S

Saint-Venant's Principle

Ionel-Dumitrel Ghiba
Department of Mathematics, "Alexandru Ioan Cuza" University of Iaşi, Iaşi, Romania
"Octav Mayer" Institute of Mathematics, Romanian Academy, Iaşi, Romania

Synonyms

Linear theory; Spatial behavior; Thermoelasticity

Overview

Saint-Venant's principle in linear elasticity was established by Toupin [20] and Knowles [15], who gave an inequality describing exponential spatial decay of effects with distance from the excited end of the right cylinder. A comprehensive review of the results concerning Saint-Venant's principle was given in [11, 13, 14]. In elastodynamics, the study of spatial decay of solutions was first suggested by Boley [2]. Boley [3–5] was also the first who studied the validity of Saint-Venant's principle for transient heat conduction.

In the dynamical theory of linear thermoelasticity, Chiriţă [7] established a spatial decay estimate of Saint-Venant type for a certain energetic measure. Then, Chiriţă and Ciarletta [8] introduced a new method, called the time-weighted surface power method, which was applied to study the spatial behavior of elastic, viscoelastic, and thermoelastic processes outside of the support of the external given data. Within the framework of thermoelasticity theory, Chiriţă and Ciarletta [8] studied the spatial behavior of dynamic processes in terms of two appropriate time-weighted surface power functions. Using the properties of these time-weighted surface power functions, for bounded domains, they established decay estimates of Saint-Venant type, while for unbounded domains, the authors established alternatives of Phragmén–Lindelöf type with both time-dependent and time-independent decay and growth rates.

On the other hand, following the research line initiated by Horgan et al. [12] (for parabolic equations), Quintanilla [17, 18] (for coupled systems of parabolic and hyperbolic equations) has shown that further exponential estimates may be established for semi-infinite cylinders. For a fixed time, the results in question prove that the mechanical and thermal effects are controlled by an exponential decay estimate in terms of the square of the distance from the support of the external given data.

The present work gives a short description of the results established in [7, 8, 17, 18].

Formulation of the Problem

Throughout this entry we shall consider a thermoelastic homogeneous material which occupies the region B of the three-dimensional Euclidian space E^3 whose boundary is the smooth surface ∂B.

R.B. Hetnarski (ed.), *Encyclopedia of Thermal Stresses*, DOI 10.1007/978-94-007-2739-7,

We refer the motion of the body to a rectangular Cartesian system of axes Ox_i. Throughout this entry, Latin subscripts take the values $1, 2, 3$; summation is carried out over repeated indices. Typical conventions for differential operations are implied such as a superposed dot or comma followed by a subscript to denote the partial derivative with respect to time or to the corresponding Cartesian coordinate, respectively.

Let us consider a given time $t_1 > 0$. In the absence of body forces and heat supplies, the equations of the linear theory of thermoelasticity consist of (see [6, 10])

– The equations of motion

$$\sigma_{ji,j} = \rho_0\, \ddot{u}_i \tag{1}$$

– The energy equation

$$-q_{i,i} - T_0 M_{ij}\, \dot{u}_{i,j} = c\dot{\theta} \tag{2}$$

– The stress–strain–temperature relation and the heat conduction equation

$$\begin{aligned} \sigma_{ij} &= C_{ijkl} e_{kl} - M_{ij}\theta \\ q_i &= -k_{ij} g_j \end{aligned} \tag{3}$$

– The strain–displacement relations and the thermal gradient–temperature relation

$$\begin{aligned} e_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}) \\ g_i &= \theta_{,i} \end{aligned} \tag{4}$$

Here we have used notation: u_i are the components of the displacement vector, θ is the variation of temperature from the reference configuration, ρ_0 is the mass density of the continuum at the initial time, σ_{ij} is the stress tensor, T_0 is the absolute temperature in the reference configuration, e_{ij} is the infinitesimal strain tensor, q_i is the heat flux vector, and C_{ijkl}, M_{ij}, k_{ij}, and c are constitutive coefficients satisfying the following properties of symmetry:

$$\begin{aligned} M_{ij} &= M_{ji} \\ C_{ijkl} &= C_{klij} = C_{jikl} \end{aligned} \tag{5}$$

In this work we suppose that C_{ijkl} is positive definite. It follows that there exist the minimum and maximum elastic moduli for C_{ijkl}, denoted by $\mu_m > 0,\ \mu_M > 0$, such that

$$\begin{aligned} \mu_m \xi_{ij}\xi_{ij} &\le C_{ijkl}\xi_{ij}\xi_{kl} \\ &\le \mu_M \xi_{ij}\xi_{ij}, \quad \text{for all symmetric} \\ &\quad \text{tensors} \quad \xi_{ij} \end{aligned} \tag{6}$$

We assume that the conductivity tensor k_{ij} is symmetric and positive definite. Thus,

$$k_{ij} = k_{ji} \tag{7}$$

and

$$\begin{aligned} k_m \xi_i \xi_i &\le k_{ij}\xi_i\xi_j \\ &\le k_M \xi_i \xi_i, \quad \text{for all vectors} \quad \xi_i \end{aligned} \tag{8}$$

where $k_m > 0$ and $k_M > 0$ denote the minimum and maximum conductivity moduli for k_{ij}.

