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Shell Models in Classical and Cosserat Nonlinear Elasticity

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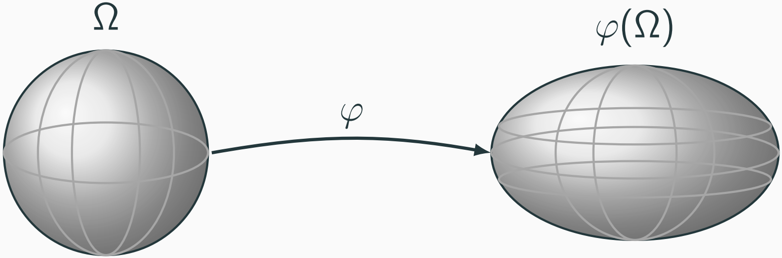
Classical theory of nonlinear 3D elasticity

Setting of nonlinear elasticity

We consider the **deformation of an elastic body**:

- $\Omega \subset \mathbb{R}^3$, Ω bounded domain, the reference configuration,
- $\varphi : \Omega \to \mathbb{R}^3$ the deformation mapping,
- $\varphi(x)$ the new position of the material point $x \in \Omega$,
- $\varphi(x) = x + u(x)$, u displacement, ∇u displacement gradient,
- $F = \nabla\varphi \in \mathrm{GL}^+(3)$ the deformation gradient.



Our focus is on three-dimensional bodies with a shell-like shape

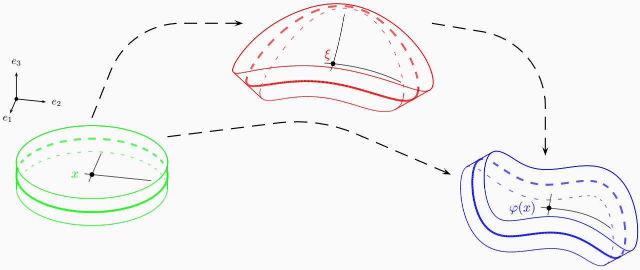


Figure 1: Kinematics of the 3D-Cosserat model.

The aim is to reduce the 3D problem to a 2D problem defined on the midsurface

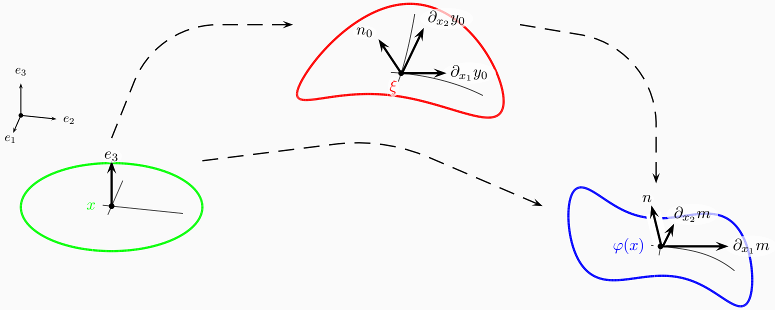


Figure 2: Kinematics of the 2D Cosserat shell model.

Setting of nonlinear 3D elasticity

Given a function $W : \mathrm{GL}^+(3) \to [0, \infty)$ (energy density), the corresponding energy I corresponding to a deformation φ is

$$I(\varphi) = \int_\Omega W(D\varphi(x))\, dx.$$

How does an energy look like?

A suitable energy function has to respond to four main requirements:

- to have a good fitting to the experimental results and to be clear what its relation with the reality,
- to have a geometrical meaning, i.e., to be clear how it measures the deformations inside the body,
- to have a form as simple as possible in order to be used in practice and
- to satisfy some minimal requirements such that the existence of the solution is assured.

Setting of nonlinear elasticity

$$I(\varphi) = \int_{\Omega} W(D\varphi(x))\, dx$$

How does an energy look like?

Since the energy has to illustrate the deformations effects, it has to be function of a measure of deformation.

How do we measure the deformation?

The usual tensor used in classical nonlinear elasticity to measure the deformation is the right stretch tensor

$$U = \sqrt{F^T F}$$

coming from the polar decomposition of

$$F = R\,U = \mathrm{polar}(F)\,\sqrt{F^T F}.$$

There are two direct choices which generalise the classical linear elasticity

- the geometrically nonlinear isotropic Biot-model

$$W_{\mathrm{Biot}}(F) = \mu \, \|U - \mathbb{1}_3\|^2 + \frac{\lambda}{2} \, [\mathrm{tr}(U - \mathbb{1}_3)]^2,$$

- the geometrically nonlinear isotropic Saint-Venant–Kirchhoff energy

$$W_{\mathrm{SVK}}(F) = \frac{\mu}{4} \, \|U^2 - \mathbb{1}_3\|^2 + \frac{\lambda}{8} [\mathrm{tr}(U^2 - \mathbb{1}_3)]^2,$$

Both energies are not rank-one convex (elliptic), which means that we do not have existence results for the nonlinear equilibrium problem.

One could argue that, even if the 3D problem is not well posed, its dimensional reduction may still lead to a well-posed 2D model

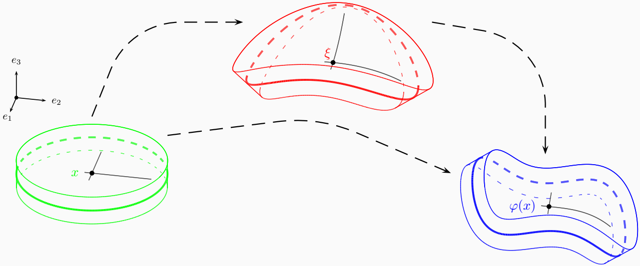


Figure 3: Kinematics of the 3D model.

This is not entirely wrong, because the reduced model may indeed be well posed even when the original 3D problem is not

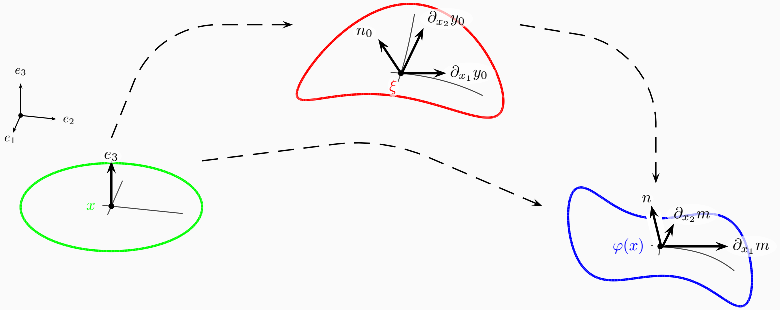


Figure 4: Kinematics of the 2D shell model.

- The chances of obtaining a well-posed shell model increase if we start from a well-posed 3D model.

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- There is no body without thickness. It could be thin but not 2D.

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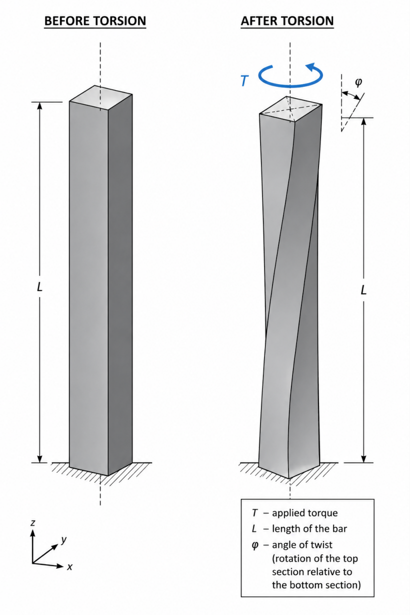
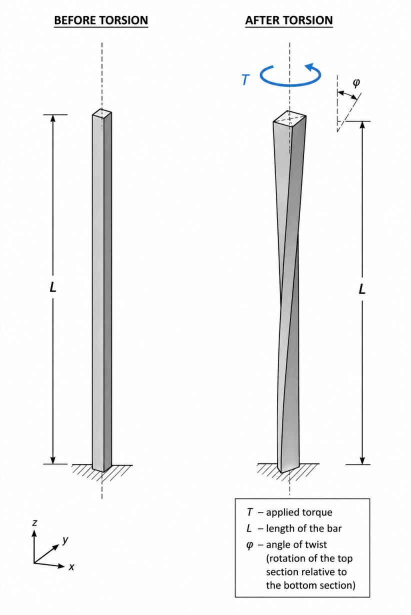
- The reduced model should preserve as many 3D effects as possible.

- The chances of obtaining a well-posed shell model increase if we start from a well-posed 3D model.
- There is no body without thickness. It could be thin but not 2D.
- The reduced model should preserve as many 3D effects as possible.
- Ideally, the shell model should remain very close to the original 3D model, in a suitable sense.


BEFORE TORSION
AFTER TORSION
T
φ
L
L
z
y
x
T – applied torque
L – length of the bar
φ – angle of twist (rotation of the top section relative to the bottom section)


BEFORE TORSION
AFTER TORSION
L
T
φ
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z
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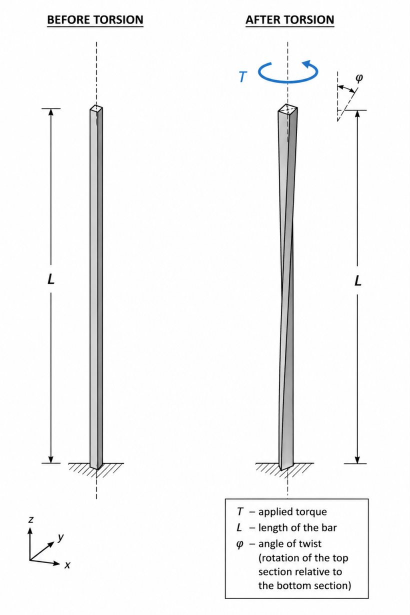
T – applied torque
L – length of the bar
φ – angle of twist
(rotation of the top
section relative to
the bottom section)


BEFORE TORSION
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T – applied torque
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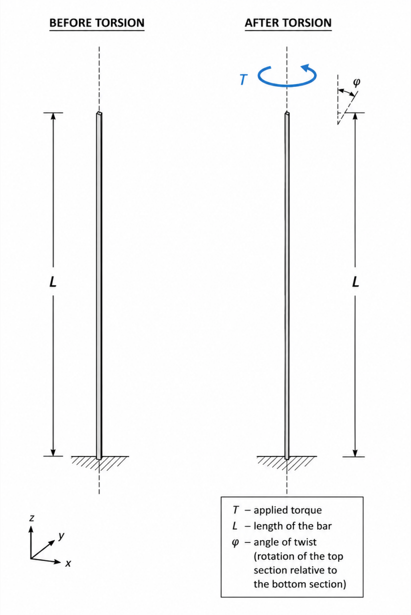
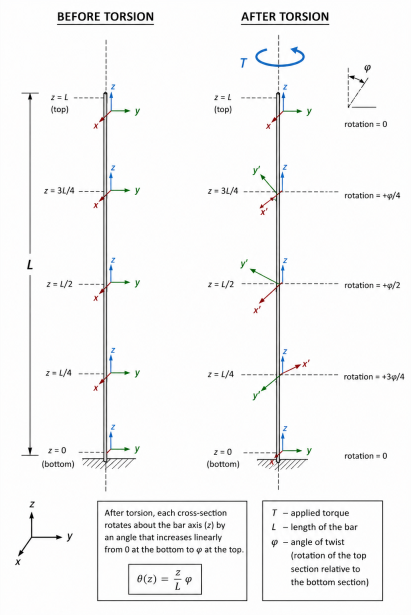
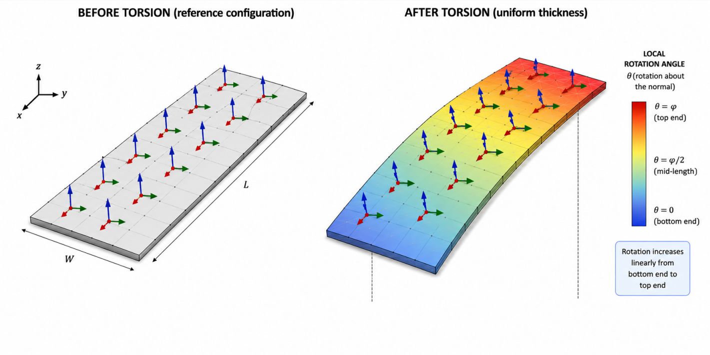

BEFORE TORSION
AFTER TORSION
T
φ
L
L
z
y
x
T – applied torque
L – length of the bar
φ – angle of twist (rotation of the top section relative to the bottom section)


BEFORE TORSION
AFTER TORSION
T
φ
z = L
(top)
z = 3L/4
z = L/2
z = L/4
z = 0
(bottom)
L
x
y
z
x'
y'
rotation = 0
rotation = +φ/4
rotation = +φ/2
rotation = +3φ/4
rotation = 0
After torsion, each cross-section rotates about the bar axis (z) by an angle that increases linearly from 0 at the bottom to φ at the top.
θ(z) = (z/L) φ
T – applied torque
L – length of the bar
φ – angle of twist (rotation of the top section relative to the bottom section)

If we attach directors to each point of the midsurface, then we may retain more three-dimensional effects



To choose, or not to choose, a 3D energy?
That is the question!

