theta = \varepsilon\theta'$, where ε is a constant small enough for square and higher powers to be neglected and u_i' and θ' are independent of ε.

If we refer all quantities to the primary state, then we have to consider u_i and θ as dependent of x_j. If we refer all the quantities to the reference configuration, then we consider these functions depending on X_K.

In the secondary state $\mathcal{B}^*$, we consider the following stress tensors: σ_{ij}^* is the Cauchy stress tensor, S_{Ki}^* and S_{KL}^* are the Piola-Kirchhoff stress tensors measured per unit area in the configuration B, and $S_{ji}^{*(1)}$ and $S_{ji}^{*(2)}$ are, respectively, the first and the second Piola-Kirchhoff stress tensors measured per unit area in the configuration $\mathcal{B}$.

In the configuration $\mathcal{B}^*$, we consider the following heat flux vectors: q_i^* is the heat flux across the planes $y_i =$ constant, measured per unit area of these surfaces and per unit time; Q_K^* is the heat flux across surfaces in $\mathcal{B}^*$ that in the reference configuration B were coordinate planes perpendicular to the X_K -axis, measured per unit area of these planes and per unit time; $\bar{Q}_i^*$ is the heat flux across surfaces in $\mathcal{B}^*$ that in the configuration $\mathcal{B}$ were coordinate planes perpendicular to the x_i -axis, measured per unit area of these planes and per unit time. These quantities are related by equations similar to (11).

The equations of motion of the secondary state are

$$S_{ji,j}^{*(1)} + \rho b_i^* = \rho \ddot{y}_i, \text{in } \mathcal{B} \times (t_0, t_I) \quad (61)$$

while the corresponding energy equation is

$$\rho T^* \dot{S}^* = -\bar{Q}_{i,i}^* + \rho r^*, \text{in } \mathcal{B} \times (t_0, t_I) \quad (62)$$

Let us consider the strain tensor

$$2E_{AB}^* = y_{i,A} y_{i,B} - \delta_{AB} \quad (63)$$

It is easy to see, in view of (58), that

$$E_{AB}^* = E_{AB} + \frac{1}{2}(x_{i,B} u_{i,A} + x_{i,A} u_{i,B}) \quad (64)$$

If we denote

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i} \quad (65)$$

then we have

$$E_{AB}^* = E_{AB} + \varepsilon_{ij} x_{i,A} x_{j,B} \quad (66)$$

Thus, in a second-order approximation, we have

$$\begin{aligned}\frac{\partial U^*}{\partial E^*_{AB}} &= \frac{\partial U}{\partial E_{AB}} + A_{ABMN}(E^*_{MN} - E_{MN}) - G_{AB}\theta \\ \frac{\partial U^*}{\partial T^*} &= \frac{\partial U}{\partial T} - G_{AB}(E^*_{AB} - E_{AB}) - A\theta\end{aligned} \tag{67}$$

where A_{ABMN}, G_{AB}, and A are constitutive coefficients depending on T_0 and $\mathbf{X}$.

Moreover, we have the following relation between $S^{*(1)}_{ij}$ and S^*_{LK}:

$$S^{*(1)}_{ij} = \frac{1}{J} x_{i,L} x_{r,K} S^*_{LK} y_{j,r} \tag{68}$$

Now, we call the usual relations between the first Piola-Kirchhoff tensors and the Cauchy tensor, the relation (67), and the form of the constitutive equations derived in the previous section. Hence, in a second-order approximation, we have [11, 12]

$$S^{*(1)}_{ij} = \sigma_{ij} + C_{ijkl}\varepsilon_{rs} - M_{ij}\theta + \sigma_{ir}u_{j,r} \tag{69}$$

where

$$\begin{aligned}C_{ijrs} &= \frac{1}{J} x_{i,K} x_{j,L} x_{r,M} x_{s,N} A_{KLMN} \\ M_{ij} &= \frac{1}{J} x_{i,K} x_{j,L} G_{KL}\end{aligned} \tag{70}$$