Since we consider the linear theory of coupled thermoelasticity, we can assume that

$$m = \left(M_{ij} M_{ij}\right)^{\frac{1}{2}} > 0 \tag{9}$$

In view of the constitutive equations (3), we have [7, 8]

$$\begin{aligned} \sigma_{ij}\sigma_{ij} &= C_{ijkl} e_{kl} \sigma_{ij} - \theta M_{ij}\sigma_{ij} \\ &\le (C_{rsmn} e_{rs} e_{mn})^{1/2} (C_{ijkl}\sigma_{ij}\sigma_{kl})^{1/2} \\ &+ |\theta| (M_{rs}M_{rs})^{1/2} (\sigma_{ij}\sigma_{ij})^{1/2} \\ &\le (\sigma_{ij}\sigma_{ij})^{1/2} \left[(\mu_M C_{rsmn} e_{rs} e_{mn})^{1/2} + m|\theta| \right] \end{aligned} \tag{10}$$

Thus, by means of the arithmetic–geometric mean inequality, we get

$$\sigma_{ij}\sigma_{ij} \le (1+\varepsilon)\mu_M C_{ijkl} e_{ij} e_{kl} + \left(1 + \frac{1}{\varepsilon}\right) m^2\theta^2, \quad \text{for all} \quad \varepsilon \in \mathbb{R} \tag{11}$$

Moreover, we have that

$$q_i q_i \le k_M k_{lm} \theta_{,l} \theta_{,m} \tag{12}$$

Throughout this entry, by an admissible process, we mean an ordered array $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ with the following properties:

1. $u_i, \dot{u}_i, \ddot{u}_i$, $(u_{i,j} + u_{j,i})$, and $(\dot{u}_{i,j} + \dot{u}_{j,i})$ are continuous on $\overline{B} \times [0, t_1)$.
2. e_{ij} is continuous symmetric tensor field on $\overline{B} \times [0, t_1)$.
3. σ_{ij} and $\sigma_{ji,j}$ are continuous on $\overline{B} \times [0, t_1)$.
4. $\theta, \dot{\theta}$ and $\theta_{,i}$ are continuous on $\overline{B} \times [0, t_1)$.
5. g_i are continuous on $\overline{B} \times [0, t_1)$.
6. q_i and $q_{i,i}$ are continuous on $\overline{B} \times [0, t_1)$.

By a thermoelastic process we mean an admissible process $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ that meets the fundamental system of field equations (1)–(4), listed at the beginning of this section.

The corresponding surface traction s_i and heat flux q are defined at every regular point of the boundary surface ∂B by

$$s_i = \sigma_{ji} n_j, \quad q = q_i n_i \tag{13}$$

where n_i are the components of outward unit normal to the boundary surface ∂B.

To a thermoelastic process $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$, we associate the quantities

$$K = \frac{1}{2} \rho_0 \, \dot{u}_i \dot{u}_i \tag{14}$$

$$U = \frac{1}{2} C_{ijkl} e_{ij} e_{kl} \tag{15}$$

$$S = \frac{1}{2T_0} c\theta^2 \tag{16}$$

$$D = \frac{1}{T_0} k_{ij} \theta_{,i} \theta_{,j} \tag{17}$$

Saint-Venant's Principle

In this section we consider the domain B to be a closed, bounded, regular region in the three-dimensional space, whose boundary includes a plane portion S_0. We choose Cartesian coordinates so that S_0 lies in the plane $x_3 = 0$, and suppose that B lies in the half-space $x_3 > 0$.

Let S_z be the intersection with B of the plane $x_3 = z$; let

$$B_z = \{\mathbf{x} \in B | x_3 > z\} \tag{18}$$

and let $L = \max\{x_3 \in [0, \infty) | \mathbf{x} \in B\}$.

We associate with the thermoelastic process $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ the kinetic energy on B_z

$$\mathcal{K}(z,t) = \int_{B_z} K \, dv \tag{19}$$

the strain energy on B_z

$$\mathcal{U}(z,t) = \int_{B_z} U \, dv \tag{20}$$

the thermal energy on B_z

$$\mathcal{S}(z,t) = \int_{B_z} S \, dv \tag{21}$$

the dissipation energy on B_z, on the time interval $[0, t)$

$$\mathcal{D}(z,t) = \int_0^t \int_{B_z} D \, dv d\tau \tag{22}$$

and the total energy on B_z

$$\mathcal{E}(z,t) = \mathcal{K}(z,t) + \mathcal{U}(z,t) + \mathcal{S}(z,t) + \mathcal{D}(z,t) \tag{23}$$

We define the energy stored in the portion B_z of B in the time interval $[0, t]$ by

$$E^*(z,t) = \int_0^t \mathcal{E}(z,\tau) d\tau \tag{24}$$

Let the body be subject to null initial conditions so that

$$u_i(\mathbf{x}, 0) = 0, \quad \dot{u}_i(\mathbf{x}, 0) = 0, \quad \theta(\mathbf{x}, 0) = 0, \quad \text{on } \overline{B} \tag{25}$$