- The 3D problem is well-posed.

- The 3D energy should allow a dimensional reduction that retains as many three-dimensional effects as possible.

One solution

- Use local directors (so the Cosserat theory).
- Use a 3D energy in the Cosserat theory for which the 3D model is well-posed.

The parental 3D Cosserat model

The deformation of the body occupying the domain Ω_ξ is described by a map φ_ξ (*called deformation*) and by a *microrotation* $\overline{R}_\xi$,

$$\varphi_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \mathbb{R}^3, \qquad \overline{R}_\xi : \Omega_\xi \subset \mathbb{R}^3 \to \mathrm{SO}(3)\,.$$

The deformation and the microrotation is solution of the following *geometrically nonlinear minimization problem* posed on Ω_ξ:

$$I(\varphi_\xi, F_\xi, \overline{R}_\xi, \alpha_\xi) = \int_{\Omega_\xi} \left[W_{\mathrm{mp}}(\overline{U}_\xi) + W_{\mathrm{curv}}(\alpha_\xi) \right] dV(\xi),$$

where

$$\begin{aligned}
F_\xi &:= D_\xi \varphi_\xi \in \mathbb{R}^{3\times 3} && \text{(the deformation gradient)},\\
\overline{U}_\xi &:= \overline{R}_\xi^T F_\xi \in \mathbb{R}^{3\times 3} && \text{(the non-symmetric Biot-type stretch tensor)},\\
\alpha_\xi &:= \overline{R}_\xi^T \, \mathrm{Curl}_\xi \, \overline{R}_\xi \in \mathbb{R}^{3\times 3} && \text{(the second order dislocation density tensor)}
\end{aligned}$$

A nonlinear parental 3D Cosserat model

The quadratic energies are

$$W_{\mathrm{mp}}(\overline{U}_\xi) := \mu\,\|\operatorname{dev}\operatorname{sym}(\overline{U}_\xi - \mathbb{1}_3)\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}(\operatorname{sym}(\overline{U}_\xi - \mathbb{1}_3))]^2$$
$$+ \mu_{\mathrm{c}}\,\|\operatorname{skew}(\overline{U}_\xi - \mathbb{1}_3)\|^2 \quad \text{(physically linear)}\,,$$

$$W_{\mathrm{curv}}(\alpha_\xi) := \mu\,L_{\mathrm{c}}^2\left(b_1\,\|\operatorname{dev}\operatorname{sym}\alpha_\xi\|^2 + b_2\,\|\operatorname{skew}\alpha_\xi\|^2 + b_3\,[\operatorname{tr}(\alpha_\xi)]^2\right).$$

It is important that this 3D parental model is well posed (P. Neff, M. Bîrsan, F. Osterbrink, Journal of Elasticity, 2015).

Why another new shell theory?

Objective: A new shell theory motivated by good results for a similar plate theory

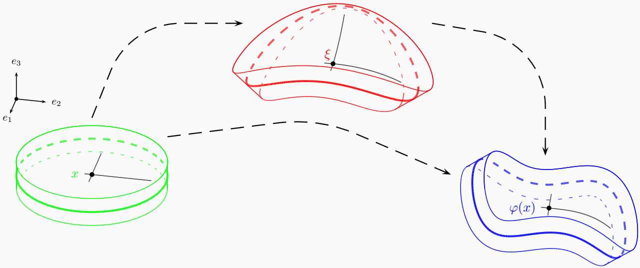


Figure 5: Kinematics of the 3D-Cosserat model.

Objective: A new shell theory motivated by good results for a similar plate theory

Figure credite to O. Sander, P. Neff, M. Bîrsan, Computational Mechanics, 2016.

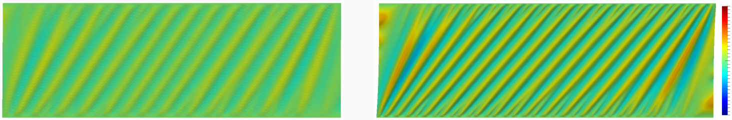


Figure 6: Simulation results of the shearing tests. The color visualizes the elevation of the wrinkles.

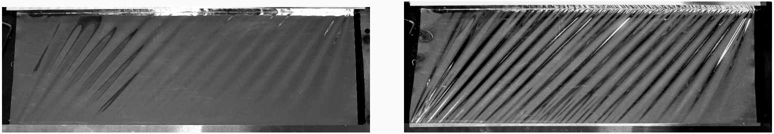
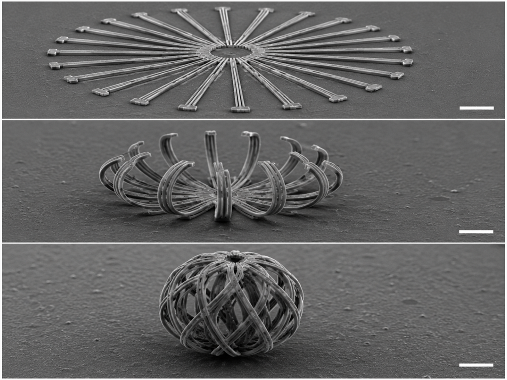


Figure 7: Experimental results of the shearing tests on plastic films or a cellophane-like sheets.

Objective: A theory up to order $O(h^5)$. Aplications to self-assembly 'micro-origami'

Among the interesting potential applications of this formulation, we mention the design control of three-dimensional objects (like coils, springs, rings, nanotubes or more sophisticated meta-materials), see Figure 25.



The construction of the Cosserat shell model up to $O(h^5)$

The Cosserat shell models up to $O(h^5)$

- I.D. Ghiba, M. Bîrsan, P. Lewintan, and P. Neff. The isotropic Cosserat shell model including terms up to $O(h^5)$. Part I: Derivation in matrix notation. *Journal of Elasticity*, 142:201–262, 2020.
- I.D. Ghiba, M. Bîrsan, P. Lewintan, and P. Neff. The isotropic elastic Cosserat shell model including terms up to order $O(h^5)$ in the shell thickness. Part II: Existence of minimizers. *Journal of Elasticity*, 142:263–290, 2020.
- I.D. Ghiba, M. Bîrsan, P. Lewintan, and P. Neff. A constrained Cosserat-shell model including terms up to $O(h^5)$. *Journal of Elasticity*, 146(1):83–141, 2021.

Different approaches to shell modelling

Derivation approach

It starts from a given three-dimensional model of the body and reduces it via physically reasonable constitutive assumptions on the kinematics to a two-dimensional model (i.e., *dimensional reduction*).

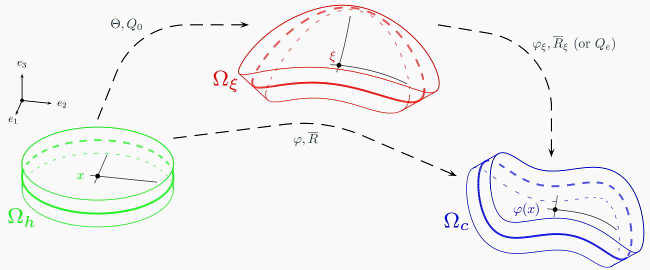
The philosophy behind the derivation approach is expressed clearly by the grandmaster W.T. Koiter as follows: "Any two-dimensional theory of thin shells is necessarily of an *approximate character*. An exact two-dimensional theory of shells cannot exist, because the actual body we have to deal with, thin as it may be, is always three-dimensional. [. . .] Since the theory we have to deal with is approximate in character, we feel that extreme rigour in its development is hardly desirable. [. . .] *Flexible bodies* like thin shells require a *flexible approach*."

Fictitious configuration

In what follows, we assume that the parameter domain $\Omega_h \subset \mathbb{R}^3$ is a right cylinder of the form

$$\Omega_h = \left\{ (x_1, x_2, x_3) \,\middle|\, (x_1, x_2) \in \omega,\ -\frac{h}{2} < x_3 < \frac{h}{2} \right\} = \omega \times \left(-\frac{h}{2}, \frac{h}{2} \right),$$

where $\omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\omega$ and the constant length $h > 0$ is the *thickness of the shell*. Thus, the domain Ω_h can be viewed as a *fictitious Cartesian configuration* of the body.



For our purpose, the diffeomorphism $\Theta : \mathbb{R}^3 \to \mathbb{R}^3$ describing the reference configuration (i.e., the curved surface of the shell), will be chosen in the specific form

$$\Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3\; n_0(x_1, x_2), \qquad n_0 = \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|}\,,$$

where $y_0 : \omega \to \mathbb{R}^3$ is a function of class $C^2(\omega)$.

This specific form of the diffeomorphism Θ maps the midsurface ω of the fictitious Cartesian configuration parameter space Ω_h onto the midsurface $\omega_\xi = y_0(\omega)$ of Ω_ξ and n_0 is the unit normal vector to ω_ξ.

Prerequisites from classical differential geometry

The map $n_0 : \omega \to S^2$ is called the *Gauß map* (where S^2 is the unit sphere in $\mathbb{R}^3$) and the moving 3-frame $(\partial_{x_1} y_0 | \partial_{x_2} y_0 | n_0)$ is called the *Gauß frame* of the surface $y_0(\omega)$, which in general is not orthonormal. The matrix representation of the *first fundamental form (metric)* on $y_0(\omega)$ is given through

$$\mathrm{I}_{y_0} := [\nabla y_0]^T \nabla y_0.$$

Because $\mathrm{rank}(\nabla y_0) = 2$, the tensor $[\nabla y_0]^T \nabla y_0$ is positive definite.

The matrix representation of the *second fundamental form* on $y_0(\omega)$ providing a measure for curvature of the surface is given by

$$\mathrm{II}_{y_0} := -[\nabla y_0]^T \nabla n_0 = -(\partial_{x_1} y_0 | \partial_{x_2} y_0)^T (\partial_{x_1} n_0 | \partial_{x_2} n_0) \,.$$

Since n_0 is orthogonal to the tangent space $T_x y_0$ of the surface y_0, the relation $0 = \partial_{x_1} \langle \partial_{x_2} y_0, n_0 \rangle = \partial_{x_2} \langle \partial_{x_1} y_0, n_0 \rangle$ shows easily that $\mathrm{II}_{y_0} \in \mathrm{Sym}(2)$.

Hence, we obtain the following alternative expression for the Weingarten map via the so called *Weingarten equations* :

$$\mathrm{L}_{y_0} = -([\nabla y_0]^T \nabla y_0)^{-1}(\nabla n_0{}^T \nabla y_0) \qquad \text{or} \qquad \mathrm{L}_{y_0} = \mathrm{I}_{y_0}^{-1}\mathrm{II}_{y_0}\,.$$

Moreover, using the symmetry of the second fundamental form we see that the Weingarten map satisfies:

$$\nabla y_0\, \mathrm{L}_{y_0} = -\nabla n_0.$$

The *Gauß curvature* K of the surface $y_0(\omega)$ is determined by

$$\mathrm{K} := \det(\mathrm{II}_{y_0}\, \mathrm{I}_{y_0}^{-1}) = \det(\mathrm{L}_{y_0})\,,$$

and the *mean curvature* H through

$$2\,\mathrm{H} := \mathrm{tr}(\mathrm{L}_{y_0})\,.$$

The principal curvatures κ_1, κ_2 are the solutions of the characteristic equation of L_{y_0}, i.e.,

$$\kappa^2 - \mathrm{tr}(\mathrm{L}_{y_0})\,\kappa + \det(\mathrm{L}_{y_0}) = \kappa^2 - 2\,\mathrm{H}\,\kappa + \mathrm{K} = 0.$$

Properties of the diffeomorphism Θ

With the help of the above lemma, we prove the following proposition.

