Seminar 16

Exerciții (curs 8)

I.6. Fie A, B două submulț. nevide ale unui spațiu normat X. Fie suma lor $A+B = \{a+b;\ a\in A, b\in B\}$. Ar. că:

i) Dacă A și B închise și cel puțin una compactă, at. $A+B$ închisă. Suma a două mulț. închise e în general mulț. închisă?

A compactă $\Leftrightarrow$ $\boxed{A \text{ secvențial compactă}}$ $(\forall x_n \subset A, \exists x_{n_k}$ conv. în $A)$

O mulțime M este închisă $\Leftrightarrow \forall y_n \subset M$ a.î. $y_n \longrightarrow y \Rightarrow y \in M$.

Fie $z_n \subset A+B \Rightarrow z_n = a_n + b_n$, $a_n \in A$, $b_n \in B$ a.î. $\exists \lim\limits_{n\to\infty} z_n = z$.

$$\left.\begin{array}{l} a_n \in A \\ A \text{ compactă} \end{array}\right\} \Rightarrow \exists a_{n_k} \text{ convergent în } A\ (\exists \lim_{k\to\infty} a_{n_k} = a \in A).$$

$$\left.\begin{array}{l} z_{n_k} = a_{n_k} + b_{n_k} \Rightarrow b_{n_k} = z_{n_k} - a_{n_k} \Rightarrow \exists \lim\limits_{k\to\infty} b_{n_k} = \lim\limits_{k\to\infty} (z_{n_k} - a_{n_k}) = z - a, \\ b_{n_k} \in B,\ B \text{ - închisă} \end{array}\right\} \Rightarrow$$

$z - a \in B \Rightarrow z - a = b$, $b \in B$.

$\Rightarrow z = a + b \in A+B$. $\Rightarrow$ $A+B$ închisă

Suma a două mulț. închise e în general mulț. închisă?

- NU.

contraexemplu: $A, B \neq \emptyset$, $A, B \subset X$

$$\left.\begin{array}{l} A = [1, +\infty) \\ B = (-\infty, 0] \end{array}\right\} \Rightarrow A+B \subset \mathbb{R}$$

$z_n = n + (1-n)$

$a_n = n \in A$

$b_n = 1-n$

ii) $\left.\begin{array}{l}A - \text{comp}\\ B - \text{comp}\end{array}\right\} \Rightarrow A+B - \text{comp}$

$z_n \in A+B \Rightarrow z_n = a_n + b_n$, $a_n \in A$, $b_n \in B$

$a_n \in A$, A - compactă $\Rightarrow \exists\, a_{n_k}$ a.î. $\exists \lim_{k\to\infty} a_{n_k} = a$, $a \in A$

$b_{n_k} \in B$, B compactă $\Rightarrow \exists\, b_{n_{k_l}}$ a.î. $\exists \lim_{l\to\infty} b_{n_{k_l}} = b \in B$.

$z_{n_{k_l}} = \underset{\substack{\downarrow\\ a\in A}}{a_{n_{k_l}}} + \underset{\substack{\downarrow\\ b\in B}}{b_{n_{k_l}}} \xrightarrow{l\to\infty} a+b \in A+B$. $\Rightarrow$ secvențial compactă

$\Rightarrow$ compactă.

I.3. $A = \{(x,y) \mid x^2+y^2 \le 4,\ y \ge x+1\}$

~~$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \{ 0, (x,y) = (0,0)$~~

$$f:\mathbb{R}^2\to\mathbb{R},\ f(x,y) = \begin{cases} \dfrac{x\sin y + y\sin x}{\sqrt{x^2+y^2}}, & (x,y)\ne(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

a) A compactă $\Leftrightarrow$ A închisă și mărginită

din $x^2+y^2 \le 2^2 \Rightarrow A \subseteq S((0,0), r=3) \Rightarrow A$ mărginită

$h: \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) = x^2+y^2$

$g: \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = y-x$

$A = \{(x,y) \mid (h(x,y) \le 4) \wedge (g(x,y) \le 1)\} = h^{-1}(-\infty, 4] \cap g^{-1}[1,+\infty)$

h, g continue (fct. elem)

închis $\cap$ închis $\Rightarrow$ închis

$\Rightarrow$ compactă

b) f cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (compunere de fct. elem)

Studiem cont. în $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{x\sin y + y\sin x}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\text{mărg}} \overset{\to 0}{\sin y} + \lim_{(x,y)\to(0,0)} \underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{\text{mărg}} \overset{\to 0}{\sin x} = 0 = f(0,0)$$

$\Rightarrow f$ cont în $(0,0)$

$\Rightarrow f$ cont pe $\mathbb{R}^2$

c) uniform cont?

Știm că o fcț. cont. pe un compact este uniform continuă pe acel compact, deci fcț. f este u.c. pe A.

d) compactă?

Știm că o fcț. cont. duce compact în compact, deci $f(A)$ compact.

(Curs 9)

22.