In this section, we suppose that the body is deformed under the action of nonzero boundary conditions only on the plane end face S_0. So, we assume that

$$\sigma_{ji}(\mathbf{x},t)n_j = 0 \text{ on } (\partial B\backslash S_0)\times(0,t_1) \quad (26)$$

$$q_i(\mathbf{x},t)n_i = 0 \text{ on } (\partial B\backslash S_0)\times(0,t_1) \quad (27)$$

The boundary conditions on the end face S_0 are

$$s_i(\mathbf{x},t) \equiv \sigma_{ji}(\mathbf{x},t)n_j = \hat{s}_i(\mathbf{x},t) \text{ on } S_0\times(0,t_1) \quad (28)$$

$$q(\mathbf{x},t) \equiv q_i(\mathbf{x},t)n_i = \hat{q}(\mathbf{x},t) \text{ on } S_0\times(0,t_1) \quad (29)$$

where $\hat{s}_i$ and $\hat{q}$ are prescribed continuous functions on $S_0\times(0,t_1)$. Further, we assume that the traction $\hat{s}_i$ is self-equilibrated at each instant, i.e.,

$$\int_{S_0} \hat{s}_i\,da = 0, \quad \int_{S_0} \varepsilon_{ijk}x_j\,\hat{s}_k\,da = 0 \quad (30)$$

Moreover, we assume that $\dot{\hat{s}}_i$ is continuous on $S_0\times[0,t_1)$.

In this entry [7], it is shown in the following result:

Theorem 1. *Let* $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ *be a thermoelastic process on* $\overline{B}$ *satisfying the initial conditions* (25) *and the boundary conditions* (26) *and* (27). *Then we have*

$$E^*(z,t) \le E^*(0,t)\exp\left\{-\frac{z}{v(t)\sqrt{t}}\right\} \quad (31)$$

for $0\le z\le L,\quad t\ge 0$, where

$$v(t) = \max\left\{2\sqrt{\frac{\mu_M t}{\rho_0}}, \frac{T_0}{2\mu_M}\sqrt{k_M c}, 2\frac{T_0 m}{c}\sqrt{\frac{t}{\rho_0\mu_M}} + \frac{\mu_M}{T_0 c}\sqrt{\frac{k_M}{c}}\right\} \quad (32)$$

Proof. We note that the function E^* is continuous differentiable with respect to z, and, moreover, we have

$$\frac{\partial E^*}{\partial z}(z,t) = -\frac{1}{2}\int_0^t\int_{S_z}\left\{\rho_0\,\dot{u}_i\dot{u}_i + C_{ijkl}e_{ij}e_{kl} + \frac{1}{T_0}c\theta^2 + \frac{2}{T_0}\int_0^s k_{ij}\theta_{,i}\theta_{,j}d\tau\right\}dads \quad (33)$$

Further, on the basis of (1), (3), and (4) and the symmetry of C_{ijkl} and M_{ij}, we get

$$\frac{1}{2}\frac{\partial}{\partial t}(\rho_0\,\dot{u}_i\dot{u}_i) = (\sigma_{ij}\,\dot{u}_j)_{,i} - C_{ijkl}e_{ij}\,\dot{e}_{kl} + M_{ij}\,\dot{e}_{ij}\,\theta \quad (34)$$

We next eliminate the last term on the right-hand side of the above equation by means of (2). Then we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}(\rho_0\,\dot{u}_i\dot{u}_i + C_{ijkl}e_{ij}e_{kl}) = \left(\sigma_{ij}\,\dot{u}_j - \frac{1}{T_0}\theta q_i\right)_{,i} - \frac{1}{T_0}c\theta\dot{\theta} + \frac{1}{T_0}q_i\theta_{,i} \quad (35)$$

Finally, we integrate relation (35) over $B_z\times[0,t]$ and use the divergence theorem, the initial conditions, and the boundary conditions in order to obtain

$$\frac{1}{2}\int_{B_z}\left\{\rho_0\,\dot{u}_i\dot{u}_i + C_{ijkl}e_{ij}e_{kl} + \frac{1}{T_0}c\theta^2 + \frac{2}{T_0}\int_0^t k_{ij}\theta_{,i}\theta_{,j}\right\}dv = \int_0^t\int_{S_z}\left\{\dot{u}_i\,s_i - \frac{1}{T_0}\theta q\right\}dads \quad (36)$$

Thus, the (36) and (24) lead to the identity

$$E^*(z,t) = \int_0^t\int_0^s\int_{S_z}\left\{\dot{u}_i\,s_i - \frac{1}{T_0}\theta q\right\}dad\tau ds \quad (37)$$

Using the Schwarz inequality and the relation (8), we get

$$E^*(z,t) \le t\left(\int_0^t\int_{S_z}\dot{u}_i\dot{u}_i\,dads\right)^{\frac{1}{2}} \times \left(\int_0^t\int_{S_z}s_is_i dads\right)^{\frac{1}{2}} \quad (38)$$

$$+\frac{1}{T_0}\left(\sqrt{t}\int_0^t\int_{S_z}\theta^2 dads\right)^{\frac{1}{2}}$$
$$\times\left(\sqrt{t}\int_0^t\int_0^s\int_{S_z}q^2 dad\tau ds\right)^{\frac{1}{2}} \tag{39}$$

Further, we use the arithmetic–geometric mean inequality and the estimates (6) and (8) in order to obtain

$$v(t)\sqrt{t}\frac{\partial E^*}{\partial z}(z,t)+E^*(z,t)\le 0 \tag{40}$$

By integrating this first-order differential inequality, we obtain the estimate (2) and the proof is complete.