Proposition

The diffeomorphism Θ has the following properties for all x_3:

i) $\det(\nabla_x \Theta(x_3)) \;=\; \det(\nabla y_0 | n_0)\Big[1 - 2\, x_3\, \mathrm{H} \,+ x_3^2\, \mathrm{K}\Big];$

ii) $\nabla_x \Theta(x_3)$ belongs to
$\mathrm{GKC} := \{X \in \mathrm{GL}^+(3) \;|\; X^T X\, e_3 \;=\; \varrho^2 e_3,\;\; \varrho \in \mathbb{R}^+\};$

iii) if $h \max\{\sup_{(x_1,x_2)\in\omega} |\kappa_1|, \sup_{(x_1,x_2)\in\omega} |\kappa_2|\} < 2$, then for all $x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right)$:
$$[\nabla_x \Theta(x_3)]^{-1} \;= \frac{1}{1 - 2\,\mathrm{H}\, x_3 + \mathrm{K}\, x_3^2}\left[\mathbb{1}_3 + x_3(\mathrm{L}^\flat_{y_0} - 2\,\mathrm{H}\,\mathbb{1}_3) + x_3^2\, \mathrm{K}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\right][\nabla_x \Theta(0)]^{-1};$$

Let us introduce the tensors[1] defined by:

$$\mathrm{A}_{y_0} := (\nabla y_0|0)\,[\nabla_x\Theta(0)]^{-1} \in \mathbb{R}^{3\times 3},$$
$$\mathrm{B}_{y_0} := -(\nabla n_0|0)\,[\nabla_x\Theta(0)]^{-1} \in \mathbb{R}^{3\times 3},$$

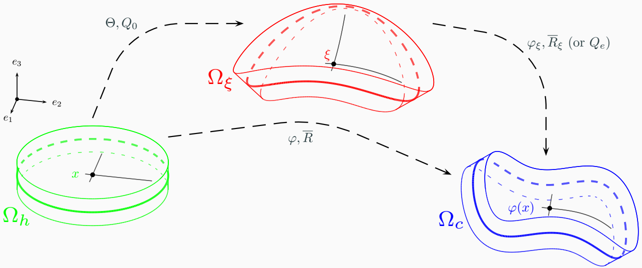
and the so-called *alternator tensor* C_{y_0} of the surface

$$\mathrm{C}_{y_0} := \det(\nabla_x\Theta(0))\,[\nabla_x\Theta(0)]^{-T}\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}[\nabla_x\Theta(0)]^{-1}.$$

The introduced tensors are essential in the derivation of the model entirely in matrix representation.

[1]These tensors are usually called the first fundamental form and the second fundamental form, respectively. However, we will not use this terminology since it may lead to some confusions. .

Transformation of the minimization problem



Transformation of the minimization problem

Now, let us define the map

$$\varphi : \Omega_h \to \Omega_c, \qquad \varphi(x_1, x_2, x_3) = \varphi_\xi(\Theta(x_1, x_2, x_3)).$$

Consider the *elastic microrotation*

$$\overline{Q}_e : \Omega_h \to \mathrm{SO}(3), \qquad \overline{Q}_e(x_1, x_2, x_3) := \overline{R}_\xi(\Theta(x_1, x_2, x_3))$$

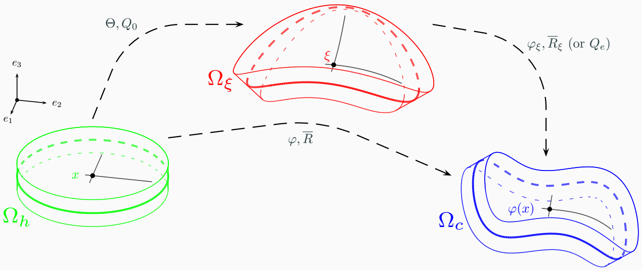
and *the elastic (non-symmetric) Biot-type stretch tensor* (the elastic first Cosserat deformation tensor)

$$\overline{U}_e : \Omega_h \to \mathrm{Sym}(3), \qquad \overline{U}_e(x_1, x_2, x_3) := \overline{U}_\xi(\Theta(x_1, x_2, x_3)) .$$

We use the polar decomposition of $\nabla_x \Theta$ and write $\nabla_x \Theta = Q_0 \, U_0$.

Corresponding to the elastic deformation process, we have the total microrotation

$$\overline{R} : \Omega_h \to \mathrm{SO}(3), \qquad \overline{R}(x_1, x_2, x_3) = \overline{Q}_e(x_1, x_2, x_3)\, Q_0(x_1, x_2, x_3).$$



Therefore, the elastic non-symmetric stretch tensor is given by

$$\overline{U}_\xi := \overline{R}_\xi^T F_\xi \to \overline{U}_e = \overline{Q}_e^T F [\nabla_x \Theta]^{-1} = Q_0 \overline{R}^T F [\nabla_x \Theta]^{-1}.$$

What about $\alpha_\xi := \overline{R}_\xi^T \operatorname{Curl}_\xi \overline{R}_\xi$?

As a Lagrangian strain measure for curvature (orientation change) one can also employ the so-called *wryness tensor* (second order tensor)

$$\Gamma_\xi := \Big(\operatorname{axl}(\overline{R}_\xi^T \partial_{\xi_1} \overline{R}_\xi) \,|\, \operatorname{axl}(\overline{R}_\xi^T \partial_{\xi_2} \overline{R}_\xi) \,|\, \operatorname{axl}(\overline{R}_\xi^T \partial_{\xi_3} \overline{R}_\xi)\Big) \in \mathbb{R}^{3\times 3},$$

since the following close relationship between the wryness tensor and the dislocation density tensor holds

$$\alpha_\xi = -\Gamma_\xi^T + \operatorname{tr}(\Gamma_\xi)\, \mathbb{1}_3.$$

For infinitesimal strains this formula is well-known under the name Nye's formula.

Ansatz for the rotation

In the following, we want to find a *reasonable approximation* of $(\varphi, \overline{R})$ involving only two-dimensional quantities. Following the formal dimensional reduction procedure for the Cosserat elastic plates given in [**?**], we consider that the rotation $\overline{R} : \Omega_h \to \mathrm{SO}(3)$ in the thin shell does not depend on the thickness variable x_3

$$\overline{R}(x_1, x_2, x_3) \,=\, \overline{R}_s(x_1, x_2)\,,$$

in line with the assumed thinness and material homogeneity of the structure. Moreover, an approximation of the elastic rotation $\overline{Q}_e : \Omega_h \to \mathrm{SO}(3)$ will be given by $\overline{Q}_{e,s}$

$$\overline{Q}_{e,s}(x_1, x_2) \,=\, \overline{R}_s(x_1, x_2)\, Q_0^T(x_1, x_2, 0)\,.$$

It follows

$$\overline{R}_s(x_1, x_2)\, e_3 \,=\, \overline{Q}_{e,s}(x_1, x_2)\, n_0\,.$$

Ansatz for the deformation

In the engineering shell community it is well known that the ansatz for the deformation over the thickness should be at least quadratic.

We consider therefore the following *8-parameter quadratic ansatz* in the thickness direction for the reconstructed total deformation $\varphi_s : \Omega_h \subset \mathbb{R}^3 \to \mathbb{R}^3$ of the shell-like structure

$$\varphi_s(x_1, x_2, x_3) = m(x_1, x_2) + \left(x_3 \varrho_m(x_1, x_2) + \frac{x_3^2}{2} \varrho_b(x_1, x_2) \right) \overline{Q}_{e,s}(x_1, x_2)\, n_0(x_1, x_2)\,,$$

where $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ takes on the role of the deformation of the midsurface of the shell viewed as a parametrized surface, the yet indeterminate functions $\varrho_m,\ \varrho_b : \omega \subset \mathbb{R}^2 \to \mathbb{R}$ allow in principal for symmetric thickness stretch ($\varrho_m \neq 1$) and asymmetric thickness stretch ($\varrho_b \neq 0$) about the midsurface.

“Zero normal stresses” on upper and lower faces

As usual in the development of shell theories, we assume that the normal stress (Piola-Kirchhoff stress tensor in the normal direction n_0) on the transverse boundaries (*upper and lower faces* ω_ξ^+ and ω_ξ^-, respectively, of the curved reference configuration Ω_ξ) are vanishing, i.e.,

$$S_1(F_\xi, \overline{R}_\xi)\Big|_{\omega_\xi^\pm} . (\pm n_0) \,=\, 0.$$

The *first Piola-Kirchhoff stress tensor* in the reference (curved) configuration Ω_ξ is given by $S_1(F_\xi, \overline{R}_\xi) = D_{F_\xi} \widetilde{W}_{\mathrm{mp}}(F_\xi, \overline{R}_\xi)$.

Next, we need to express the tensors

$$\widetilde{\mathcal{E}}_s := \overline{U}_{e,s} - \mathbb{1}_3 = \overline{Q}_{e,s}^T \widetilde{F}_{e,s} - \mathbb{1}_3,$$
$$\Gamma_s := \left(\mathrm{axl}(\overline{Q}_{e,s}^T \partial_{x_1} \overline{Q}_e) \,|\, \mathrm{axl}(\overline{Q}_{e,s}^T \partial_{x_2} \overline{Q}_e) \,|0\right)[\nabla_x \Theta(x_3)]^{-1}$$

with the help of the usual strain measures in the nonlinear 6-parameter shell theory. Therefore, we introduce the following tensor fields on the surface ω_ξ

$$\mathcal{E}_{m,s} := \overline{Q}_{e,s}^T (\nabla m | \overline{Q}_{e,s} \nabla_x \Theta(0)\, e_3)[\nabla_x \Theta(0)]^{-1} - \mathbb{1}_3$$

(the elastic shell strain tensor),

$$\mathcal{K}_{e,s} := \left(\mathrm{axl}(\overline{Q}_{e,s}^T \partial_{x_1} \overline{Q}_e) \,|\, \mathrm{axl}(\overline{Q}_{e,s}^T \partial_{x_2} \overline{Q}_e) \,|0\right)[\nabla_x \Theta(0)]^{-1}$$

(elastic shell bending–curvature tensor).

The Cosserat shell model: I.D. Ghiba, M. Bîrsan, P. Lewintan, and P. Neff., Journal of Elasticity (2020a)

Gathering our results, we have obtained the following two-dimensional minimization problem for the deformation of the midsurface $m : \omega \to \mathbb{R}^3$ and the microrotation of the shell $\overline{Q}_{e,s} : \omega \to \mathrm{SO}(3)$ solving on $\omega \subset \mathbb{R}^2$: minimize with respect to $(m, \overline{Q}_{e,s})$ the functional

$$I = \int_\omega \Big[W_{\mathrm{memb}}\big(\mathcal{E}_{m,s}\big) + W_{\mathrm{memb,bend}}\big(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}\big) + W_{\mathrm{bend,curv}}\big(\mathcal{K}_{e,s}\big) \Big] \mathrm{det}(\nabla y_0 | n_0) \mathrm{d}a,$$

where the membrane part $W_{\mathrm{memb}}\big(\mathcal{E}_{m,s}\big)$, the membrane–bending part $W_{\mathrm{memb,bend}}\big(\mathcal{E}_{m,s}, \mathcal{K}_{e,s}\big)$ and the bending–curvature part $W_{\mathrm{bend,curv}}\big(\mathcal{K}_{e,s}\big)$ of the shell energy density are given by

$$
\begin{aligned}
W_{\mathrm{memb}}\big(\mathcal{E}_{m,s}\big) &= \Big(h + \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\big),\\
W_{\mathrm{memb,bend}}\big(\mathcal{E}_{m,s},\, \mathcal{K}_{e,s}\big) &= \Big(\frac{h^3}{12} - \mathrm{K}\,\frac{h^5}{80}\Big)\, W_{\mathrm{shell}}\big(\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s}\big)\\
&\quad - \frac{h^3}{3}\mathrm{H}\,\mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s}, \mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s}\big)\\
&\quad + \frac{h^3}{6}\,\mathcal{W}_{\mathrm{shell}}\big(\mathcal{E}_{m,s}, (\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\,\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big)\\
&\quad + \frac{h^5}{80}\, W_{\mathrm{mp}}\big((\mathcal{E}_{m,s}\,\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s})\mathrm{B}_{y_0}\big),\\
W_{\mathrm{bend,curv}}\big(\mathcal{K}_{e,s}\big) &= \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\big)\\
&\quad + \Big(\frac{h^3}{12} - \mathrm{K}\,\frac{h^5}{80}\Big)\, W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}\big)\\
&\quad + \frac{h^5}{80}\, W_{\mathrm{curv}}\big(\mathcal{K}_{e,s}\mathrm{B}_{y_0}^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
W_{\mathrm{shell}}(X) &= \mu\,\|\mathrm{sym}\,X\|^2 + \mu_{\mathrm{c}}\,\|\mathrm{skew}\,X\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\,\big[\mathrm{tr}(X)\big]^2,\\
\mathcal{W}_{\mathrm{shell}}(X,Y) &= \mu\,\big\langle \mathrm{sym}\,X, \mathrm{sym}\,Y\big\rangle + \mu_{\mathrm{c}}\big\langle \mathrm{skew}\,X, \mathrm{skew}\,Y\big\rangle + \frac{\lambda\,\mu}{\lambda+2\mu}\mathrm{tr}(X)\mathrm{tr}(Y),\\
W_{\mathrm{mp}}(X) &= \mu\,\|\mathrm{sym}\,X\|^2 + \mu_{\mathrm{c}}\,\|\mathrm{skew}\,X\|^2 + \frac{\lambda}{2}\,\big[\mathrm{tr}(X)\big]^2,\\
W_{\mathrm{curv}}(X) &= \mu\,L_{\mathrm{c}}^2\left(b_1\,\|\mathrm{dev}\,\mathrm{sym}\,X\|^2 + b_2\,\|\mathrm{skew}\,X\|^2 + 4\,b_3\,\big[\mathrm{tr}(X)\big]^2\right).
\end{aligned}
$$

In this formulation, all the constitutive coefficients are deduced from the three-dimensional formulation, without using any a posteriori fitting of some two-dimensional constitutive coefficients.

