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Thermoelastic Waves

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Synonyms

Isotropic materials; Linear theory; Orthotropic materials; Progressive waves; Rayleigh's waves; Thermoelasticity

Overview

The study of wave propagation in thermoelastic or elastic solids has a long history (see the books by Love [1] and by Achenbach [2]). Our entry is based on the books by Nowacki [3], Eringen [4], and Ieşan [5] and the paper by Chiriţă [6].

In the first part of this entry, we study the propagation of progressive waves in isotropic thermoelastic materials. We study both the uncoupled and coupled case. For uncoupled case, we point out that we have a purely elastic wave which propagates in the both directions of the axis Ox_1 and, in addition, we have a new wave, called the thermal wave, which propagates in the same direction. Since the phase velocity of the thermal wave depends on the frequency, these waves are dispersive. Moreover, the waves are also attenuated. In the coupled case, we have that there exist two types of waves, and we can see how the coupled term affects the waves. These waves are also dispersive and are also attenuated. This problem was studied by Chadwick and Sneddon [7] and by Chadwick [8].

The second purpose of this entry is to present explicit inhomogeneous plane wave solutions of the equations of motion within the framework of the linear theory of thermoelastic isotropic materials. The Rayleigh waves are studied in a thermoelastic half-space, and the equation for the Rayleigh wave speed is presented. This problem was studied by Lockett [9]. We have to outline that, in classical elasticity, the solution of the secular equation for Rayleigh surface waves was established numerically by Rayleigh [10], and it was also studied by Hayes and Rivlin [11] (see also the book [2]).

The third objective is to discuss the effect of temperature and strain on the propagation of Rayleigh waves in an orthotropic thermoelastic half-space by transposing the so-called Stroh [15, 17, 18] formulation. For the general case of anisotropic thermoelastic materials, this problem was studied by Chiriţă [6]. Chiriţă has showed that the Stroh formulation of the problem leads to a first-order linear partial differential system with constant coefficients. The author has also illustrated these results on the case of an orthotropic homogeneous thermoelastic material half-space and of an isotropic homogeneous thermoelastic material half-space. The last paragraph of this present entry is based on the analysis developed by Chiriţă [6].

The present mathematical results can be used in laboratories for studying the specific properties of thermoelastic materials.

Basic Equations of the Linear Thermoelasticity

Throughout this entry, we shall consider an isotropic or an orthotropic homogeneous thermoelastic solid which at the initial time occupies the region Ω of the three-dimensional Euclidean space E^3. The body is referred to a fixed system of rectangular Cartesian axes Ox_i $(i = 1, 2, 3)$. Throughout this entry, Latin indices have the range 1, 2, and 3; Greek indices have the range 1 and 2; and the usual summation convention is employed. Typical conventions for differential operations are implied such as a superposed dot or comma followed by a subscript to denote the partial derivative with respect to time or to the corresponding Cartesian coordinate, respectively.

We use the following notations:

1. $\mathbf{u}$ is the displacement vector.
2. θ describes the changes of the temperature from the reference configuration.
3. ρ_0 is the mass density of the continuum at the initial time.
4. σ_{ij} is the stress tensor.
5. T_0 is the absolute temperature in the reference configuration.
6. e_{ij} is the infinitesimal strain tensor.
7. q_i is the heat flux vector.
8. S is the entropy per unit mass.
9. $\mathrm{Re}[v]$ denotes the real part of the complex number v, while $\mathrm{Im}[v]$ denotes the coefficient of the imaginary part of the complex number v.

In the absence of the body force and heat supply field, the fundamental system of equations of the linear thermoelasticity consists of [12, 13, 16]:

- The equations of motion

$$\sigma_{ji,j} = \rho_0 \ddot{u}_i \tag{1}$$

- The energy equation

$$\rho_0 T_0 \dot{S} + q_{i,i} = 0 \tag{2}$$

- The constitutive equations:
  - for isotropic materials

$$\begin{aligned} \sigma_{ij} &= \lambda e_{rr}\delta_{ij} + \mu e_{ij} - m\theta\delta_{ij} \\ \rho_0 S &= m e_{rr} + a\theta \\ q_i &= -k\theta_{,i} \end{aligned} \tag{3}$$

  - for orthotropic materials

$$\begin{aligned} \sigma_{11} &= c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} - m_{11}\theta \\ \sigma_{22} &= c_{12}e_{11} + c_{22}e_{22} + c_{23}e_{33} - m_{22}\theta \\ \sigma_{33} &= c_{13}e_{11} + c_{23}e_{22} + c_{33}e_{33} - m_{33}\theta \\ \sigma_{12} &= 2c_{66}e_{12}, \quad \sigma_{23} = 2c_{44}e_{23}, \quad \sigma_{31} = 2c_{55}e_{31} \\ \rho_0 S &= m_{11}e_{11} + m_{22}e_{22} + m_{33}e_{33} + a\theta \\ q_1 &= -k_{11}\theta_{,1} \\ q_2 &= -k_{22}\theta_{,2} \\ q_3 &= -k_{33}\theta_{,3} \end{aligned} \tag{4}$$

- The geometrical equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

Because we have considered homogeneous materials, all constitutive coefficients are constants. The scalars λ and μ are called Lamé moduli; μ is the shear modulus; m is the stress-temperature modulus; k is the conductivity, while $c_{11}, c_{22}, \ldots, c_{66}$ represent the standard notation for the components of the orthotropic elasticity tensor; and k_{11}, k_{22}, and k_{33} are the components of the conductivity tensor for orthotropic materials.