In the following we give an explicit upper bound for $E^*(0,t)$ in terms of the given data [7]

Theorem 2. *Let* $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ *be a thermoelastic process on* $\overline{B}$ *satisfying the initial conditions* (25) *and the boundary conditions* (26)–(29). *We assume that* $\hat{s}_i$, $\dot{\hat{s}}_i$ *and* $\hat{q}$ *are continuous on* $S_0\times[0,t_1)$ *and* $\hat{s}_i$ *is self-equilibrated at each instant. Then we have*

$$E^*(0,t)\le 16b^{-1}\left[\int_0^t\left(\int_0^s\int_{S_0}\dot{\hat{s}}_i\dot{\hat{s}}_i dad\tau\right)^{1/2}ds\right]^2$$
$$+\frac{2}{T_0^2}C\max\left(t,\frac{k_M\rho_0}{\mu_M c}\right)\int_0^t\int_0^s\int_{S_0}\hat{q}^2 dad\tau ds \tag{41}$$

where b and C are positive constants depending on the domain B, on S_0, T_0, and the thermoelastic moduli.

Proof. Using the integration over B for (1) and by means of the boundary conditions (26) and (28) and the initial conditions (25), we get

$$\int_B u_i dv=0,\quad \int_B \varepsilon_{ijk}x_j u_k dv=0 \tag{42}$$

Then, we have the inequalities [1, 7, 9]

$$\int_{S_0}u_iu_i da\le b^{-1}\int_B C_{ijkl}e_{ij}e_{kl}dv \tag{43}$$

$$\int_{\partial B}\theta^2 d\sigma\le\frac{C}{2}\int_B\left(\frac{1}{T_0}c\theta^2+\frac{k_M\rho_0}{\mu_M c}\frac{2}{T_0}k_{ij}\theta_{,i}\theta_{,j}\right)dv \tag{44}$$

where b and C are positive constants depending on B, S_0, T_0, and the thermoelastic coefficients. It follows that

$$\int_0^t\int_{S_0}u_iu_i dads\le 2b^{-1}E^*(0,t) \tag{45}$$

$$\int_0^t\int_0^s\int_{S_0}\theta^2 dad\tau ds\le C\max\left(t,\frac{k_M\rho_0}{\mu_M c}\right)E(0,t) \tag{46}$$

On the other hand, from the relation (8) and using the initial conditions, we obtain

$$E^*(0,t)=\int_0^t\int_{S_0}u_i\,\hat{s}_i\,dads$$
$$-\int_0^t\int_0^s\int_{S_0}u_i\,\dot{\hat{s}}_i dad\tau ds$$
$$-\int_0^t\int_0^s\int_{S_0}\frac{1}{T_0}\theta\hat{q}dad\tau ds \tag{47}$$

Moreover, due to the compatibility between the initial conditions (25) and the end conditions (28), we have the inequality

$$\frac{d}{dt}\left(\int_0^t\int_{S_0}\hat{s}_i\hat{s}_i dads\right)\le 2\left(\int_0^t\int_{S_0}\hat{s}_i\hat{s}_i dads\right)^{1/2}\left(\int_0^t\int_{S_0}\dot{\hat{s}}_i\dot{\hat{s}}_i dads\right)^{1/2} \tag{48}$$

which implies that

$$\left(\int_0^t\int_{S_0}\hat{s}_i\hat{s}_i\,dads\right)^{1/2}$$
$$\le\int_0^t\left(\int_0^s\int_{S_0}\dot{\hat{s}}_i\dot{\hat{s}}_i\,dad\tau\right)^{1/2}ds \tag{49}$$

Now, we use the relations (45), (46), (47), and the Schwarz inequality in order to obtain

$$E^{*2}(0,t) \leq \left\{ 16\, b^{-1} \left[\int_0^t \left(\int_0^s \int_{S_0} \dot{\hat{s}}_i \dot{\hat{s}}_i da d\tau \right)^{1/2} ds \right]^2 + \frac{2}{T_0^2} C \max\left(t, \frac{k_M \rho_0}{\mu_M c} \right) \int_0^t \int_0^s \int_{S_0} \hat{q}^2\, da d\tau ds \right\} E^*(0,t) \tag{50}$$

which leads to the estimate (12), and the proof is complete.