We consider the following boundary conditions for the midsurface deformation m and rotation field $\overline{R}_s$ on the Dirichlet part of the lateral boundary $\gamma_0 \subset \partial\omega$:

$m\,|_{\gamma_0} = m_0,$ simply supported (fixed, welded) $\overline{R}_s\,|_{\gamma_0} = \hat{R},$

It is possible to use the referential fundamental forms I_{y_0}, II_{y_0} and L_{y_0} instead of the matrices A_{y_0}, B_{y_0} and C_{y_0}, and to rewrite all the arguments of the energy terms as

$$\begin{aligned}
\mathcal{E}_{m,s} =&[\nabla_x\Theta(0)]^{-T}\left(\begin{array}{c|c} \mathcal{G} & 0 \\ \mathcal{T} & 0 \end{array}\right)[\nabla_x\Theta(0)]^{-1} \\
\mathrm{C}_{y_0}\mathcal{K}_{e,s} =& [\nabla_x\Theta(0)]^{-T}\left(\begin{array}{c|c} -\mathcal{R} & 0 \\ 0 & 0 \end{array}\right)[\nabla_x\Theta(0)]^{-1}, \\
\mathcal{E}_{m,s}\mathrm{B}_{y_0} + \mathrm{C}_{y_0}\mathcal{K}_{e,s} =& [\nabla_x\Theta(0)]^{-T}\left(\begin{array}{c|c} \mathcal{G}\,\mathrm{L}_{y_0} - \mathcal{R} & 0 \\ \mathcal{T}\,\mathrm{L}_{y_0} & 0 \end{array}\right)[\nabla_x\Theta(0)]^{-1},
\end{aligned}$$

where

$$\mathcal{G} := (\overline{Q}_{e,s}\nabla y_0)^T\nabla m - \mathrm{I}_{y_0} \notin \mathrm{Sym}(2) \qquad \textit{the change of metric tensor},$$
$$\mathcal{T} := (\overline{Q}_{e,s} n_0)^T(\nabla m) \qquad \textit{the transverse shear deformation vector},$$
$$\mathcal{R} := -(\overline{Q}_{e,s}\nabla y_0)^T\nabla(\overline{Q}_{e,s} n_0) - \mathrm{II}_{y_0} \notin \mathrm{Sym}(2) \qquad \textit{the bending strain tensor}.$$

Existence results

Existence result for the theory including terms up to order $O(h^5)$: I.D. Ghiba, M. Bîrsan, P. Lewintan, and P. Neff., Journal of Elasticity (2020b)

Assume that the boundary data satisfy the conditions

$$m^* \in \mathrm{H}^1(\omega, \mathbb{R}^3), \qquad \overline{Q}^*_{e,s} \in \mathrm{H}^1(\omega, \mathrm{SO}(3)).$$

Assume that the following conditions concerning the initial configuration are satisfied: $y_0 : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a continuous injective mapping and

$$y_0 \in \mathrm{H}^1(\omega, \mathbb{R}^3), \qquad Q_0(0) \in \mathrm{H}^1(\omega, \mathrm{SO}(3)),$$
$$\nabla_x \Theta(0) \in \mathrm{L}^\infty(\omega, \mathbb{R}^{3\times 3}), \qquad \det[\nabla_x \Theta(0)] \geq a_0 > 0\,,$$

where a_0 is a constant. Then, for sufficiently small values of the thickness h such that

$$h \max\{\sup_{x\in\omega} |\kappa_1|, \sup_{x\in\omega} |\kappa_2|\} < \alpha \qquad \text{with} \qquad \alpha < \sqrt{\frac{2}{3}(29 - \sqrt{761})} \simeq 0.97083$$

Existence result for the theory including terms up to order $O(h^5)$

and for constitutive coefficients such that $\mu > 0$, $\mu_{\rm c} > 0$, $2\,\lambda + \mu > 0$, $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$, the minimization problem admits at least one minimizing solution pair $(m, \overline{Q}_{e,s}) \in \mathcal{A}$, where the admissible set $\mathcal{A}$ of solutions is defined by

$$\mathcal{A} = \left\{(m, \overline{Q}_{e,s}) \in {\rm H}^1(\omega, \mathbb{R}^3) \times {\rm H}^1(\omega, {\rm SO}(3)) \;\middle|\; m\big|_{\gamma_d} = m^*,\ \overline{Q}_{e,s}\big|_{\gamma_d} = \overline{Q}^*_{e,s}\right\},$$

and the boundary conditions are to be understood in the sense of traces.

Existence result for the theory including terms up to order $O(h^3)$

Then, if the thickness h satisfies at least one of the following conditions:

i) $h\max\{\sup_{x\in\omega}|\kappa_1|, \sup_{x\in\omega}|\kappa_2|\} < \alpha$ and
$h^2 < \frac{(5-2\sqrt{6})(\alpha^2-12)^2}{4\,\alpha^2}\frac{c_2^+}{C_1^+}$ with $0<\alpha<2\sqrt{3}$;

ii) $h\max\{\sup_{x\in\omega}|\kappa_1|, \sup_{x\in\omega}|\kappa_2|\} < \frac{1}{a}$ with

$$a > \max\left\{1+\frac{\sqrt{2}}{2}, \frac{1+\sqrt{1+3\frac{C_1^+\}}{c_1^+\}}}}{2}\right\},$$

where c_2^+ denotes the smallest eigenvalue of $W_{\rm curv}(S)$, and c_1^+ and $C_1^+>0$ denote the smallest and the biggest eigenvalues of the quadratic form $W_{\rm shell}(S)$ and for constitutive coefficients such that $\mu>0$, $\mu_{\rm c}>0$, $2\,\lambda+\mu>0$, $b_1>0$, $b_2>0$ and $b_3>0$, the minimization problem admits at least one minimizing solution pair $(m,\overline{Q}_{e,s})\in\mathcal{A}$, where the admissible set $\mathcal{A}$ of solutions is defined by

$$\mathcal{A}=\left\{(m,\overline{Q}_{e,s})\in{\rm H}^1(\omega,\mathbb{R}^3)\times{\rm H}^1(\omega,{\rm SO}(3))\;\middle|\; m\big|_{\gamma_d}=m^*,\ \overline{Q}_{e,s}\big|_{\gamma_d}=\overline{Q}^*_{e,s}\right\},$$

and the boundary conditions are to be understood in the sense of traces.

The limit problem for infinite Cosserat couple modulus

$\mu_{\mathrm{c}} \to \infty$

Constrained elastic Cosserat shell models

For a given matrix $M \in \mathbb{R}^{2\times 2}$ we define the *3D-lifted quantities*

$$\widehat{M} = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \qquad \text{and}$$

$$M^\flat = \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}, \qquad M^\flat = \widehat{M}\, \mathbb{1}_2^\flat.$$

In the constrained elastic Cosserat shell models we consider

$$\mu_c \to \infty \qquad \Longrightarrow$$

$$Q_\infty = \operatorname{polar}\big((\nabla m|n)[\nabla_x\Theta]^{-1}\big) = (\nabla m|n)[\nabla_x\Theta]^{-1}\sqrt{[\nabla_x\Theta]\,\widehat{\mathrm{I}}_m^{-1}\,[\nabla_x\Theta]^T}$$

A new set of admissible functions

The set $\mathcal{A}^{\mathrm{mod}}$ of admissible functions is accordingly defined by

$$\mathcal{A}^{\mathrm{mod}} = \Big\{ (m, Q_\infty) \in \mathrm{H}^1(\omega, \mathbb{R}^3) \times \mathrm{H}^1(\omega, \mathrm{SO}(3)) \;\Big|\; m\big|_{\gamma_d} = m^*,$$
$$Q_\infty Q_0.e_3\big|_{\gamma_d} = \frac{\partial_{x_1} m^* \times \partial_{x_2} m^*}{\|\partial_{x_1} m^* \times \partial_{x_2} m^*\|}$$
$$U := Q_\infty^T (\nabla m | Q_\infty Q_0.e_3)[\nabla_x \Theta]^{-1} \in \mathrm{L}^2(\omega, \mathrm{Sym}^+(3)) \Big\},$$

which incorporates a weak reformulation of the imposed symmetry constraint $\mathcal{E}_{m,s} \in \mathrm{Sym}(3)$.

Constrained elastic Cosserat shell models

The variational problem for the constrained Cosserat $O(h^5)$-shell model is now to find a deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ minimizing on ω:

$$
\begin{aligned}
I = \int_\omega \Big[& \Big(h + \mathrm{K}\,\frac{h^3}{12}\Big)\, W^{\infty}_{\mathrm{shell}}\big(\sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_m\,\mathbb{1}_2^\flat\,[\nabla_x\Theta]^{-1}} - \sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_{y_0}\,\mathbb{1}_2^\flat\,[\nabla_x\Theta]^{-1}}\big), \\
& + \Big(\frac{h^3}{12} - \mathrm{K}\,\frac{h^5}{80}\Big)\, W^{\infty}_{\mathrm{shell}}\Big(\mathrm{sym}\big[\sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_m\,[\nabla_x\Theta]^{-1}}\,[\nabla_x\Theta]\Big(\mathrm{L}^\flat_{y_0} - \mathrm{L}^\flat_m\Big)[\nabla_x\Theta]^{-1}\big]\Big) \\
& - \frac{h^3}{3}\mathrm{H}\,\mathcal{W}^{\infty}_{\mathrm{shell}}\Big(\sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_m\,\mathbb{1}_2^\flat\,[\nabla_x\Theta]^{-1}} - \sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_{y_0}\,\mathbb{1}_2^\flat\,[\nabla_x\Theta]^{-1}}, \\
& \qquad\qquad \mathrm{sym}\big[\sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_m\,[\nabla_x\Theta]^{-1}}\,[\nabla_x\Theta]\Big(\mathrm{L}^\flat_{y_0} - \mathrm{L}^\flat_m\Big)[\nabla_x\Theta]^{-1}\big]\Big) \\
& + \frac{h^3}{6}\,\mathcal{W}^{\infty}_{\mathrm{shell}}\Big(\sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_m\,\mathbb{1}_2^\flat\,[\nabla_x\Theta]^{-1}} - \sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_{y_0}\,\mathbb{1}_2^\flat\,[\nabla_x\Theta]^{-1}}, \\
& \qquad\qquad \mathrm{sym}\big[\sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_m\,[\nabla_x\Theta]^{-1}}\,[\nabla_x\Theta]\Big(\mathrm{L}^\flat_{y_0} - \mathrm{L}^\flat_m\Big)\mathrm{L}^\flat_{y_0}[\nabla_x\Theta]^{-1}\big]\Big) \\
& + \frac{h^5}{80}\,W^{\infty}_{\mathrm{mp}}\Big(\mathrm{sym}\big[\sqrt{[\nabla_x\Theta]^{-T}\,\widehat{\mathrm{I}}_m\,[\nabla_x\Theta]^{-1}}\,[\nabla_x\Theta]\Big(\mathrm{L}^\flat_{y_0} - \mathrm{L}^\flat_m\Big)\mathrm{L}^\flat_{y_0}[\nabla_x\Theta]^{-1}\big]\Big) \\
& + \Big(h - \mathrm{K}\,\frac{h^3}{12}\Big)\, W_{\mathrm{curv}}(\mathcal{K}_\infty) + \Big(\frac{h^3}{12} - \mathrm{K}\,\frac{h^5}{80}\Big)\, W_{\mathrm{curv}}(\mathcal{K}_\infty \mathrm{B}_{y_0}) \\
& + \frac{h^5}{80}\,W_{\mathrm{curv}}(\mathcal{K}_\infty \mathrm{B}^2_{y_0})\Big]\,\det[\nabla_x\Theta]\,\mathrm{d}a - \overline{\Pi}(m, Q_\infty).
\end{aligned}
$$

Gamma-convergence, a useful tool for dimensional reduction

The sequence $I_j : X \to \overline{\mathbb{R}}$ in the metric space X is called Γ-*convergent* to $I_0 : X \to \overline{\mathbb{R}}$, if for all $x \in X$ we have the two following conditions:

Gamma-convergence: convergence of minimizers

Gamma-convergence, a useful tool for dimensional reduction

The sequence $I_j : X \to \overline{\mathbb{R}}$ in the metric space X is called Γ-*convergent* to $I_0 : X \to \overline{\mathbb{R}}$, if for all $x \in X$ we have the two following conditions:

- (**liminf inequality**) $\forall x_j \to x, \quad I_0(x) \leq \liminf_j I_j(x_j)$,
- (**limsup inequality**) $\exists x_j \to x, \quad I_0(x) \geq \limsup_j I_j(x_j)$.