Let us consider an isotropic and homogeneous thermoelastic material. If we introduce the relations (3) and (5) in (1) and (2), we obtain the basic equations in terms of the displacement u_i and temperature variation θ, that is,

$$\begin{aligned} &\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} - m\theta_{,i} - \rho_0 \ddot{u}_i = 0 \\ &k\theta_{,ii} - T_0 m \dot{u}_{j,j} - c\dot{\theta} = 0 \end{aligned} \tag{6}$$

where we substituted $T_0 a$ by the specific heat c.

The above system consists of the hyperbolic equations of motion coupled with the parabolic equation of energy.

We suppose that the material is strongly elliptic [14]. For isotropic materials, the entropy inequality implies that k is a positive constant. On the other hand, for orthotropic materials, the Clausius-Duhem inequality assures that $k_{11} \geq 0$, $k_{22} \geq 0$, and $k_{33} \geq 0$.

Progressive Waves

In this section, we study the plane progressive waves in thermoelastic isotropic materials. By a plane harmonic wave in a thermoelastic material, we mean a set of functions (u_j, θ) on $\mathbb{R}^3 \times (-\infty, \infty)$ of the following form:

$$\begin{aligned} u_j &= \mathrm{Re}\{A_j \exp[i(\varkappa m_n x_n - \omega t)]\} \\ \theta &= \mathrm{Re}\{B \exp[i(\varkappa m_n x_n - \omega t)]\} \end{aligned} \tag{7}$$

where $i = \sqrt{-1}$, $\mathbf{A} = (A_1, A_2, A_3)$ denotes the direction of motion; $B \neq 0$ is a complex constant; $\mathbf{m} = (m_1, m_2, m_3)$ is the direction of propagation and $(A_1, A_2, A_3) \neq 0$, $\mathrm{Re}\left[\frac{\omega}{\varkappa}\right]$ is the speed of propagation; $\frac{2\pi}{\mathrm{Re}(\varkappa)}$ is the wavelength; and $\mathrm{Re}\left[\frac{\omega}{2\pi}\right]$ is the frequency. If $\mathrm{Im}\left[\frac{\omega}{\varkappa}\right] \neq 0$, then the waves are damped, whereas if $\frac{\omega}{\varkappa}$ is a real number, the waves are undamped.

As we already mentioned, $\mathbf{m}$ is a unit vector in the direction of propagation. We can assume that the Ox_1 axis is in the direction of propagation (i.e., $m_1 = 1, m_2 = m_3 = 0$), and so we seek a solution of the equations (6) of the form

$$\begin{aligned} u_j &= \mathrm{Re}[u_j^*(x_1, \omega) \exp(-i\omega t)] \\ \theta &= \mathrm{Re}[\theta^*(x_1, \omega) \exp(-i\omega t)] \end{aligned} \tag{8}$$

Thus, (u_1^*, θ^*) has to be solution of the following differential system:

$$\begin{aligned} &\left(\frac{d^2}{dx_1^2} + \sigma^2\right) u_1^* - \frac{m}{\rho_0 c_1^2} \frac{d\theta^*}{dx_1} = 0 \\ &\left(\frac{d^2}{dx_1^2} + q\right) \theta^* + \frac{iT_0 m\omega}{\varkappa} \frac{du_1^*}{dx_1} = 0 \end{aligned} \tag{9}$$

while

$$u_2^* = u_3^* = 0 \tag{10}$$

where

$$\begin{aligned} c_1^2 &= \frac{\lambda + 2\mu}{\rho_0}, \quad c_2^2 = \frac{\mu}{\rho_0} \\ \sigma^2 &= \frac{\omega^2}{c_1^2}, \quad q = \frac{i\omega c}{k} \end{aligned} \tag{11}$$

In the following, we assume that ω is a given positive constant.

We distinguish two cases:

Uncoupled waves. If we take $m = 0$, the solution of the equation (9) has the form

$$\begin{aligned} \hat{u}_1^* &= C_1 \exp[i\lambda_1 x_1] + C_2 \exp[-i\lambda_1 x_1] \\ \hat{\theta}^* &= D_1 \exp[i\lambda_2 x_1] + D_2 \exp[-i\lambda_2 x_1] \end{aligned} \tag{12}$$

where

$$\lambda_1 = \sigma = \frac{\omega}{c_1}, \quad \lambda_2 = \sqrt{q} = (1+i)\sqrt{\frac{\omega c}{2k}} \tag{13}$$

and C_α and D_α are arbitrary constants.