Estimates for Arbitrary Domains

In this section, we consider a domain B with an arbitrary shape. Given the thermoelastic process $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ on $\overline{B}$, we define $\hat{D}_{t_1}$ to be the set of all $\mathbf{x} \in \overline{B}$ such that:

1. If $\mathbf{x} \in B$, then

$$u_i(\mathbf{x}, 0) \neq 0 \text{ or } \dot{u}_i(\mathbf{x}, 0) \neq 0 \text{ or } \theta(\mathbf{x}, 0) \neq 0 \tag{51}$$

or

2. If $\mathbf{x} \in \partial B$, then

$$s_i(\mathbf{x}, \tau)\, \dot{u}_i(\mathbf{x}, \tau) \neq 0 \quad \text{or} \tag{52}$$

$$q(\mathbf{x}, \tau)\theta(\mathbf{x}, \tau) \neq 0 \text{ for some } \tau \in [0, t_1]$$

Roughly speaking, $\hat{D}_{t_1}$ represents the support of the initial and boundary data on the time interval $[0, t_1]$. In what follows we shall assume that $\hat{D}_{t_1}$ is a bounded set. We consider next a nonempty set $\hat{D}^*_{t_1}$ so that $\hat{D}_{t_1} \subset \hat{D}^*_{t_1} \subset \overline{B}$ and such that:

1. If $\hat{D}_{t_1} \cap B \neq \emptyset$, then we choose $\hat{D}^*_{t_1}$ to be the smallest bounded regular region in $\overline{B}$ that includes $\hat{D}_{t_1}$; in particular, we set $\hat{D}^*_{t_1} = \hat{D}_{t_1}$ if $\hat{D}_{t_1}$ also happens to be a regular region.
2. If $\emptyset \neq \hat{D}_{t_1} \subset \partial B$, then we choose $\hat{D}^*_{t_1}$ to be the smallest regular subsurface of ∂B that includes $\hat{D}_{t_1}$; in particular, we set $\hat{D}^*_{t_1} = \hat{D}_{t_1}$ if $\hat{D}_{t_1}$ is a regular subsurface of ∂B.
3. If $\hat{D}_{t_1} = \emptyset$, then we choose $\hat{D}_{t_1}$ to be an arbitrary nonempty regular subsurface of ∂B.

On this basis, we define the set D_r, $r \geq 0$, by

$$D_r = \left\{ \mathbf{x} \in \overline{B} : \hat{D}^*_{t_1} \cap \overline{\Sigma(\mathbf{x}, r)} \neq \emptyset \right\} \tag{53}$$

where $\Sigma(\mathbf{x}, r)$ is the open ball with radius r and center at $\mathbf{x}$. Further, we shall use the notation B_r, for the part of B contained in $B \backslash D_r$, and we set $B(r_1, r_2) = B_{r_1} \backslash B_{r_2}$, $r_1 > r_2$. Moreover, we shall denote by S_r the subsurface of ∂B_r contained into inside of B and whose outward unit normal vector is forwarded to the exterior of D_r.

We associate with the thermoelastic process $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ the following time-weighted surface power function [8]:

$$Q(r,t) = -\int_0^t \int_{S_r} e^{-\lambda s} \left[s_i(s)\, \dot{u}_i(s) - \frac{1}{T_0} q(s)\theta(s) \right] da ds \tag{54}$$

for all $r \geq 0$ and $0 \leq t \leq t_1$, where λ is a prescribed strictly positive parameter at our disposal. Moreover, we consider the function

$$\tilde{Q}(r,t) = \int_0^t Q(r,s) ds \tag{55}$$

In the paper [8], the following result has been established:

Theorem 3. *Let $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ be a thermoelastic process on $\overline{B}$ corresponding to given data with the bounded support $\hat{D}_{t_1}$ on the time interval $[0, t_1]$. Then the corresponding time-weighted surface power function $Q(r,t)$ has the following properties:*

(Q1) For each $t \in [0, t_1]$

$$Q(r_1,t) - Q(r_2,t) = -\frac{1}{2} \int_{B(r_1,r_2)} e^{-\lambda t} \left[\rho_0\, \dot{u}_i \dot{u}_i + C_{ijkl} e_{ij} e_{kl} + \frac{c}{T_0} \theta^2 \right] dv - \int_0^t \int_{B(r_1,r_2)} e^{-\lambda s} \left\{ \frac{\lambda}{2} \left[\rho_0\, \dot{u}_i \dot{u}_i + C_{ijkl} e_{ij} e_{kl} + \frac{c}{T_0} \theta^2 \right] + \frac{1}{T_0} k_{ij} \theta_{,i} \theta_{,j} \right\} dv ds, \; 0 \leq r_2 \leq r_1 \tag{56}$$

(Q2) $Q(r,t)$ is a continuous differentiable function on $r \geq 0$, $0 \leq t \leq t_1$, and

$$\frac{\partial}{\partial r}Q(r,t) = -\frac{1}{2}\int_{S_r} e^{-\lambda t}\left[\rho_0 \dot{u}_i \dot{u}_i + C_{ijkl}e_{ij}e_{kl} + \frac{c}{T_0}\theta^2\right]dv - \int_0^t \int_{S_r} e^{-\lambda s}\left\{\frac{\lambda}{2}\left[\rho_0 \dot{u}_i \dot{u}_i + C_{ijkl}e_{ij}e_{kl} + \frac{c}{T_0}\theta^2\right] + \frac{1}{T_0}k_{ij}\theta_{,i}\theta_{,j}\right\}dads \leq 0 \quad (57)$$

(Q3) For each fixed $t \in [0, t_1]$, $Q(r,t)$ is a nonincreasing function with respect to r.