Scaling

Nonlinear scaling

The nonlinear or natural scaling for a vector field $z : \Omega_h \longrightarrow \mathbb{R}^3$ is $z^\natural : \Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}] \longrightarrow \mathbb{R}^3$, where only the independent variables will be scaled:

$$x_1 = \eta_1\,, \quad x_2 = \eta_2\,, \quad x_3 = h\,\eta_3\,,$$

$$z^\natural\Big(x_1, x_2, \frac{1}{h}x_3\Big) := z(x_1, x_2, x_3)\,, \qquad \text{nonlinear scaling}\,.$$

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The gradient of z with respect to $x = (x_1, x_2, x_3)$ is

$$\begin{aligned}\nabla_x z(x_1, x_2, x_3) &= \Big(\partial_{\eta_1} z^\natural(\eta_1, \eta_2, \eta_3) \mid \partial_{\eta_2} z^\natural(\eta_1, \eta_2, \eta_3) \mid \frac{1}{h}\partial_{\eta_3} z^\natural(\eta_1, \eta_2, \eta_3)\Big) \\ &= \begin{pmatrix} \partial_{\eta_1} z_1^\natural(\eta) & \partial_{\eta_2} z_1^\natural(\eta) & \frac{1}{h}\partial_{\eta_3} z_1^\natural(\eta) \\ \partial_{\eta_1} z_2^\natural(\eta) & \partial_{\eta_2} z_2^\natural(\eta) & \frac{1}{h}\partial_{\eta_3} z_2^\natural(\eta) \\ \partial_{\eta_1} z_3^\natural(\eta) & \partial_{\eta_2} z_3^\natural(\eta) & \frac{1}{h}\partial_{\eta_3} z_3^\natural(\eta) \end{pmatrix} := \nabla_\eta^h z^\natural(\eta)\,.\end{aligned}$$

Nonlinear scaling for deformation and microrotation

After defining the following scaling transformation, we apply the nonlinear scaling for both deformation and microrotation. We define the domain $\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^3$, with fixed thickness.

Scaling transformation

$$\zeta : \ \eta \in \Omega_1 \mapsto \mathbb{R}^3 \,, \qquad \zeta(\eta_1, \eta_2, \eta_3) := (\eta_1, \eta_2, h\,\eta_3) \,,$$

$$\zeta^{-1} : \ x \in \Omega_h \mapsto \mathbb{R}^3 \,, \qquad \zeta^{-1}(x_1, x_2, x_3) := (x_1, x_2, \frac{x_3}{h}) \,,$$

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where $\zeta(\Omega_1) = \Omega_h$.

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Scaling transformation

$$\zeta : \ \eta \in \Omega_1 \mapsto \mathbb{R}^3 , \qquad \zeta(\eta_1, \eta_2, \eta_3) := (\eta_1, \eta_2, h\,\eta_3) ,$$
$$\zeta^{-1} : \ x \in \Omega_h \mapsto \mathbb{R}^3 , \qquad \zeta^{-1}(x_1, x_2, x_3) := (x_1, x_2, \frac{x_3}{h}) ,$$

where $\zeta(\Omega_1) = \Omega_h$. As well as,

$$\varphi(x_1, x_2, x_3) = \varphi^\natural(\zeta^{-1}(x_1, x_2, x_3)) \quad \forall x \in \Omega_h ; \quad \varphi^\natural(\eta) = \varphi(\zeta(\eta)) \quad \forall \eta \in \Omega_1 ,$$

$$\nabla_x \varphi(x_1, x_2, x_3) = \begin{pmatrix} \partial_{\eta_1} \varphi_1^\natural(\eta) & \partial_{\eta_2} \varphi_1^\natural(\eta) & \frac{1}{h} \partial_{\eta_3} \varphi_1^\natural(\eta) \\ \partial_{\eta_1} \varphi_2^\natural(\eta) & \partial_{\eta_2} \varphi_2^\natural(\eta) & \frac{1}{h} \partial_{\eta_3} \varphi_2^\natural(\eta) \\ \partial_{\eta_1} \varphi_3^\natural(\eta) & \partial_{\eta_2} \varphi_3^\natural(\eta) & \frac{1}{h} \partial_{\eta_3} \varphi_3^\natural(\eta) \end{pmatrix} = \nabla_\eta^h \varphi^\natural(\eta) = F_h^\natural,$$

Nonlinear scaling for deformation and microrotation

The other nonlinear scaled parameters:

- $\overline{Q}_e(x_1, x_2, x_3) = \overline{Q}_e^{\natural}(\zeta^{-1}(x_1, x_2, x_3))\,\forall x \in \Omega_h\,;$
  $\overline{Q}_e^{\natural}(\eta) = \overline{Q}_e(\zeta(\eta)), \forall \eta \in \Omega_1\,,$
- $(\nabla_x \Theta)^{\natural}(\eta) = (\nabla_x \Theta)(\zeta(\eta)),\ Q_0^{\natural}(\eta) = Q_0(\zeta(\eta)),\ U_0^{\natural}(\eta) = U_0(\zeta(\eta))\,,$
- $\overline{R}(x_1, x_2, x_3) = \overline{R}^{\natural}(\zeta^{-1}(x_1, x_2, x_3))\,\forall x \in \Omega_h\,;$
  $\overline{R}^{\natural}(\eta) = \overline{R}(\zeta(\eta))\,,\ \forall \eta \in \Omega_1\,,$
- $\overline{U}_e^{\natural} = \overline{Q}_e^{\natural, T} F_h^{\natural}[(\nabla_x \Theta)^{\natural}]^{-1} = \overline{Q}_e^{\natural, T} \nabla_\eta^h \varphi^{\natural}(\eta)[(\nabla_x \Theta)^{\natural}]^{-1}\,,$
- $\Gamma_h^{\natural} = (\mathrm{axl}(\overline{Q}_{e,h}^{\natural, T} \partial_{\eta_1} \overline{Q}_{e,h}^{\natural})|\mathrm{axl}(\overline{Q}_{e,h}^{\natural, T}\, \partial_{\eta_2} \overline{Q}_{e,h}^{\natural})|\tfrac{1}{h}\mathrm{axl}(\overline{Q}_{e,h}^{\natural, T} \partial_{\eta_3} \overline{Q}_{e,h}^{\natural}))[(\nabla_x \Theta)^{\natural}]^{-1},$

Nonlinear scaling for deformation and microrotation

The other nonlinear scaled parameters:

- $\overline{Q}_e(x_1,x_2,x_3) = \overline{Q}_e^\natural(\zeta^{-1}(x_1,x_2,x_3))\,\forall x \in \Omega_h\,;$
  $\overline{Q}_e^\natural(\eta) = \overline{Q}_e(\zeta(\eta)), \forall \eta \in \Omega_1\,,$
- $(\nabla_x\Theta)^\natural(\eta) = (\nabla_x\Theta)(\zeta(\eta)),\ Q_0^\natural(\eta) = Q_0(\zeta(\eta)),\ U_0^\natural(\eta) = U_0(\zeta(\eta))\,,$
- $\overline{R}(x_1,x_2,x_3) = \overline{R}^\natural(\zeta^{-1}(x_1,x_2,x_3))\,\forall x \in \Omega_h\,;$
  $\overline{R}^\natural(\eta) = \overline{R}(\zeta(\eta))\,,\ \forall \eta \in \Omega_1\,,$
- $\overline{U}_e^\natural = \overline{Q}_e^{\natural,T} F_h^\natural[(\nabla_x\Theta)^\natural]^{-1} = \overline{Q}_e^{\natural,T}\nabla_\eta^h\varphi^\natural(\eta)[(\nabla_x\Theta)^\natural]^{-1}\,,$
- $$\Gamma_h^\natural = (\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\partial_{\eta_1}\overline{Q}_{e,h}^\natural)|\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_2}\overline{Q}_{e,h}^\natural)|\tfrac{1}{h}\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\partial_{\eta_3}\overline{Q}_{e,h}^\natural))[(\nabla_x\Theta)^\natural]^{-1},$$

where we recall that

$$\mathrm{axl}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathrm{Anti}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

and its inverse is denoted by Anti : $\mathbb{R}^3 \to \mathfrak{so}(3)$.

Transformation of the problem from Ω_h to Ω_1

In order to apply the Γ-convergence techniques, we need to transform the minimization problem to the fixed domain $\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}]$.

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$$
\begin{aligned}
I_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) &= \int_{\Omega_1} \Big(W_{\mathrm{mp}}(\overline{U}_h^\natural) + \widetilde{W}_{\mathrm{curv}}(\Gamma_h^\natural) \Big) \det(\nabla_\eta \zeta(\eta)) \det((\nabla_x \Theta)^\natural) dV_\eta \\
&= \int_{\Omega_1} \; h \left[\Big(W_{\mathrm{mp}}(\overline{U}_h^\natural) + \widetilde{W}_{\mathrm{curv}}(\Gamma_h^\natural) \Big) \det((\nabla_x \Theta)^\natural) \right] dV_\eta \\
&\qquad \mapsto \min \ \text{w.r.t} \ (\varphi^\natural, \overline{Q}_e^\natural) \, ,
\end{aligned}
$$

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$$= \int_{\Omega_1} h \left[\Big(W_{\mathrm{mp}}(\overline{U}_h^\natural) + \widetilde{W}_{\mathrm{curv}}(\Gamma_h^\natural) \Big) \det((\nabla_x \Theta)^\natural) \right] dV_\eta$$

$$\mapsto \min \ \text{w.r.t} \ (\varphi^\natural, \overline{Q}_e^\natural) \,,$$

where

$$W_{\mathrm{mp}}(\overline{U}_h^\natural) = \mu \, \|\mathrm{sym}(\overline{U}_h^\natural - \mathbb{1}_3)\|^2 + \mu_c \, \|\, \mathrm{skew}(\overline{U}_h^\natural - \mathbb{1}_3)\|^2 + \frac{\lambda}{2} [\mathrm{tr}(\mathrm{sym}(\overline{U}_h^\natural - \mathbb{1}_3))]^2 \,,$$

$$\widetilde{W}_{\mathrm{curv}}(\Gamma_h^\natural) = \mu \, L_c^2 \left(a_1 \, \|\mathrm{dev}\,\mathrm{sym}\, \Gamma_h^\natural\|^2 + a_2 \, \|\mathrm{skew}\, \Gamma_h^\natural\|^2 + a_3 \, [\mathrm{tr}(\Gamma_h^\natural)]^2 \right) .$$

Energy functionals

The family of energy functionals

$$\mathcal{I}_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) = \begin{cases} \dfrac{1}{h} I_h^\natural(\varphi^\natural, \nabla_\eta^h \varphi^\natural, \overline{Q}_e^\natural, \Gamma_h^\natural) & \text{if } (\varphi^\natural, \overline{Q}_e^\natural) \in \mathcal{S}', \\ +\infty & \text{else in } X. \end{cases}$$

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where

Admissible sets

$$\begin{aligned} X &:= \{(\varphi^\natural, \overline{Q}_e^\natural) \in \mathrm{L}^2(\Omega_1, \mathbb{R}^3) \times \mathrm{L}^2(\Omega_1, \mathrm{SO}(3))\}, \\ \mathcal{S}' &:= \{(\varphi, \overline{Q}_e) \in \mathrm{H}^1(\Omega_1, \mathbb{R}^3) \times \mathrm{H}^1(\Omega_1, \mathrm{SO}(3)) \mid \varphi|_{\Gamma_1}(\eta) = \varphi_d^\natural(\eta)\}. \end{aligned}$$