Therefore, the displacement and the temperature are given by

$$\begin{aligned} u_1 &= \mathrm{Re}\left\{C_1 \exp\left[-i\omega\left(t - \frac{x_1}{c_1}\right)\right] + C_2 \exp\left[-i\omega\left(t + \frac{x_1}{c_1}\right)\right]\right\} \\ u_2 &= u_3 = 0 \\ \theta &= \mathrm{Re}\left\{D_1 \exp\left(-\sqrt{\frac{\omega c}{2k}} x_1\right) \times \exp\left[-i\omega\left(t - x_1 \Big/ \sqrt{\frac{2k\omega}{c}}\right)\right] + D_2 \exp\left(\sqrt{\frac{\omega c}{2k}} x_1\right) \times \exp\left[-i\omega\left(t + x_1 \Big/ \sqrt{\frac{2k\omega}{c}}\right)\right]\right\} \end{aligned} \tag{14}$$

From the above expressions, we can remark that there is a purely elastic wave which propagates in both directions of the axis Ox_1 with the speed c_1 and, in addition, there is a new wave, called the thermal wave, which propagates in the same direction with the velocity c_T given by

$$c_T = \sqrt{\frac{2k\omega}{c}} \tag{15}$$

Since the phase velocity of the thermal wave depends on the frequency, these waves are dispersive. They are also attenuated with the attenuating coefficient $\mathrm{Im}[\lambda_2]$.

Coupled waves. In this case, $m \neq 0$ and hence u_1^* and θ^* satisfy the equation

$$\left[\left(\frac{d^2}{dx_1^2}+\sigma^2\right)\left(\frac{d^2}{dx_1^2}+q\right)+\varepsilon q\frac{d^2}{dx_1^2}\right]y=0 \quad (16)$$

where

$$\varepsilon=\frac{T_0m^2}{c\rho_0c_1^2} \quad (17)$$

We seek a solution of this equation of the form $\exp[i\eta x_1]$ where η is solution of the equation

$$\eta^4-\eta^2(\sigma^2+q+\varepsilon q)+q\sigma^2=0 \quad (18)$$

In fact, we have

$$\eta_{1,2}^2=\frac{1}{2}\left\{\sigma^2+q+\varepsilon q\pm\left[(\sigma^2+q+\varepsilon q)^2-4q\sigma^2\right]^{1/2}\right\} \quad (19)$$

We can view η as function of ε, and we remark that

$$\eta_1^2(0)=\sigma^2,\quad \eta_2^2(0)=q \quad (20)$$

Further, we choose η_α in such way to satisfy

$$\eta_1(0)=\frac{\omega}{c_1},\quad \eta_2(0)=(1+i)\sqrt{\frac{\omega c}{2k}} \quad (21)$$

Since u_1^* and θ^*, solutions of differential equation (16), has to satisfy (9), it follows from (8) that

$$\begin{aligned}u_1^*=Re\Bigg\{&A_1\exp(-\vartheta_1x_1)\exp\left[-i\omega\left(t-\frac{x_1}{v_1}\right)\right]\\&+A_2\exp(\vartheta_1x_1)\exp\left[-i\omega\left(t+\frac{x_1}{v_1}\right)\right]\\&+\frac{i\eta_2m}{\rho0c_1^2(\sigma^2-\eta_2^2)}\left[B_1\exp(-\vartheta_2x_1)\exp\left[-i\omega\left(t-\frac{x_1}{v_2}\right)\right]\right.\\&\left.-B_2\exp(\vartheta_1x_1)\exp\left[-i\omega\left(t+\frac{x_1}{v_2}\right)\right]\right]\Bigg\}\\\theta^*=Re\Bigg\{&B_1\exp(-\vartheta_2x_1)\exp\left[-i\omega\left(t-\frac{x_1}{v_2}\right)\right]\\&+B_2\exp(\vartheta_2x_1)\exp\left[-i\omega\left(t+\frac{x_1}{v_2}\right)\right]\\&+\frac{T_0m\omega\eta_1}{k(q-\eta_1^2)}\left[A_1\exp(-\vartheta_1x_1)\exp\left[-i\omega\left(t-\frac{x_1}{v_1}\right)\right]\right.\\&\left.-A_2\exp(\vartheta_1x_1)\exp\left[-i\omega\left(t+\frac{x_1}{v_1}\right)\right]\right]\Bigg\}\end{aligned} \quad (22)$$

where A_α and B_α are arbitrary constants and v_α and ϑ_α are real numbers so that

$$\eta_\alpha=\frac{\omega}{v_\alpha}+i\vartheta_\alpha \quad (23)$$

From the above expressions, we can see that there exist two types of waves propagating with the velocities v_1 and v_2. These waves are dispersive since v_1 and v_2 are depending on ω and they are also attenuated according to the values of q_1 and q_2. The terms

$$\begin{aligned}&A_1\exp(-\vartheta_1x_1)\exp\left[-i\omega\left(t-\frac{x_1}{v_1}\right)\right]\\&A_2\exp(\vartheta_1x_1)\exp\left[-i\omega\left(t+\frac{x_1}{v_1}\right)\right]\end{aligned} \quad (24)$$

are called quasi-elastic terms, while the terms

$$\begin{aligned}&B_1\exp(-\vartheta_2x_1)\exp\left[-i\omega\left(t-\frac{x_1}{v_2}\right)\right]\\&B_2\exp(\vartheta_2x_1)\exp\left[-i\omega\left(t+\frac{x_1}{v_2}\right)\right]\end{aligned} \quad (25)$$

are called quasi-thermal terms.