(Q4) $Q(r,t)$ satisfies the following first-order differential inequality

$$\frac{\lambda}{\kappa}|Q(r,t)| + \frac{\partial}{\partial r}Q(r,t) \leq 0, \quad r \geq 0, \ 0 \leq t \leq t_1 \quad (58)$$

where

$$\kappa = \sqrt{\frac{(1+\varepsilon)\mu_M}{\rho_0}} \quad (59)$$

and ε is the positive root of the algebraic equation

$$\varepsilon^2 + \varepsilon\left(1 - \frac{mT_0}{c\mu_M} - \frac{\lambda\rho_0 k_M}{2c\mu_M}\right) - \frac{mT_0}{c\mu_M} = 0 \quad (60)$$

(Q5) $\tilde{Q}(r,t)$ satisfies the following first-order differential inequality

$$t\chi(t)\frac{\partial}{\partial r}\tilde{Q}(r,t) + |\tilde{Q}(r,t)| \leq 0, \ r \geq 0, \ 0 \leq t \leq t_1 \quad (61)$$

where

$$\chi(t) = \sqrt{\frac{[1+\delta(t)]\mu_M}{\rho_0}} \quad (62)$$

and $\delta(t)$ is the positive root of the algebraic equation

$$\delta^2 + \delta\left(1 - \frac{mT_0}{c\mu_M} - \frac{\rho_0 k_M}{2tc\mu_M}\right) - \frac{mT_0}{c\mu_M} = 0 \quad (63)$$

(Q6) For a bounded body, $Q(r,t)$ and $\tilde{Q}(r,t)$ are positive functions.

The analysis of spatial behavior of thermoelastic processes requests a separate discussion for bounded and unbounded domains. Using the above theorem, in [8] there were established two spatial decay estimates of Saint-Venant type, one of which predicts a time-independent decay rate, while the other predicts a time-dependent decay rate.

We first consider the case of a bounded domain, that is, r ranges on $[0, l]$, $l < \infty$, where

$$l = \max\left\{\min\left\{[(x_i - y_i)(x_i - y_i)]^{1/2} | \mathbf{y} \in \hat{D}^*_{t_1}\right\} | \mathbf{x} \in \overline{B}\right\} \quad (64)$$

Theorem 4. (Spatial behavior for bounded domains). *Let* $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ *be a thermoelastic process on the bounded regular region* $\overline{B}$*, corresponding to given data with the bounded support* $\hat{D}_{t_1}$ *on the time interval* $[0, t_1]$*, and let* $Q(r,t)$ *be the corresponding time-weighted surface power function. Then, for each fixed* $t \in [0, t_1]$*, either*

$$Q(r,t) \leq Q(0,t)\exp\left(-\frac{\lambda}{\kappa}r\right), \quad 0 \leq r \leq l \quad (65)$$

where κ is given by (59) or

$$\tilde{Q}(r,t) \leq \tilde{Q}(0,t)\exp\left(-\frac{1}{t\chi(t)}r\right), \quad 0 \leq r \leq l \quad (66)$$

where $\chi(t)$ is given by (62).

Let us discuss the spatial behavior of thermoelastic processes on unbounded domains. For such bodies, a Phragmèn–Lindelöf-type result has been established in [8].

Theorem 5. (Spatial behavior for unbounded domains). *Let* $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$ *be a thermoelastic process on the unbounded regular region* $\overline{B}$*, corresponding to given data with the bounded support* $\hat{D}_{t_1}$ *on the time* *interval* $[0, t_1]$, *and let* $Q(r,t)$ *be the corresponding time-weighted surface power function. Then, for each fixed* $t \in [0, t_1]$, *the following alternative holds:*

1. Either $Q(r,t) \geq 0$ for all $r \geq 0$ and then

$$Q(r,t) \leq Q(0,t) \exp\left(-\frac{\lambda}{\kappa} r\right), \quad r \geq 0 \qquad (67)$$

or

2. There exists a value $r_t \geq 0$ so that $Q(r_t, t) < 0$ and then $Q(r,t) < 0$ for all $r \geq r_t$ and

$$-Q(r,t) \geq -Q(r_t,t) \exp\left[\frac{\lambda}{\kappa}(r - r_t)\right], \quad r \geq r_t \qquad (68)$$

Estimates with time-dependent rates are given by the following theorem:

Theorem 6. *Assume the hypotheses of Theorem 3 hold. Then, for each fixed* $t \in [0, t_1]$, *the following alternative holds:*

1. Either $\tilde{Q}(r,t) \geq 0$ for all $r \geq 0$ and then

$$\tilde{Q}(r,t) \leq \tilde{Q}(0,t) \exp\left(-\frac{1}{t\chi(t)} r\right), \quad r \geq 0 \qquad (69)$$

or

2. There exists a value $r_t^* \geq 0$ so that $\tilde{Q}(r_t^*, t) < 0$ and then $\tilde{Q}(r,t) < 0$ for all $r \geq r_t^*$ and

$$-\tilde{Q}(r,t) \geq -\tilde{Q}(r_t,t) \exp\left[\frac{1}{t\chi(t)}(r - r_t^*)\right], \quad r \geq r_t^* \qquad (70)$$

Further Estimates for Unbounded Domains

In this section, we consider an unbounded domain, and we present an estimate in the sense described by Horgan, Payne, and Wheeler [12] and by Quintanilla [17, 18].