By letting $h \to 0$, we are searching for a Γ-limit of $\mathcal{I}_h$

$$\mathcal{I}_0(m, \overline{Q}_{e,0}) = \begin{cases} \int_\omega \left[W_{\mathrm{mp}}^{\mathrm{hom}}(\mathcal{E}_{m,\overline{Q}_{e,\mathbf{0}}}) + \widetilde{W}_{\mathrm{curv}}^{\mathrm{hom}}(\mathcal{K}_{e,s})\right] \det(\nabla y_0 | n_0)\, d\omega & \text{if } (m, \overline{Q}_{e,0}) \in \mathcal{S}'_\omega \\ +\infty & \text{else in } X_\omega, \end{cases}$$

where

Admissible sets

$$X_\omega := \{(\varphi, \overline{Q}_e) \in \mathrm{L}^2(\omega, \mathbb{R}^3) \times \mathrm{L}^2(\omega, \mathrm{SO}(3))\},$$
$$\mathcal{S}'_\omega := \{(\varphi, \overline{Q}_e) \in \mathrm{H}^1(\omega, \mathbb{R}^3) \times \mathrm{H}^1(\omega, \mathrm{SO}(3)) \,\big|\, \varphi|_{\partial\omega}(\eta_1, \eta_2) = \varphi_d^\natural(\eta_1, \eta_2, 0)\}\,.$$

The Cosserat shell model as Γ-limit

- M.M. Saem, E. Bulgariu, I.D. Ghiba, and P. Neff. Explicit formula for the Gamma-convergence homogenized quadratic curvature energy in isotropic Cosserat shell models. *Zeitschrift für angewandte Mathematik und Physik*, 76(2):76, 2025.
- M.M. Saem, I.D. Ghiba, and P. Neff. A geometrically nonlinear Cosserat (micropolar) curvy shell model via Gamma convergence. *Journal of Nonlinear Science*, 33(5):70, 2023.

Theorem

Assume that the initial configuration of the curved shell is defined by a continuous injective mapping $y_0 : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ which admits an extension to $\overline{\omega}$ into $C^2(\overline{\omega}; \mathbb{R}^3)$ such that for $\Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3 n_0(x_1, x_2)$ we have $\det[\nabla_x \Theta(0)] \geq a_0 > 0$ on $\overline{\omega}$, where a_0 is a constant, and assume that the boundary data satisfy the conditions

$$\varphi_d^\natural = \varphi_d\big|_{\Gamma_1} \quad \text{(in the sense of traces) for } \varphi_d \in \mathrm{H}^1(\Omega_1; \mathbb{R}^3).$$

Let the constitutive parameters satisfy

$$\mu > 0, \quad \kappa > 0, \quad \mu_c > 0, \quad a_1 > 0, \quad a_2 > 0, \quad a_3 > 0.$$

Then, for any sequence $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in X$ such that $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \to (\varphi_0, \overline{Q}_{e,0})$ as $h_j \to 0$, the sequence of functionals $\mathcal{I}_{h_j} : X \to \overline{\mathbb{R}}$ Γ-*converges* to the limit energy functional $\mathcal{I}_0 : X \to \overline{\mathbb{R}}$

Main result

$$\mathcal{I}_0(m,\overline{Q}_{e,0}) = \begin{cases} \int_\omega [W_{\rm mp}^{\rm hom}(\mathcal{E}_{m,\overline{Q}_{e,\mathbf{0}}}) + \widetilde{W}_{\rm curv}^{\rm hom}(\mathcal{K}_{e,s})] \det(\nabla y_0|n_0)\; d\omega; \\ \hfill (m,\overline{Q}_{e,0}) \in \mathcal{S}'_\omega\,, \\ +\infty \hfill \text{else in } X_\omega. \end{cases}$$

where

$$W_{\rm mp}^{\rm hom}(\mathcal{E}_{m,\overline{Q}_{e,\mathbf{0}}}) = W_{\rm shell}\big(\mathcal{E}^{\parallel}_{m,\overline{Q}_{e,\mathbf{0}}}\big) + \frac{2\,\mu\;\mu_{\rm c}}{\mu_c\;+\mu}\|\mathcal{E}^{\perp}_{m,\overline{Q}_{e,\mathbf{0}}}\|^2,$$

$$\widetilde{W}_{\rm curv}^{\rm hom}(\mathcal{K}_{e,s}) = \mu L_c^2\Big(b_1\|{\rm sym}\mathcal{K}^{\parallel}_{e,s}\|^2 + b_2\|\,{\rm skew}\,\mathcal{K}^{\parallel}_{e,s}\|^2 + \frac{b_1\,b_3}{(b_1+b_3)}{\rm tr}(\mathcal{K}^{\parallel}_{e,s})^2 + \frac{2\,b_1\,b_2}{b_1+b_2}\|\mathcal{K}^{\perp}_{e,s}\|\Big)\,,$$

Main result

$$\mathcal{I}_0(m, \overline{Q}_{e,0}) = \begin{cases} \int_\omega [W^{\rm hom}_{\rm mp}(\mathcal{E}_{m,\overline{Q}_{e,\mathbf{0}}}) + \widetilde{W}^{\rm hom}_{\rm curv}(\mathcal{K}_{e,s})] \det(\nabla y_0 | n_0)\ d\omega; \\ \qquad\qquad\qquad\qquad\qquad\qquad (m, \overline{Q}_{e,0}) \in \mathcal{S}'_\omega\ , \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad \text{else in } X_\omega. \end{cases}$$

where

$$W^{\rm hom}_{\rm mp}(\mathcal{E}_{m,\overline{Q}_{e,\mathbf{0}}}) = W_{\rm shell}\big(\mathcal{E}^{\parallel}_{m,\overline{Q}_{e,\mathbf{0}}}\big) + \frac{2\,\mu\ \mu_{\rm c}}{\mu_c\ + \mu}\|\mathcal{E}^{\perp}_{m,\overline{Q}_{e,\mathbf{0}}}\|^2,$$

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with $m(x_1, x_2) := \varphi_0(x_1, x_2) = \lim_{h_j\to 0} \varphi^{\natural}_{h_j}(x_1, x_2, \frac{1}{h_j}x_3)$

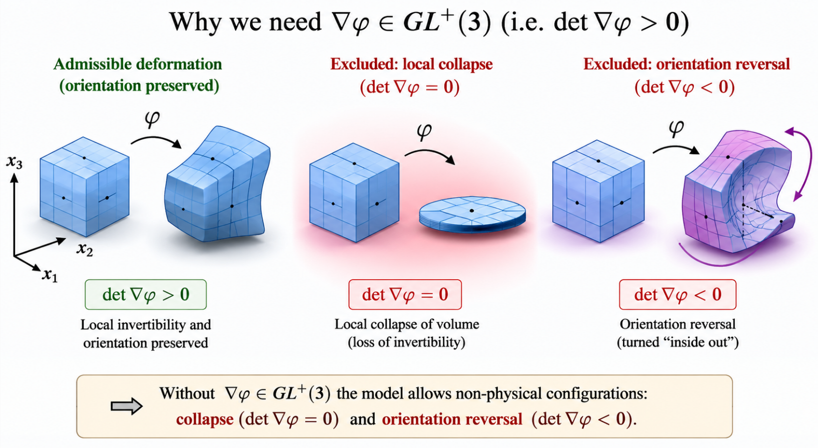
$$\overline{Q}_{e,0}(x_1, x_2) = \lim_{h_j\to 0} \overline{Q}^{\natural}_{e,h_j}(x_1, x_2, \frac{1}{h_j}x_3),\ \mathcal{E}_{m,\overline{Q}_{e,\mathbf{0}}} = (\overline{Q}^{T}_{e,0}\nabla m - \nabla y_0|0)[\nabla_x\Theta(0)]^{-1}$$

$$\mathcal{K}_{e,s} = \Big({\rm axl}(\overline{Q}^{T}_{e,0}\,\partial_{x_1}\overline{Q}_{e,0})\,|\,{\rm axl}(\overline{Q}^{T}_{e,0}\,\partial_{x_2}\overline{Q}_{e,0})\,|0\Big)[\nabla_x\Theta(0)\,]^{-1} \not\in {\rm Sym}(3)\,.$$

A limitation of the model

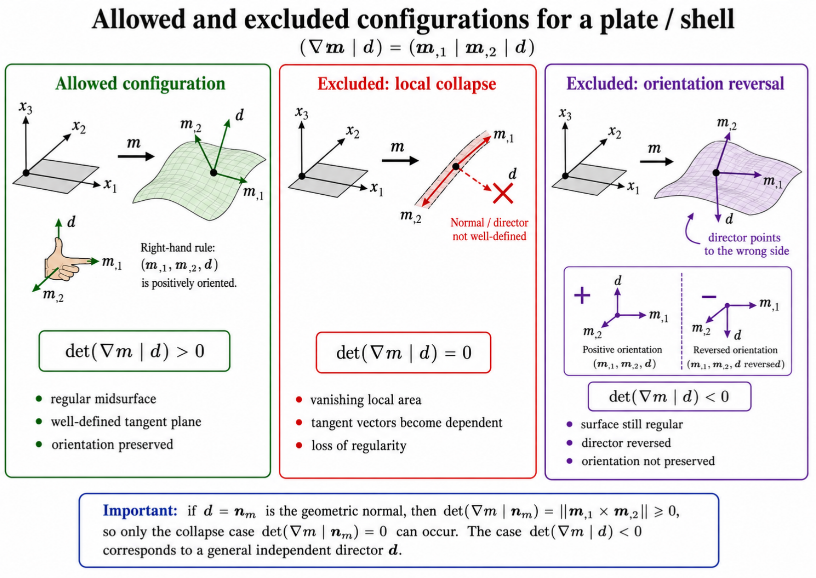
What we would like to have

The three-dimensional problem does not guarantee that $\nabla\varphi \in \mathrm{GL}^+(3)$.



Orientation preserving for shells

Similarly, the shell model does not guarantee that $(\nabla m \mid n_m) \in \mathrm{GL}^+(3)$.

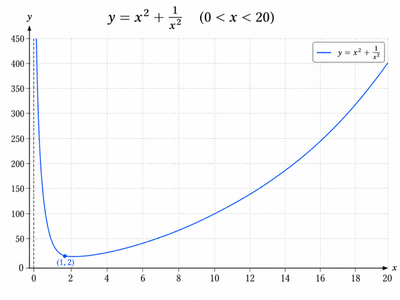


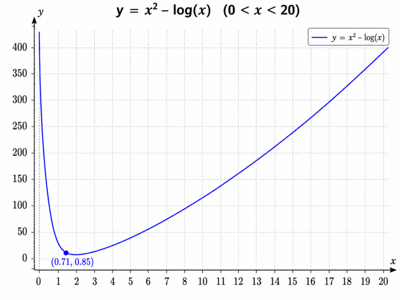
An idea to overcome this shortcoming

One could add some ad hoc terms to the final functional, such as

$$\frac{1}{\det(\nabla m \mid n_m)^2} + \det(\nabla m \mid n_m)^2 \qquad \text{or} \qquad [\det(\nabla m \mid n_m)]^2 - \log\det(\nabla m \mid n_m),$$

in order to penalize the loss of orientation.





Adding an ad hoc term could help for a particular fitting, but not as a general model

However, this is not the approach we pursue because we do not like the ad hoc terms not well motivated.

We first tried to understand how this idea works in classical nonlinear elasticity.

As a work in progress, we are extending the approach to the Cosserat case.

Another solution to the modelling problems

Use a 3D well-posed model which leads to a solution in $\mathrm{GL}^+(3)$.

There are not many!

One good choice is the Ciarlet-Geymonat energy

$$W_{\mathrm{CG}}(F_\xi) := \frac{\mu}{2}\left[\|F_\xi\|^2 - 2\log(\det F_\xi) - 3\right] + \frac{\lambda}{4}\left[(\det F_\xi)^2 - 2\log(\det F_\xi) - 1\right]$$

which is

- rich enough to retain essential three-dimensional effects, but
- still simple enough to permit a rigorous mathematical analysis.