It is known that the experimental studies prove that the coupling parameter ε is small for a large class of materials. For such small values of ε, the effect of thermal coupling can be studied by expanding v_1, v_2, ϑ_1 and ϑ_2 into power series of ε. For these results, we refer the reader to the article by Chadwick [8].

Rayleigh's Waves in an Isotropic Homogeneous Half-Space

In this section, we consider that the material occupying the half-space $x_2\geq0$ is homogeneous and isotropic. We assume that the body force and the heat supply are absent, and also we assume that the boundary $x_2=0$ is traction free and the half-space changes temperature with the atmosphere $x_2<0$ (the case when the half-space does not change temperature with the atmosphere will be discussed at the end of the work as a special case of orthotropic materials). That means we assume

$$\sigma_{i2} = 0, \quad \frac{\partial \theta}{\partial x_2} + h\theta = 0 \quad \text{for} \quad x_2 = 0 \tag{26}$$

where h is a positive constant.

Moreover, we require that the solutions be attenuated in the direction x_2 so that they are decaying with distance from the plane surface $x_2 = 0$, that is, we require that

$$\lim_{x_2 \to \infty} (u_1, u_2, u_2, \theta) = (0, 0, 0, 0) \tag{27}$$

Without loss in generality, we will study the waves propagating along the x_1 -axis.

We assume that the components of the displacement are of the form (see, e.g., [9])

$$u_1 = \varphi_{,1} - \psi_{,2}, \quad u_2 = \varphi_{,2} + \psi_{,1}, \quad u_3 = 0 \tag{28}$$

where φ, ψ are depending on x_1, x_2, and t only. If φ, ψ, and θ satisfy the following equations

$$c_1^2 \Delta\varphi - \ddot{\varphi} - \frac{m}{\varrho_0}\theta = 0 \tag{29}$$

$$c_2^2 \Delta\psi - \ddot{\psi} = 0 \tag{30}$$

$$k\Delta\theta - T_0 m \Delta\dot{\varphi} - c\dot{\theta} = 0 \tag{31}$$

with

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

then the (6) are identically satisfied.

In order to satisfy the (29)–(31), we try to find the functions φ, ψ, and θ in the form

$$[\varphi, \psi, \theta] = \mathrm{Re}\{[\Phi(x_2), \Psi(x_2), \Lambda(x_2)] \exp[i(\varkappa x_1 - \omega t)]\} \tag{32}$$

In view of the asymptotic condition (4), we find that

$$\begin{aligned}
\Phi &= A \exp\left(-x_2\sqrt{\varkappa^2 - z_1^2}\right) + B \exp\left(-x_2\sqrt{\varkappa^2 - z_2^2}\right) \\
\Lambda &= \frac{\rho_0 c_1^2}{m}\left[A\left(\frac{\omega^2}{c_1^2} - z_1^2\right)\exp\left(-x_2\sqrt{\varkappa^2 - z_1^2}\right)\right. \\
&\quad \left. + B\left(\frac{\omega^2}{c_1^2} - z_2^2\right)\exp\left(-x_2\sqrt{\varkappa^2 - z_2^2}\right)\right] \\
\Psi &= C \exp\left(-x_2\sqrt{\varkappa^2 - z_3^2}\right)
\end{aligned} \tag{33}$$

where A, B, C are arbitrary constants and z_1^2 and z_2^2 are roots of the equation

$$z^4 - \left[\frac{\omega^2}{c_1^2} + \frac{i\omega c}{k}(1+\varepsilon)\right] z^2 + \frac{i\omega^3 c}{k c_1^2} = 0 \tag{34}$$

and

$$z_3^2 = \frac{\omega^2}{c_2^2} \tag{35}$$

The explicit expressions of z_1^2 and z_2^2 are given by the right terms of the relations (19).

With the aid of relations (3), we find that the boundary conditions (26) are satisfied if and only if the unknown constants A, B and C are solutions of the following algebraic system:

$$\begin{aligned}
(A+B)\left(2 - \tfrac{\omega^2}{\varkappa^2 c_2^2}\right) - 2ib_3 C &= 0 \\
2i(b_1 A + b_2 B) + \left(2 - \tfrac{\omega^2}{\varkappa^2 c_2^2}\right) C &= 0 \\
A\left(\tfrac{h}{\varkappa} - b_1\right)\left(b_1^2 - 1 + \tfrac{\omega^2}{\varkappa^2 c_1^2}\right) & \\
+ B\left(\tfrac{h}{\varkappa} - b_2\right)\left(b_2^2 - 1 + \tfrac{\omega^2}{\varkappa^2 c_1^2}\right) &= 0
\end{aligned} \tag{36}$$

where we have used the notation

$$b_i^2 = 1 - \frac{z_i^2}{\varkappa^2} \tag{37}$$

The above system has a nontrivial solution if and only if the following equation is satisfied:

$$\begin{aligned}
&\left(2 - \tfrac{v^2}{c_2^2}\right)^2 \left(b_1^2 + b_1 b_2 + b_2^2 - 1 + \tfrac{v^2}{c_1^2}\right) - 4 b_1 b_2 b_3 (b_1 + b_2) \\
&\quad = \tfrac{hv}{c}\left[(b_1 + b_2)\left(2 - \tfrac{v^2}{c_2^2}\right) - 4 b_3 \left(b_1 b_2 + 1 - \tfrac{v^2}{c_1^2}\right)\right]
\end{aligned} \tag{38}$$

where

$$v = \frac{\omega}{\varkappa} \tag{39}$$

Moreover, using the Vièta's formulas, the (38) becomes an equation in v. The quantity $1/\mathrm{Re}[v^{-1}]$ is the speed, while $\omega\mathrm{Im}[v^{-1}]$ is the attenuation coefficient. We have a dispersive wave because these two quantities depend on the frequency ω.

Rayleigh Surface Waves in an Orthotropic Thermoelastic Half-Space

In this section, we consider the problem of Rayleigh surface waves in the class of rhombic thermoelastic materials.

We consider steady wave solutions propagating with wave speed v in the x_1- direction in the orthotropic homogeneous thermoelastic half-space $x_2 \geq 0$. Therefore, we consider solutions of (1), (2), and (4) in the form

$$\{u_1, u_2, u_3, \theta\}(x_1, x_2, x_3, t) = \{u_1, u_2, u_3, \theta\}(x_1 - vt, x_2) \tag{40}$$

and hence we can write

$$\sigma_{r1} = Q_{rs}u_{s,1} + R_{rs}u_{s,2} - m_{r1}\theta \tag{41}$$

$$\sigma_{r2} = R_{sr}u_{s,1} + T_{rs}u_{s,2} - m_{r2}\theta \tag{42}$$

where the matrices **Q**, **R**, and **T** are given by

$$\mathbf{Q} = \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{66} & 0 \\ 0 & 0 & c_{55} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 & c_{12} & 0 \\ c_{66} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\mathbf{T} = \begin{pmatrix} c_{66} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{44} \end{pmatrix} \tag{43}$$

and $m_{12} = m_{21} = 0$, $m_{23} = m_{32} = 0$, and $m_{31} = m_{13} = 0$.

On the other hand, in view of relation (40), the differential system (1), (2), and (4) can be written as

$$\left(\sigma_{1r} - \rho_0 v^2 u_{r,1}\right)_{,1} + \sigma_{2r,2} = 0 \tag{44}$$

$$\left[q_1 - cv\theta - vT_0\left(m_{r1}u_{r,1} + m_{r2}u_{r,2}\right)\right]_{,1} + q_{2,2} = 0 \tag{45}$$

and thus it follows that there exists a stress function vector $\phi = (\phi_1, \phi_2, \phi_3)^T$, such that

$$\sigma_{1r} - \rho_0 v^2 u_{r,1} = -\phi_{r,2} \tag{46}$$

$$\sigma_{2r} = \phi_{r,1} \tag{47}$$

Moreover, there exists a heat function ψ such that

$$q_1 - cv\theta - vT_0\left(m_{11}u_{1,1} + m_{22}u_{2,2}\right) = -\psi_{,2}$$

$$q_2 = \psi_{,1} \tag{48}$$

Then, we are led to the following first-order partial differential system with constant coefficients

$$\mathcal{U}_{,2} = \mathcal{A}\mathcal{U}_{,1} + \mathcal{B}\mathcal{U} \tag{49}$$

where

$$\mathcal{A} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{N}_1 \\ \mathcal{P}_1 & \mathcal{Q}_1 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \mathcal{M}_2 & \mathcal{N}_2 \\ \mathcal{P}_2 & \mathcal{Q}_2 \end{pmatrix} \tag{50}$$

$$\mathcal{M}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\frac{c_{12}}{c_{22}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{N}_1 = \begin{pmatrix} \frac{1}{c_{66}} & 0 & 0 & 0 \\ 0 & \frac{1}{c_{22}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_{44}} & 0 \\ 0 & 0 & 0 & -\frac{1}{k_{22}} \end{pmatrix}$$

$$\mathcal{P}_1 = \begin{pmatrix} \rho_0 v^2 - c_{11} + \frac{c_{12}^2}{c_{22}} & 0 & 0 & 0 \\ 0 & \rho_0 v^2 & 0 & 0 \\ 0 & 0 & -(c_{55} - \rho_0 v^2) & 0 \\ T_0 v\left(-\frac{c_{12}}{c_{22}} m_{22} + m_{11}\right) & 0 & 0 & k_{11} \end{pmatrix}$$

$$\mathcal{Q}_1 = \begin{pmatrix} 0 & -\frac{c_{12}}{c_{22}} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & T_0 v \frac{m_{22}}{c_{22}} & 0 & 0 \end{pmatrix} \tag{51}$$

$$\mathcal{M}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{m_{22}}{c_{22}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{N}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\mathcal{P}_2 = \begin{pmatrix} 0 & 0 & 0 & -\frac{c_{12}}{c_{22}} m_{22} + m_{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & cv\left(1 + \frac{T_0 m_{22}^2}{c c_{22}}\right) \end{pmatrix},$$
$$\mathcal{Q}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{52}$$