Similarly, we have that

$$\begin{aligned}\bar{Q}^*_i &= q_i - h_{ijk}\varepsilon_{jk} - r_i\theta - k_{ij}\theta_{,j} \\ \gamma &\equiv \rho_0(S^* - S) = J(M_{ij}\varepsilon_{ij} + a\theta)\end{aligned} \tag{71}$$

In Section 2, we have given the basic equations related to the primary state, in terms of the quantities referred to the initial configuration B. If we refer the motion to the configuration $\mathcal{B}$, then the equations are

$$\begin{aligned}\sigma_{ji,j} + \rho b_i &= \rho \ddot{x}_i \\ -q_{i,i} + \rho r &= \rho T\dot{S}, \text{in } \mathcal{B} \times (t_0, t_1)\end{aligned} \tag{72}$$

In order to study the motion between the secondary state and primary state, we subtract (61) and (62) from $(72)_1$ and $(72)_2$, respectively, and we obtain

$$\begin{aligned}\lambda_{ji,j} + \rho B_i &= \rho \ddot{u}_i \\ -\psi_{i,i} + \rho R &= \frac{1}{J} T\dot{\gamma} + \rho\theta\dot{S}, \text{in } \mathcal{B} \times (t_0, t_1)\end{aligned} \tag{73}$$

where $B_i = b^*_i - b_i$, $R = r^* - r$, $\lambda_{ij} = S^{*(1)}_{ij} - \sigma_{ij}$, and $\psi_i = \bar{Q}^*_i - q_i$.

Now we consider that the primary configuration $\mathcal{B}$ is identical with the reference configuration B. We suppose that B is subjected to an initial stress and is at a nonuniform temperature T. So, $\mathbf{u}$ denotes the displacement vector, θ denotes the temperature variation, $J = 1, \rho_0 = \rho, \overline{T} \equiv T$, and $E_{AB} = 0$. The quantities σ_{ij}, S, C_{ijkl}, M_{ij}, and a must be evaluated for $E_{AB} = 0$ and $T = \overline{T}$. The functions σ_{ij} correspond to the initial stresses, and q_i corresponds to the initial heat flux.

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In view of the above results, the equations of linearized theory of materials with initial stresses and initial heat flux are

$$\begin{aligned}\lambda_{ji,j} + \rho_0 B_i &= \rho_0 \ddot{u}_i \\ -\psi_{i,i} + \rho r &= T\dot{\gamma}, \text{in } \mathcal{B} \times (t_0, t_1)\end{aligned} \tag{74}$$

where

$$\begin{aligned}\lambda_{ij} &= (C_{ijrs} + \sigma_{is}\delta_{jr})u_{r,s} - M_{ij}\theta \\ \gamma &= M_{ij}u_{i,j} + a\theta \\ \psi_i &= -h_{ijk}u_{j,k} - r_i\theta - k_{ij}\theta_{,j}\end{aligned} \tag{75}$$

If $\overline{T}$ is constant, then $q_i = 0, h_{ijk} = 0$ and $r_i = 0$. To this system, we have to adjoin boundary conditions (Neumann or Dirichlet conditions) and initial conditions.

Cross-References

▶ Structural Stability in Linear Thermoelasticity

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Linear Thermoelasticity

► Well-Posed Problems

Linear Thermoelasticity Without Energy Dissipation

► Hamilton–Kirchhoff Principle

Lines of Strong and Weak Discontinuity

► Transient Thermoelastic Rayleigh Waves on the Surfaces of Bodies of Revolution

Local Gradient Thermomechanics

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Synonyms

Interface and structural nonhomogeneity; Interface phenomena

Definitions

Local gradient models are the models describing locally heterogeneous, namely, structured deformable systems. They are based on principles of irreversible thermodynamics and solid mechanics. The local gradientality means that a chemical potential gradient is used for describing the equilibrium state of physically small subsystem of a one-component solid body. The parameter conjugated to the gradient of chemical potential is the density.

Real solids are locally heterogeneous. They feature a structure as, for example, internal areas of grains and their surfaces. In massive solid homogeneous bodies, the relative part of interfacial component in internal energy and other energetic characteristics usually is small as compared with a volume component, and the interfacial effects are used to be ignored in classic mathematical models of thermomechanics.

Near-surface heterogeneity is especially important in solids whose characteristic size is