The most important point is that the energy is polyconvex and therefore yields an existence result

- Ph.G. Ciarlet. *Quelques remarques sur les problèmes d'existence en élasticité non linéaire*. PhD thesis, INRIA, 1982.
- Ph.G. Ciarlet. *Three-Dimensional Elasticity.*, volume 1 of *Studies in Mathematics and its Applications*. Elsevier, Amsterdam, first edition, 1988.
- Ph.G. Ciarlet and G. Geymonat. Sur les lois de comportement en élasticité non linéaire compressible. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 295:423–426, 1982.

A Kirchhoff–Love shell model in the classical nonlinear elasticity

The three-dimensional parental problem and geometry of the referential configuration

The three dimensional deformation φ_ξ is a solution of the following *geometrically nonlinear minimization problem* posed on Ω_ξ:

$$\mathcal{I}(\varphi_\xi, F_\xi) = \int_{\Omega_\xi} W_{\rm CG}(F_\xi)\mathrm{d}\xi - \Pi(\varphi_\xi) \quad \to \ \text{min.} \quad \text{w.r.t.} \quad \varphi_\xi \,,$$

where

$$\begin{aligned}
F_\xi &:= \nabla_\xi \varphi_\xi \in \mathbb{R}^{3\times 3}, \\
W_{\rm CG}(F_\xi) &:= \frac{\mu}{2}\left[\|F_\xi\|^2 - 2\,\log(\det F_\xi) - 3\right] + \frac{\lambda}{4}\left[\,(\det F_\xi)^2 - 2\,\log(\det F_\xi) - 1\right] \\
\Pi(\varphi_\xi) &:= \Pi_f(\varphi_\xi) + \Pi_t(\varphi_\xi) = \text{the external loading potential}, \\
\Pi_f(\varphi_\xi) &:= \int_{\Omega_\xi} \langle f, u\rangle\,\mathrm{d}\xi \ = \text{potential of external applied body forces } f, \\
\Pi_t(\varphi_\xi) &:= \int_{\partial\Omega_t} \langle t, u\rangle\,\mathrm{d}\xi \ = \text{potential of external applied boundary forces } t\,,
\end{aligned}$$

and $\mathrm{d}\xi$ denotes the volume element or the area element, respectively in the reference configuration, $u = \varphi_\xi - \xi$ is the displacement vector.

Objective: A new shell theory motivated by good results for a similar plate theory

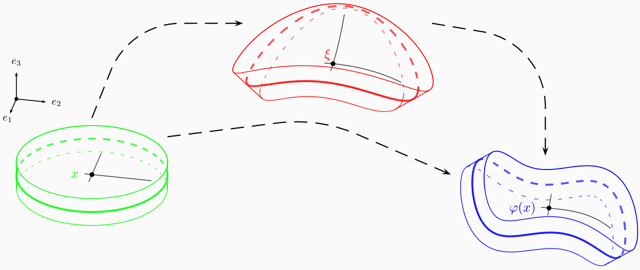


Figure 8: Kinematics of the 3D elastic model.

Following the classical Kirchhoff–Love kinematics, we assume that the deformation admits the representation

$$\varphi(x_1, x_2, x_3) = m(x_1, x_2) + x_3 \, n_m(x_1, x_2), \quad n_m = \frac{\partial_{x_1} m \times \partial_{x_2} m}{\|\partial_{x_1} m \times \partial_{x_2} m\|},$$

where $m : \omega \to \mathbb{R}^3$ is a regular mapping which defines the midsurface of the deformed configuration Ω_c. In the rest of the paper, the subscript m will indicate quantities associated with the deformed midsurface $m(\omega)$.

We have

$$\int_{-h/2}^{h/2} \|\nabla_\xi \varphi\|^2 \det \nabla\Theta(x', x_3)\mathrm{d}x_3$$
$$= \Big(h + \frac{h^3}{12}(-\mathrm{K}_{y_0}) + \frac{h^5}{80}(\mathrm{K}^2_{y_0})\Big)\mathcal{F}_0(\mathrm{I}_m) + \Big(-\frac{h^3}{3}\ \mathrm{H}_{y_0}\Big)\mathcal{F}_0(\mathrm{II}_m)$$
$$+ \Big(\frac{h^3}{12} - \frac{h^5}{80}\mathrm{K}_{y_0}\Big)\mathcal{F}_0(\mathrm{III}_m) + \Big(-\frac{h^5}{40}\mathrm{H}_{y_0}\mathrm{K}_{y_0}\Big)\mathcal{F}_1(\mathrm{I}_m)$$
$$+ \Big(\frac{h^3}{6} - \frac{h^5}{40}\mathrm{K}_{y_0}\Big)\mathcal{F}_1(\mathrm{II}_m) + \Big(\frac{h^3}{12} + \frac{h^5}{80}(4\,\mathrm{H}^2_{y_0} - \mathrm{K}_{y_0})\Big)\mathcal{F}_2(\mathrm{I}_m)$$
$$+ \Big(\frac{h^5}{20}\mathrm{H}_{y_0}\Big)\mathcal{F}_2(\mathrm{II}_m) + \frac{h^5}{80}\mathcal{F}_2(\mathrm{III}_m) + \Big(h + \frac{h^3}{12}\mathrm{K}_{y_0}\Big),$$

which provides an asymptotic direct approximation of the energy term $\|\nabla_\xi \varphi_\xi\|^2$ appearing in the three-dimensional Ciarlet-Geymonat energy, where the three linear operators $\mathcal{F}_i : \mathbb{R}^{2\times 2} \to \mathbb{R}$, $i = 0, 1, 2$ are

$$\mathcal{F}_0(Q) := \langle Q, \mathrm{I}^{-1}_{y_0}\rangle,$$
$$\mathcal{F}_1(Q) := \langle Q, \mathrm{L}_{y_0}\mathrm{I}^{-1}_{y_0} + \mathrm{I}^{-1}_{y_0}\mathrm{L}_{y_0}\rangle,$$
$$\mathcal{F}_2(Q) := \langle Q, \mathrm{L}^T_{y_0}\mathrm{I}^{-1}_{y_0}\mathrm{L}_{y_0}\rangle, \qquad Q \in \mathbb{R}^{2\times 2}.$$

We now turn to the approximation of the purely volumetric terms of the three-dimensional Ciarlet-Geymonat energy. To this end, we define

$$\mathrm{a}_m := \det(\nabla m | n_m), \qquad \mathrm{a}_{y_0} := \det(\nabla y_0 | n_{y_0})$$

and the differences between the curvature of the referential (undeformed) mid-surface defined by y_0 and the current (deformed) mid-surface defined by m

$$\Delta \mathrm{H}_m := \mathrm{H}_m - \mathrm{H}_{y_0}, \qquad \Delta \mathrm{K}_m := \mathrm{K}_m - \mathrm{K}_{y_0}.$$

Volumetric terms

Then, after multiplication with the Jacobian $\det \nabla\Theta(x', x_3) = \det(\nabla y_0 | n_{y_0})\, b(x_3)$ and integration through the thickness we obtain

$$\int_{-h/2}^{h/2} (\det \nabla_\xi \varphi)^2 \det \nabla\Theta(x', x_3)\, \mathrm{d}x_3$$

$$= \frac{\mathrm{a}_m^2}{\mathrm{a}_{y_0}} \Bigg[h + \frac{h^3}{12}\Big(\mathrm{K}_{y_0} + 4(\Delta \mathrm{H}_m)^2 + 2\Delta\mathrm{K}_m \Big) + \frac{h^5}{80}\Big(16\, \mathrm{H}_{y_0}^2 (\Delta\mathrm{H}_m)^2$$

$$- 8\, \mathrm{H}_{y_0} \Delta\mathrm{H}_m\, \Delta\mathrm{K}_m - 4\, \mathrm{K}_{y_0} \Delta\mathrm{H}_m)^2 + (\Delta\mathrm{K}_m)^2 \Big) + O(h^6) \Bigg],$$

which is the dimensional reduction of the pure volumetric term $(\det \nabla_\xi \varphi_\xi)^2$ appearing in the three-dimensional Ciarlet-Geymonat energy.

This destroys an essential mathematical structure

Although this approximation will not be used in the final model, we briefly present the dimensional reduction of the pure volumetric term $\log(\det F_\xi)$ appearing in the three-dimensional Ciarlet-Geymonat energy. This intermediate computation is useful for understanding the structure of the reduced energy and for assessing the lower semicontinuity properties of the resulting functional. More precisely, we would have

$$\int_{-h/2}^{h/2} \log(\det F_\xi) \det \nabla\Theta(x', x_3)\, \mathrm{d}x_3$$

$$= \mathrm{a}_{y_0}\Bigg[h \log\left(\frac{\mathrm{a}_m}{\mathrm{a}_{y_0}}\right) + \frac{h^3}{12}\left(\mathrm{K}_{y_0} \log\left(\frac{\mathrm{a}_m}{\mathrm{a}_{y_0}}\right) + \Delta\mathrm{K}_m - 2(\Delta\mathrm{H}_m)^2\right)$$

$$+ \frac{h^5}{80}\Bigg(-4(\Delta\mathrm{H}_m)^4 - \frac{32}{3}\,\mathrm{H}_{y_0}(\Delta\mathrm{H}_m)^3 + (2\,\mathrm{K}_{y_0} - 8\,\mathrm{H}_{y_0}^2)(\Delta\mathrm{H}_m)^2$$

$$+ 4\,\mathrm{H}_{y_0}\Delta\mathrm{H}_m\,\Delta\mathrm{K}_m + 4(\Delta\mathrm{H}_m)^2\Delta\mathrm{K}_m - \frac{1}{2}(\Delta\mathrm{K}_m)^2\Bigg)\Bigg].$$

This preserves an essential mathematical structure.

In contrast, these difficulties do not arise when Simpson's rule is used. Indeed, Simpson's integration rule leads to

$$\int_{-h/2}^{h/2} \log(\det F_\xi) \det \nabla\Theta(x', x_3)\, \mathrm{d}x_3$$

$$\approx \frac{h}{6}\Big[\mathrm{A}^-_{y_0}(\log(\mathrm{a}_m\, \mathrm{A}^-_m) - \log(\mathrm{a}_{y_0} \mathrm{A}^-_{y_0})) \; + 4(\log \mathrm{a}_m - \log \mathrm{a}_{y_0})$$

$$+ \mathrm{A}^+_{y_0}(\log(\mathrm{a}_m\, \mathrm{A}^+_m) - \log(\mathrm{a}_{y_0} \mathrm{A}^+_{y_0}))\Big],$$

where

$$\mathrm{A}^{\pm}_m = 1 \mp h\, \mathrm{H}_m + \frac{h^2}{4}\, \mathrm{K}_m, \qquad \mathrm{A}^{\pm}_{y_0} = 1 \mp h\, \mathrm{H}_{y_0} + \frac{h^2}{4}\, \mathrm{K}_{y_0}.$$

This formula requires only three evaluations: on the midsurface and on the upper and lower faces of the shell. It is exact for all polynomials of degree ≤ 3, and its error is of the order $\mathrm{O}(h^5)$. We further remark that the reduced shell energy obtained from $\log\det F_\xi$ by means of Simpson's rule can be expressed in terms of the quantities a_m and $\mathrm{a}_m\, \mathrm{A}^{\pm}_m$, which will make the functional lower semicontinuous.

For the sake of a unified treatment, Simpson's rule may also be applied to all volumetric terms, including $(\det F_\xi)^2$, leading to the approximation

$$\int_{-h/2}^{h/2} (\det \nabla_\xi \varphi)^2 \det \nabla\Theta(x', x_3)\,\mathrm{d}x_3$$
$$\approx \mathrm{a}_{y_0}\Big[\mathrm{A}^-_{y_0}\Big(\frac{\mathrm{a}_m \mathrm{A}^-_m}{\mathrm{a}_{y_0}\mathrm{A}^-_{y_0}}\Big)^2 + 4\,\frac{\mathrm{a}_m^2}{\mathrm{a}_{y_0}^2} + \mathrm{A}^+_{y_0}\Big(\frac{\mathrm{a}_m \mathrm{A}^+_m}{\mathrm{a}_{y_0}\mathrm{A}^+_{y_0}}\Big)^2\Big],$$

I.D. Ghiba, T.H. Giang, C. Ureche, Nonlinear Kirchhoff-Love shell models derived from the Ciarlet-Geymonat energy: modelling and well-posedness - arXiv preprint arXiv:2603.18164, 2026.