Considering now the propagation of a wave with speed v and wave number $\varkappa$ in the x_1- direction in the orthotropic homogeneous thermoelastic half-space $x_2 \geq 0$, that is, by considering solutions of (49) in the form

$$\mathcal{U}(x_1, x_2) = \mathcal{V}(x_2) \exp[i\varkappa(x_1 - vt)]$$
$$\mathcal{V}(x_2) = \{v_1, v_2, v_3, \chi, \Phi_1, \Phi_2, \Phi_3, \Psi\}^T (x_2) \tag{53}$$

then the first-order partial differential system (49) reduces to the following first-order differential system:

$$\mathcal{V}'(x_2) = i\varkappa\Lambda\mathcal{V}(x_2) \tag{54}$$

where

$$\Lambda = \mathcal{A} - \frac{i}{\varkappa}\mathcal{B} \tag{55}$$

Finally, we seek solutions of the differential system (44) in exponential evanescent form

$$\mathcal{V}(x_2) = \mathcal{W} \exp[i\varkappa p x_2] \tag{56}$$

where $\mathcal{W} = \{V_1, V_2, V_3, \vartheta, \varphi_1, \varphi_2, \varphi_3, \varsigma\}^T$ is a constant vector and p is a complex scalar, and we choose solutions for which

$$\lim_{x_2 \to \infty} \{u_1, u_2, u_3, \theta\}(x_1, x_2, t) = 0 \tag{57}$$

$$\lim_{x_2 \to \infty} \{\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \sigma_{23}, q_1, q_2\}(x_1, x_2, t) = 0 \tag{58}$$

for all $x_1 \in \mathbb{R}$ and $t \geq 0$, and

$$\lim_{x_2 \to \infty} \mathcal{V}(x_2) = 0 \tag{59}$$

Concluding, we see that $\mathcal{W}$ and p have to satisfy the eigenvalue problem

$$\Lambda\mathcal{W} = p\mathcal{W} \tag{60}$$

and the characteristic equation is

$$\left(p^2 + C_3\right)\left(p^6 + P_1 p^4 + P_2 p^2 + P_3\right) = 0 \tag{61}$$

where

$$P_1 = \frac{k_{11}}{k_{22}} + \frac{c_{11}}{c_{66}} C_1 + \frac{c_{66}}{c_{22}} C_2 - \frac{(c_{12} + c_{66})^2}{c_{22} c_{66}} - \frac{icv}{\varkappa k_{22}}\left(1 + \frac{T_0 m_{22}^2}{c_{22} c}\right) \tag{62}$$

$$P_2 = \frac{k_{11}}{k_{22}} + \left[\frac{c_{11}}{c_{66}} C_1 + \frac{c_{66}}{c_{22}} C_2 - \frac{(c_{12} + c_{66})^2}{c_{22} c_{66}}\right] + \frac{c_{11}}{c_{22}} C_1 C_2 - \frac{icv}{\varkappa k_{22}}\left\{\frac{c_{11}}{c_{66}} C_1 + \frac{c_{66}}{c_{22}} C_2 - \frac{(c_{12} + c_{66})^2}{c_{22} c_{66}} + \frac{T_0}{c}\left[\frac{c_{11} C_1}{c_{66}} \frac{m_{22}^2}{c_{22}} - \frac{2 m_{11} m_{22}}{c_{22} c_{66}}(c_{12} + c_{66}) + \frac{m_{11}^2}{c_{66}}\right]\right\} \tag{63}$$

$$P_3 = \frac{k_{11}}{k_{22}} \frac{c_{11}}{c_{22}} C_1 C_2 - \frac{icv}{\varkappa k_{22}}\left(\frac{c_{11}}{c_{22}} C_1 C_2 + \frac{T_0 m_{11}^2}{c_{22} c} C_2\right) \tag{64}$$

and

$$C_1 = 1 - \frac{\rho_0 v^2}{c_{11}}, \quad C_2 = 1 - \frac{\rho_0 v^2}{c_{66}}, \quad C_3 = \frac{c_{55} - \rho_0 v^2}{c_{44}} \tag{65}$$

Thus, the propagation condition (61) is a quartic in p^2 with complex coefficients, and it has solution

$$p_4^2 = -C_3 \tag{66}$$

while the other three solutions p_1^2, p_2^2, and p_3^2 satisfy the following equation with complex coefficients

$$p^6 + P_1 p^4 + P_2 p^2 + P_3 = 0 \tag{67}$$

In [6], it is proved that all the roots p_r^2, $r = 1, 2, 3$, of the propagation condition (67) are non-real complex numbers, and moreover, we have the following form of the eigenvalues:

$$p_1 = B_1 + iA_1, \quad p_2 = B_2 + iA_2, \quad p_3 = B_3 + iA_3 \tag{68}$$

where A_1, A_2, and A_3 are strictly positive real numbers and B_1, B_2, and B_3 are real numbers.