To this end, we assume that the support $\hat{D}_{t_1}$ of the initial and boundary data, defined in the previous section, is enclosed in the half-space $x_3 < 0$. We introduce the notation S_z for the open cross section of B for which $x_3 = z,\ z \geq 0$ and whose unit normal vector is $(0,0,1)$. We assume that the unbounded set B is so that S_z is bounded for all finite $z \in [0, \infty)$. We denote by B_z that portion of B for which $x_3 > z$.

In the works by Quintanilla [17, 18], with the $[u_i, e_{ij}, \sigma_{ij}, \theta, g_i, q_i]$, the following function was associated:

$$H(z,t) = -\int_0^t \int_{S_z} \left[\sigma_{3i}\,\dot{u}_i - \frac{1}{T_0} q_3 \theta\right] da d\tau \qquad (71)$$

We assume further the following asymptotic behavior for $H(z,t)$ (see [17, 18]):

$$H(z,\ t) \to 0 \qquad (72)$$

uniformly in t, as $z \to \infty$.

Using (1), the divergence theorem and having in mind the definition of S_z and B_z, we deduce that the function $H(z,t)$ can be written in the form

$$\begin{aligned} H(z,t) &= \int_0^t \int_{B_z} \left[\sigma_{3i}\,\dot{u}_i - \frac{1}{T_0} q_3 \theta\right]_{,3} d a d\tau \\ &= \int_{B_z} [K(t) + S(t) + U(t)] dv + \int_0^t \int_{B_z} D(\tau) dv d\tau \end{aligned} \qquad (73)$$

Further, we introduce the function

$$E(z,t) = \int_z^\infty H(p,t) dp \qquad (74)$$

and note that

$$\frac{\partial E}{\partial z} = -H(z,t) \qquad (75)$$

and

$$\begin{aligned} \frac{\partial^2 E}{\partial z^2} &= \int_{S_z} [K(t) + S(t) + U(t)] da + \\ &+ \int_0^t \int_{S_z} D(\tau) da d\tau \end{aligned} \qquad (76)$$

On the other hand, from the constitutive equations (3) and the relations (71) and (74), we have

$$\frac{\partial E}{\partial t} = -\int_{B_z}\left(\sigma_{i3}\,\dot{u}_i - \frac{1}{T_0}q_3\theta\right)dv \tag{77}$$

We remark that

$$\begin{aligned}\int_{B_z} q_3\theta dv &= -\int_z^{\infty}\int_{S_z} k_{33}\theta_{,3}\theta dadz\\ &= \frac{1}{2}\int_{S_z} k_{33}\theta^2 da\end{aligned} \tag{78}$$

The next step is to estimate the time derivative of E in terms of the two first spatial derivatives of E. In view of the relations (75), (76), (77), and (78), we can conclude that there exists the positive constants A_1 and A_2 such that

$$\frac{\partial E}{\partial t} \le -A_1\frac{\partial E}{\partial z} + A_2\frac{\partial^2 E}{\partial z^2} \tag{79}$$

Following Horgan et al. [12], we use the change of variable

$$w(z,t) = \exp(b_2 t - b_1 z)E(z,t) \tag{80}$$

with

$$b_1 = \frac{A_1}{2A_2}, \quad b_2 = b_1^2 A_2 \tag{81}$$

Thus, the inequality (79) becomes

$$\frac{\partial w}{\partial t} \le A_2\frac{\partial^2 w}{\partial z^2} \tag{82}$$

We have that $w(z,t)$ satisfies the relations

$$\begin{aligned}&A_2\frac{\partial^2 w}{\partial z^2}(z,t) - \frac{\partial w}{\partial t}(z,t) \ge 0\,, \quad z\in[0,\infty)\,, t\in[0,t_1)\\ &w(z,0)=0\,,\quad z\in[0,\infty)\\ &w(0,t) = \exp(b_2 t)E(0,t)\ge 0\,,\quad t\in[0,t_1)\\ &w(z,t)\to 0 \text{ (uniformly in } t) \text{ for } z\to\infty\end{aligned} \tag{83}$$

Using the maximum principle for parabolic partial differential equations, we obtain

$$w(z,t) \le J(z,t)\,,\quad z\in[0,\infty)\,,\ t\in[0,t_1)$$

where $J(z,t)$ is solution of the following problem

$$\begin{aligned}&A_2\frac{\partial^2 J}{\partial z^2}(z,t) - \frac{\partial J}{\partial t}(z,t) = 0\,,\quad z\in[0,\infty)\,,\ t\in[0,\ t_1)\\ &J(z,\ 0)=0\,,\quad z\in[0,\infty)\\ &J(0,\ t) = \exp(b_2 t)E(0,\ t)\ge 0\quad t\in[0,\ t_1)\\ &J(z,\ t)\to 0 \text{ (uniformly in } t) \quad \text{for } z\to\infty\end{aligned} \tag{84}$$

The solution of the above problem is (c.f. Tikhonov and Samarskii [19])

$$\begin{aligned}J(z,t) = &\frac{A_2}{2\sqrt{\pi}}\int_0^t \frac{z}{[A_2(t-\tau)]^{3/2}}\exp\left(-\frac{z^2}{4A_2(t-\tau)}\right)\\ &\times\exp(b_2\tau)E(0,\tau)d\tau\end{aligned}$$