Collecting the results of the previous subsections, we arrive at the following two-dimensional minimization problem for the deformation of the middle surface $m : \omega \to \mathbb{R}^3$. Let $\omega \subset \mathbb{R}^2$, we consider the variational problem of minimizing the functional

$$\mathcal{J}_1(m) := \int_\omega \Big[W^{(1)}_{\rm shell}(\mathrm{I}_m, \mathrm{II}_m, \mathrm{III}_m) + W_{\rm curv}(\mathrm{a}_m, \mathrm{a}_m \mathrm{A}^{\pm}_m) \Big] \det(\nabla y_0 | n_{y_0})\, \mathrm{d}x' - \mathcal{L}(m, n_m)\,,$$

where

$$\begin{aligned} W^{(1)}_{\rm shell}(\mathrm{I}_m, \mathrm{II}_m, \mathrm{III}_m) := \frac{\mu}{2} \Big\{ & \Big(h + \frac{h^3}{12}(-\mathrm{K}_{y_0}) + \frac{h^5}{80}(\mathrm{K}^2_{y_0})\Big)\,\mathcal{F}_0(\mathrm{I}_m) + \Big(-\frac{h^3}{3}\,\mathrm{H}_{y_0}\Big)\,\mathcal{F}_0(\mathrm{II}_m) \\ & + \Big(\frac{h^3}{12} - \frac{h^5}{80}\,\mathrm{K}_{y_0}\Big)\,\mathcal{F}_0(\mathrm{III}_m) + \Big(-\frac{h^5}{40}\,\mathrm{H}_{y_0}\,\mathrm{K}_{y_0}\Big)\,\mathcal{F}_1(\mathrm{I}_m) \\ & + \Big(\frac{h^3}{6} - \frac{h^5}{40}\,\mathrm{K}_{y_0}\Big)\,\mathcal{F}_1(\mathrm{II}_m) + \Big(\frac{h^3}{12} + \frac{h^5}{80}(4\,\mathrm{H}^2_{y_0} - \mathrm{K}_{y_0})\Big)\,\mathcal{F}_2(\mathrm{I}_m) \\ & + \Big(\frac{h^5}{20}\,\mathrm{H}_{y_0}\Big)\,\mathcal{F}_2(\mathrm{II}_m) + \Big(\frac{h^5}{80}\Big)\,\mathcal{F}_2(\mathrm{III}_m) + \Big(h + \frac{h^3}{12}\,\mathrm{K}_{y_0}\Big)\Big\} - \frac{3\,\mu}{2}, \end{aligned}$$

and

$$W_{\mathrm{curv}}(\mathrm{a}_m, \mathrm{a}_m \mathrm{A}_m^{\pm}) := -\frac{\lambda + 2\,\mu}{4}\left[\frac{h}{6}\Big[\mathrm{A}_{y_0}^{-}\big(\log(\mathrm{a}_m \mathrm{A}_m^{-}) - \log(\mathrm{a}_{y_0}\mathrm{A}_{y_0}^{-})\big)\right.$$
$$\left. + 4\log\frac{\mathrm{a}_m}{\mathrm{a}_{y_0}} + \mathrm{A}_{y_0}^{+}\big(\log(\mathrm{a}_m \mathrm{A}_m^{+}) - \log(\mathrm{a}_{y_0}\mathrm{A}_{y_0}^{+})\big)\right]$$
$$+ \frac{\lambda}{4}\Big\{\mathrm{a}_{y_0}\Big[\mathrm{A}_{y_0}^{-}\left(\frac{\mathrm{a}_m \mathrm{A}_m^{-}}{\mathrm{a}_{y_0}\mathrm{A}_{y_0}^{-}}\right)^2 + 4\,\frac{\mathrm{a}_m^2}{\mathrm{a}_{y_0}^2} + \mathrm{A}_{y_0}^{+}\left(\frac{\mathrm{a}_m \mathrm{A}_m^{+}}{\mathrm{a}_{y_0}\mathrm{A}_{y_0}^{+}}\right)^2\Big] - \frac{\lambda}{4}.$$

Model II up to order $\mathrm{O}(h^5)$

Again, we would like to have a model which retains terms up to order $\mathrm{O}(h^5)$. In particular, we present another variant of the model, in which the exact Taylor expansion is used for the volumetric term $(\det F_\xi)^2$ instead of Simpson's rule. The corresponding model is given by the following variational formulation: minimize with respect to m the functional

$$\mathcal{J}_3(m) := \int_\omega \Big[W^{(1)}_{\mathrm{shell}}\big(\mathrm{I}_m, \mathrm{II}_m, \mathrm{III}_m\big) + W^{(1)}_{\mathrm{curv}}(\mathrm{a}_m, \mathrm{a}_m \mathrm{A}^{\pm}_m) + W^{(2)}_{\mathrm{curv}}(\mathrm{a}_m, \mathrm{a}_m \Delta\,\mathrm{H}_m, \mathrm{a}_m \Delta\,\mathrm{K}_m)\Big] \ \det(\nabla y_0 | n_{y_0})\,\mathrm{d}x' - \mathcal{L}(m, n_m)\,,$$

where $W^{(1)}_{\mathrm{shell}}\big(\mathrm{I}_m, \mathrm{II}_m, \mathrm{III}_m\big)$ is the same and

$$W^{(2)}_{\mathbf{curv}}(\mathrm{a}_m, \mathrm{a}_m \Delta\mathrm{H}_m, \mathrm{a}_m \Delta\mathrm{K}_m) := \frac{\lambda}{4}\Big\{ \mathrm{a}_{y_0}\Big(\frac{\mathrm{a}_m}{\mathrm{a}_{y_0}}\Big)^2 \Big[h + \frac{h^3}{12}(\,\mathrm{K}_{y_0} + 4(\Delta\mathrm{H}_m)^2 + 2\Delta\mathrm{K}_m) + \frac{h^5}{80}\Big(16\,\mathrm{H}^2_{y_0}(\Delta\mathrm{H}_m)^2 - 8\,\mathrm{H}_{y_0}\Delta\mathrm{H}_m\Delta\mathrm{K}_m - 4\,\mathrm{K}_{y_0}(\Delta\mathrm{H}_m)^2 + (\Delta\mathrm{K}_m)^2\Big)\Big] - 1\Big\}.$$

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Lemma: Weak convergences
S. Anicic, Discrete and Continuous Dynamical Systems S (2019)

Let $\omega \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz boundary. Let $y_0 \in C^2(\bar{\omega}, \mathbb{R}^3)$ be such that $\partial_{x_1} y_0(x')$ and $\partial_{x_2} y_0(x')$ are linearly independent for every $x' \in \bar{\omega}$ and let γ_{y_0} be a nonempty relatively open subset of $\partial\omega$. For $h > 0$, $p \geq 2$ and $q > 1$, where h satisfies (**??**), we define

$$V^h := \Big\{ m \in W^{1,p}(\omega;\mathbb{R}^3) \,\Big|\, \mathrm{a}_m \in L^q(\omega),\ \ \mathrm{a}_m > 0 \text{ a.e. in } \omega,$$

$$n_m \in W^{1,p}(\omega;\mathbb{R}^3), \quad \frac{h}{2}\max\{1/R_1(m),\, 1/R_2(m)\} < 1 \text{ a.e. in } \omega,$$

$$m = y_0 \quad \text{and} \quad n_m = n_{y_0}\ ds\text{-a.e. in } \gamma_{y_0} \Big\},$$

where $R_1(m)$ and $R_2(m)$ denote the principal radii of curvature[2] of the deformed surface $m(\omega)$.

[2] The condition $\frac{h}{2}\max\{1/R_1(m),\, 1/R_2(m)\} < 1$ a.e. in ω implies that $A_m^{\pm} > 0$ a.e. in ω

Lemma: Weak convergences

Assume that (m_k) is a sequence with $m_k \in V^h$ for all k, for which there exist $m \in W^{1,p}(\omega;\mathbb{R}^3)$, $\varkappa \in W^{1,p}(\omega;\mathbb{R}^3)$, $(\xi_1,\xi_2,\xi_3) \in (L^q(\omega;\mathbb{R}^3))^3$ and $(\alpha_1,\alpha_2,\alpha_3) \in (L^q(\omega))^3$ such that

$$\begin{aligned}
m_k &\rightharpoonup m \quad \text{in } W^{1,p}(\omega;\mathbb{R}^3), & n_{m_k} &\rightharpoonup \varkappa \quad \text{in } W^{1,p}(\omega;\mathbb{R}^3),\\
\partial_1 m_k \wedge \partial_2 m_k &\to \xi_1 \quad \text{in } L^q(\omega;\mathbb{R}^3), & \mathrm{a}_{m_k} &\to \alpha_1 \quad \text{in } L^q(\omega),\\
\mathrm{H}_{m_k}\partial_1 m_k \wedge \partial_2 m_k &\rightharpoonup \xi_2 \quad \text{in } L^q(\omega;\mathbb{R}^3), & \mathrm{H}_{m_k}\mathrm{a}_{m_k} &\rightharpoonup \alpha_2 \quad \text{in } L^q(\omega),\\
\mathrm{K}_{m_k}\partial_1 m_k \wedge \partial_2 m_k &\rightharpoonup \xi_3 \quad \text{in } L^q(\omega;\mathbb{R}^3), & \mathrm{K}_{m_k}\mathrm{a}_{m_k} &\rightharpoonup \alpha_3 \quad \text{in } L^q(\omega).
\end{aligned}$$

Assume further that

$$\mathrm{a}_m > 0 \quad \text{a.e. in} \quad \omega.$$

Then, almost everywhere in ω, we have

$$\begin{aligned}
\varkappa &= n_m, & \xi_1 &= \partial_1 m \wedge \partial_2 m,\\
\xi_2 &= \mathrm{H}_m \partial_1 m \wedge \partial_2 m, & \xi_3 &= \mathrm{K}_m \partial_1 m \wedge \partial_2 m,
\end{aligned}$$

$$\alpha_1 = \mathrm{a}_m, \qquad \alpha_2 = \mathrm{H}_m\mathrm{a}_m \qquad \text{and} \qquad \alpha_3 = \mathrm{K}_m\mathrm{a}_m.$$

Polyconvexity for shells

1. **Polyconvexity:** For almost all $x' \in \omega$, there exists a convex function $\mathbb{W}(x', \cdot) : \mathrm{M} \to \mathbb{R}$ where
$$\mathrm{M} := \left\{ (A, B, a, b, c) \in (\mathbb{M}^{3\times 2})^2 \times \mathbb{R}^3;\ a - \frac{h}{2}|b| > 0 \text{ and } a - h|b| + \frac{h^2}{4} c > 0 \right\}$$
such that for almost all $x \in \omega$
$$W(x', m) = \mathbb{W}(x', \nabla m(x'), \nabla n_m(x'), (1,\ \mathrm{H}_m(x'),\ \mathrm{K}_m(x'))\mathrm{a}_m(x')).$$
2. **Measurability:** The function $\mathbb{W}(\cdot, A, B, a, b, c) : \omega \to \mathbb{R}$ is measurable for all $(A, B, a, b, c) \in \mathrm{M}$.
3. **Coerciveness:** There exist constants $C_1 > 0$ and C_2 such that
$$W(x', m) \geq C_1\{|\nabla m|^p + |\nabla n_m|^q + \mathrm{a}_m^q\} + C_2$$
for all $\psi \in V^\varepsilon$ and almost all $x' \in \omega$.
4. **Orientation-preserving condition:**
$$\begin{aligned} W(x', m) &\to +\infty \quad \text{as} \quad \mathrm{a}_m(x') \to 0^+, \\ W(x', m) &\to +\infty \quad \text{as} \quad \mathrm{A}_m^+(x') \to 0^+, \\ W(x', m) &\to +\infty \quad \text{as} \quad \mathrm{A}_m^-(x') \to 0^+ \end{aligned}$$
for all $m \in V^h$ and almost all $x' \in \omega$.

Assume that $\inf_{m \in V^h} \mathcal{I}(m) < +\infty$, then there exists at least one function $m^* \in V^\varepsilon$ such that $\mathcal{I}(m^*) = \inf_{m \in V^h} \mathcal{I}(m)$.

Existence for the constructed models
I.D. Ghiba, T.H. Giang, C. Ureche (2026)

Theorem

Let $\omega \subset \mathbb{R}^2$ *be an open bounded domain with Lipschitz boundary. Let* $y_0 \in C^2(\bar{\omega}, \mathbb{R}^3)$ *be such that* $\partial_{x_1} y_0(x')$ *and* $\partial_{x_2} y_0(x')$ *are linearly independent for every* $x' \in \bar{\omega}$*. Then, there exists* $h_0 > 0$ *such that for every* $0 < h < h_0$*, the functionals defining our models have minimizers in* V^h*.*

Thank you for your attention

The slide are already available at
https://www.math.uaic.ro/ ghiba/activities.html