The admissible eigenvalue p_4 has to satisfy relations (71) and (66), and hence the wave speed v is so that

$$v^2 < c_4 \tag{69}$$

where

$$c_4 = \frac{c_{55}}{\rho_0} \tag{70}$$

In consequence, the eigenvalues have positive imaginary part, that is,

$$\mathrm{Im}(p_n) > 0 \tag{71}$$

so that the asymptotic conditions (57)–(59) are satisfied. Let $\mathcal{W}^{(n)} = \left\{V_1^{(n)},\ V_2^{(n)}, V_3^{(n)}, \vartheta^{(n)},\ \varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)},\ \varsigma^{(n)}\right\}^T$ be the eigenvector corresponding to the eigenvalue $p_n, n = 1, 2, 3, 4$.

The associated eigenvector corresponding to the eigenvalue p_4 is proportional to $W^{(4)} = \{0, 0, 1, 0,\ 0, 0, c_{44}p_4, 0\}^T$.

For a surface wave propagating in the direction of the x_1 -axis in the half-space $x_2 \geq 0$, the surface traction and the heat flux at $x_2 = 0$ must vanish, that is,

$$\sigma_{2r} = 0, \quad q_2 = 0 \quad \text{at} \quad x_2 = 0 \tag{72}$$

We seek a solution $\mathcal{U}(x_1, x_2, t)$ of the problem by superposing the four eigensolutions $\mathcal{U}^{(n)}(x_1, x_2, t) = \mathcal{W}^{(n)} \exp[i\varkappa(x_1 - vt + p_n x_2)]$, $n = 1, 2, 3, 4$, that is, we set

$$\mathcal{U}(x_1, x_2, t) = \sum_{n=1}^{4} \gamma_n \mathcal{W}^{(n)} \exp[i\varkappa(x_1 - vt + p_n x_2)] \tag{73}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)^T$ is a nonzero constant vector to be determined in order the boundary conditions (72) to be satisfied. Then the boundary conditions (72) imply that

$$\begin{aligned}
\gamma_1\varphi_1^{(1)} + \gamma_2\varphi_1^{(2)} + \gamma_3\varphi_1^{(3)} + \gamma_4\varphi_1^{(4)} &= 0\\
\gamma_1\varphi_2^{(1)} + \gamma_2\varphi_2^{(2)} + \gamma_3\varphi_2^{(3)} + \gamma_4\varphi_2^{(4)} &= 0\\
\gamma_1\varphi_3^{(1)} + \gamma_2\varphi_3^{(2)} + \gamma_3\varphi_3^{(3)} + \gamma_4\varphi_3^{(4)} &= 0\\
\gamma_1\varsigma^{(1)} + \gamma_2\varsigma^{(2)} + \gamma_3\varsigma^{(3)} + \gamma_4\varsigma^{(4)} &= 0
\end{aligned} \tag{74}$$

and hence, for a nontrivial solution for γ, we obtain

$$\Delta(v) \equiv \begin{vmatrix} \varphi_1^{(1)} & \varphi_1^{(2)} & \varphi_1^{(3)} & \varphi_1^{(4)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \varphi_2^{(3)} & \varphi_2^{(4)} \\ \varphi_3^{(1)} & \varphi_3^{(2)} & \varphi_3^{(3)} & \varphi_3^{(4)} \\ \varsigma^{(1)} & \varsigma^{(2)} & \varsigma^{(3)} & \varsigma^{(4)} \end{vmatrix} = 0 \tag{75}$$

which represents the secular equation for the wave speed v.

To derive the secular equation (75) in an explicit form, we need to know that the corresponding eigenvectors $\mathcal{W}^{(r)} = \left\{V_1^{(r)}, V_2^{(r)},\right.$

$V_3^{(r)}, \vartheta^{(r)}, \varphi_1^{(r)}, \varphi_2^{(r)}, \varphi_3^{(r)}, \varsigma^{(r)}\}^T, \quad r = 1, 2, 3,$ are given by

$$\begin{aligned}
V_1^{(r)} &= \frac{i}{\varkappa}\{[-c_{22}m_{11} + (c_{12} + c_{66})m_{22}]p_r^2 - c_{66}m_{11}C_2\} \\
V_2^{(r)} &= \frac{i}{\varkappa}[-c_{66}m_{22}p_r^2 - c_{11}m_{22}C_1 + m_{11}(c_{12} + c_{66})]p_r \\
V_3^{(r)} &= 0 \\
\vartheta^{(r)} &= c_{22}c_{66}p_r^4 + \left[c_{11}c_{22}C_1 + c_{66}^2C_2 - (c_{12} + c_{66})^2\right]p_r^2 \\
&\quad + c_{11}c_{66}C_1C_2 \\
\varphi_1^{(r)} &= \frac{ic_{66}}{\varkappa}\{(-c_{22}m_{11} + c_{12}m_{22})p_r^3 \\
&\quad + [-m_{22}(c_{11} - \rho_0 v^2) + m_{11}(c_{12} + \rho_0 v^2)]p_r\} \\
\varphi_2^{(r)} &= \frac{ic_{66}}{\varkappa}\left\{[c_{22}m_{11} - (c_{12} + \rho_0 v^2)m_{22}]p_r^2 \right. \\
&\quad \left. + C_2(-c_{12}m_{11} + c_{11}C_1m_{22})\right\} \\
\varphi_3^{(r)} &= 0 \\
\varsigma^{(r)} &= -k_{22}\left\{c_{22}c_{66}p_r^5 \right. \\
&\quad + \left[c_{11}c_{22}C_1 + c_{66}^2C_2 - (c_{12} + c_{66})^2\right]p_r^3 \\
&\quad \left. + c_{11}c_{66}C_1C_2p_r\right\}
\end{aligned} \tag{76}$$

for each $r = 1, 2, 3$.