From (80) and the above relation, we deduce that

$$E(z,t) \le \left(\max_{\tau\in[0,t]} E(0,\tau)\right)\exp\left(\frac{A_1 z}{2A_2}\right)G(z,t) \tag{85}$$

where

$$\begin{aligned}G(z,t) = &\frac{1}{2\sqrt{A_2\pi}}\\ &\times\int_0^t z\tau^{-3/2}\exp\left(-\frac{z^2}{4A_2\tau} - \frac{A_1^2}{4A_2}\tau\right)d\tau\end{aligned} \tag{86}$$

Using the estimate discussed by Pompei and Scalia [16], we have

$$\begin{aligned}G(z,t) \le &\frac{2z(A_2 t/\pi)^{1/2}\exp\left(-\frac{A_1^2}{4A_2}t\right)}{z^2 - A_1^2 t^2}\\ &\times\exp\left(-\frac{z^2}{4A_2 t}\right)\end{aligned} \tag{87}$$

for $z > A_1 t$.

Thus, we find the estimate

$$E(z,t) \leq \left(\max_{\tau\in[0,t]} E(0,\tau)\right) \frac{1}{z^2 - A_1^2 t^2} 2z \left(\frac{A_2 t}{\pi}\right)^{\frac{1}{2}} \times \exp\left(-\frac{A_1^2}{4A_2} t\right) \exp\left(\frac{A_1 z}{2A_2} - \frac{z^2}{4A_2 t}\right) \tag{88}$$

for $z > A_1 t$.

The estimate (88) proves that, for a fixed time, at large distance to the support $\hat{D}_{t_1}$ of the given data, the dominant term is $\exp\left(-\frac{z^2}{4A_2 t}\right)$. Moreover, A_2 depends only on the thermal coefficients (see [17, 18]). We can conclude that at large distance from the support of the external given data, the spatial decay of processes is influenced only by the thermal effect.

References

1. Berdichevskii VL (1975) On the proof of the Saint–Venant principle for bodies of arbitrary shape. J Appl Math Mech 38:799–813
2. Boley BA (1955) Application of Saint–Venant's principle in dynamical problems. J Appl Mech 22:204–206
3. Boley BA (1958) Some observations on Saint–Venant's principle. In: Proceedings of the 3rd U.S. National Congress of Applied Mechanics, 259–264
4. Boley BA (1960) Upper bounds and Saint–Venant's principle in transient heat conduction. Quart Appl Math 18:205–207
5. Boley BA (1960) Theory of thermal stresses. Wiley, New York
6. Carlson DE (1972) Linear thermoelasticity. In: Truesdell C (ed) Handbuch der physik VI a/2. Springer, Berlin/Heidelberg/New York, pp 297–345
7. Chiriţă S (1995) Saint–Venant's principle in linear thermoelasticity. J Therm Stresses 18:485–496
8. Chiriţă S, Ciarletta M (1999) Time-weighted surface power function method for the study of spatial behaviour in dynamics of continua. Eur J Mech A Solids 18:915–933
9. Fichera G (1972) Existence theorems in elasticity. In: Truesdell C (ed) Handbuch der physik VI a/2. Springer, Berlin/Heidelberg/New York, pp 391–424
10. Hetnarski RB, Eslami MR (2009) Thermal stresses. Advanced theory and applications. Springer-Verlag, New York
11. Horgan CO, Knowles JK (1982) Recent developments concerning Saint–Venant's principle. In: Wu TY, Hutchinson JW (eds) Advances in applied mechanics 23. Academic, New York, pp 179–269
12. Horgan CO, Payne LE, Wheeler LT (1984) Spatial decay estimates in transient heat conduction. Quar Appl Math 42:119–127
13. Horgan CO (1989) Recent developments concerning Saint-Venant's principle: an update. Appl Mech Rev 42:295–303
14. Horgan CO (1996) recent developments concerning Saint-venant's principle: an second update. Appl Mech Rev 49:S101–S111
15. Knowles JK (1966) On Saint–Venant's principle in the two-dimensional linear theory of elasticity. Arch Rational Mech Anal 21:1–22
16. Pompei A, Scalia A (2002) On the asymptotic spatial behaviour in linear thermoelasticity of materials with voids. J Therm Stresses 25:183–193
17. Quintanilla R (2001) End effects in thermoelasticity. Math Methods Appl Sci 24:93–102
18. Quintanilla R (2001) Damping of end effects in a thermoelastic theory. Appl Math Lett 14:137–141
19. Tikhonov AN, Samarskii AA (1964) Partial differential equations of mathematical physics. Holden–Day, New York
20. Toupin RA (1965) Saint-Venant' principle. Arch Rational Mech Anal 18:83–96

Saint-Venant's Problem

► Saint-Venant's Problem for Cosserat Elastic Shells

Saint-Venant's Problem for Cosserat Elastic Shells

Mircea Bîrsan
Department of Mathematics, University "A.I. Cuza" of Iaşi, Iaşi, Romania
Faculty of Mathematics, University Duisburg–Essen, Essen, Germany

Synonyms

Bending; Cosserat surfaces; Elastic cylindrical shells; Extension; Flexure; Saint-Venant's problem; Torsion