A substitution of these values in (76) and (75) leads to the secular equation for an orthotropic thermoelastic half-space. For a specific thermoelastic material, we can determine the three roots p_r, $r = 1, 2, 3$ (see [6]).

If we now consider the half-space $x_2 \geq 0$ to be occupied by an isotropic homogeneous thermoelastic material, we can ignore the anti-plane deformation furnished by $\mathcal{W}^{(4)} = \{0, 0, 1, 0, 0, 0, c_{44}p_4, 0\}^T$. Thus, in what follows we will consider on the earth's surface the following boundary conditions:

$$\begin{aligned}
\sigma_{12}(x_1, 0, x_3, t) &= 0 \\
\sigma_{22}(x_1, 0, x_3, t) &= 0 \\
q_2(x_1, 0, x_3, t) &= 0
\end{aligned} \tag{77}$$

for all $x_1, x_3 \in \mathbb{R},\ t \geq 0$.

For an isotropic homogeneous thermoelastic material, we can see that

$$\begin{aligned}
&c_{11} = c_{22} = c_{33} = \lambda + 2\mu, \quad c_{66} = c_{44} = c_{55} = \mu, \\
&c_{12} = c_{23} = c_{31} = \lambda \\
&k_{11} = k_{22} = k_{33} = k, \quad m_{11} = m_{22} = m_{33} = m
\end{aligned} \tag{78}$$

Then, the propagation condition (67) becomes

$$\begin{aligned}
(p^2 + C_2)\{p^4 + \left[1 + C_1 - \frac{icv}{\varkappa k}(1 + \varepsilon)\right]p^2 \\
+ C_1 - \frac{icv}{\varkappa k}(C_1 + \varepsilon)\right\} = 0
\end{aligned} \tag{79}$$

where

$$\varepsilon = \frac{T_0 m^2}{c c_{11}} \tag{80}$$

and the dispersion relation is given by

$$\begin{aligned}
(1 + C_2)^2(p_1^2 + p_2^2 + p_1p_2 + C_1) \\
+ 4p_1p_2p_3(p_1 + p_2) = 0
\end{aligned} \tag{81}$$

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Thermoelastic Waves at an Interface Between Two Solids

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Synonyms

Thermomechanical waves

Overview

Thermoelasticity includes the coupled interaction between the deformation and the temperature field. Thermoelasticity based on irreversible thermodynamics was developed by Biot [1–3]. It was assumed that the elastic solid is initially in a stress-free equilibrium state at uniform temperature. The strains, the rotations, the displacements, and the temperature variations are taken small of first order to analyze the linearized perturbations in the vicinity of the equilibrium state. Thermoelasticity has been an intensive topic of discussions in the literature for over a century among others like Duhamel [4], Neumann [5], and Voigt [6]. Finite thermal conductivity effect on the propagation of elastic waves in the presence of boundaries is studied by Deresiewicz [7], Lockett [8], and Chadwick [9]. The classical theory of thermoelasticity is based on Fourier's law of heat conduction, which with other laws of mechanics and thermodynamics gives rise to the displacement-temperature field equations of hyperbolic-parabolic type that imply an infinite speed of propagation of thermoelastic waves. Lord and Shulman [10] and Green and Lindsay [11] extended the classical dynamical coupled theory of thermoelasticity to generalized thermoelasticity theories, which treat heat propagation as a wave phenomenon rather than a diffusion phenomenon and predict a finite speed of heat propagation. Ignaczak and Ostoja-Starzewski [12] explained in detail the above theories in their recent book. The generalized theory of thermoelasticity without energy dissipation was formulated by Green and Naghdi [13], which includes the isothermal displacement gradients among its independent constitutive variables and differs from the previous theories in that it does not accommodate dissipation of thermal energy. Chandrasekharaiah [14] proposed a dual-phase-lag thermoelasticity. Hetnarski and Ignaczak [15] reviewed the representative theories in the range of generalized thermoelasticity. The reflection and refraction of generalized thermoelastic waves at an interface was initially studied by Sinha and Elsibai [16, 17] and Singh [18].

Governing Equations of Linear Thermoelasticity

The governing equations for an isotropic, homogeneous, and linear thermoelastic medium are
(a) The constitutive equations

$$\sigma_{ij} = \left[\lambda\, e - \gamma\left(1 + v_0 \frac{\partial}{\partial t}\right)\Theta\right]\delta_{ij} + 2\mu\, e_{ij} \quad (1)$$