$f:(a,b)\to\mathbb{R}$ cont pe (a,b)

$\exists \lim\limits_{x\to a, x>a} f(x)\in\mathbb{R}$

$\exists \lim\limits_{x\to a, x<a} f(x)\in\mathbb{R}$

Ar. că f este uniform continuă pe (a,b).

caz I: $a,b\in\mathbb{R}$.

putem prelungi prin continuitate f la $[a,b]$

$$\tilde{f}=\begin{cases}\lim\limits_{x\to a} f(x), & x=a\\ f(x), & x\in(a,b)\\ \lim\limits_{x\to b} f(x), & x=b\end{cases}$$

$$\left.\begin{array}{l}\tilde{f}:[a,b]\to\mathbb{R}\\ \tilde{f} \text{ cont pe } [a,b]\\ [a,b] \text{ compactă}\end{array}\right\}\Rightarrow \begin{array}{c}\tilde{f} \text{ u.c. pe } [a,b]\\ \Downarrow\\ f \text{ e u.c. pe } (a,b).\end{array}$$

caz II: $(a,b)=(-\infty,b)$, $b\in\mathbb{R}$

~ Dreapta reală încheiată se notează cu $\overline{\mathbb{R}}=[-\infty,+\infty]$.

$\overline{\mathbb{R}}$ - mulț. compactă

putem prelungi f la $\tilde{f}:[-\infty,b]\to\mathbb{R}$ → compactă

$$\tilde{f}=\begin{cases}\lim\limits_{x\to-\infty} f(x), & x=-\infty\\ f(x), & x\in(-\infty,b)\\ \lim\limits_{x\to b} f(x), & x=b.\end{cases}$$

$$\left.\begin{array}{l}\tilde{f} \text{ cont. pe } [-\infty, b] \\ [-\infty, b] \text{ compactă}\end{array}\right\} \Rightarrow \tilde{f} \text{ uniform cont.}$$

$\Rightarrow f$ e u.c. pe $(-\infty, b)$.

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$f: \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

1) $f(x,y) = (x^2+y^2) \operatorname{arctg} \frac{x}{y}$

i) $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[(x^2+y^2) \cdot \frac{\partial}{\partial x} \operatorname{arctg} \frac{x}{y}\right] = \frac{\partial}{\partial x}(x^2+y^2) \operatorname{arctg} \frac{x}{y} + (x^2+y^2) \frac{\partial}{\partial x} \operatorname{arctg} \frac{x}{y}$$

$$= 2x \operatorname{arctg} \frac{x}{y} + (x^2+y^2) \cdot \frac{\frac{\partial}{\partial x} \frac{x}{y}}{1+\left(\frac{x}{y}\right)^2}$$

$$= 2x \operatorname{arctg} \frac{x}{y} + (x^2+y^2) \frac{\frac{1}{y}}{\frac{x^2+y^2}{y^2}} = 2x \operatorname{arctg} \frac{x}{y} + y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[(x^2+y^2) \operatorname{arctg} \frac{x}{y}\right] = \frac{\partial}{\partial y}(x^2+y^2) \operatorname{arctg} \frac{x}{y} + (x^2+y^2) \cdot \frac{\partial}{\partial y} \operatorname{arctg} \frac{x}{y}$$

$$= 2y \operatorname{arctg} \frac{x}{y} + (x^2+y^2) \cdot \frac{\frac{\partial}{\partial y}\left(\frac{x}{y}\right)}{1+\left(\frac{x}{y}\right)^2}$$

$$= 2y \operatorname{arctg} \frac{x}{y} + (x^2+y^2) \cdot \frac{(-x)}{y^2} \cdot \frac{y^2}{x^2+y^2} = 2y \operatorname{arctg} \frac{x}{y} - x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x \operatorname{arctg} \frac{x}{y} + y\right)$$

$$= \frac{\partial}{\partial x}(2x) \operatorname{arctg} \frac{x}{y} + 2x \frac{\partial}{\partial x} \operatorname{arctg} \frac{x}{y} + 0$$

$$= 2 \operatorname{arctg} \frac{x}{y} + 2x \frac{\frac{\partial}{\partial x}\left(\frac{x}{y}\right)}{1+\left(\frac{x}{y}\right)^2}$$

$$= 2 \operatorname{arctg} \frac{x}{y} + 2y \frac{\frac{-x}{y^2}}{\frac{x^2+y^2}{y^2}}$$

$$= 2 \operatorname{arctg} \frac{x}{y} - \frac{2xy}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2y \operatorname{arctg}\frac{x}{y} - x\right)$$

$$= 2y \, \frac{\frac{\partial}{\partial x} \frac{x}{y}}{1+\left(\frac{x}{y}\right)^2} - \frac{\partial}{\partial x} x = 2y \cdot \frac{\frac{1}{y}}{\frac{x^2+y^2}{y^2}} - 1$$

$$= \frac{2y^2}{x^2+y^2} - 1 = \frac{-x^2+y^2}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x \operatorname{arctg}\frac{x}{y} + y\right) = 2x \frac{\partial}{\partial x} \operatorname{arctg}\frac{x}{y} + \frac{\partial}{\partial y} y$$

$$= 2x \, \frac{\frac{\partial}{\partial y} \cdot \frac{x}{y}}{1+\left(\frac{x}{y}\right)^2} + 1 = \ldots = \frac{-x^2+y^2}{x^2+y^2}$$

ii) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy}{\sqrt{x^2+y^2}}\right) = \frac{\frac{\partial}{\partial x}(xy) \cdot \frac{1}{\sqrt{x^2+y^2}} - xy \cdot \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$= \frac{y\sqrt{x^2+y^2} - xy \cdot \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{x^2+y^2}$$

$$= \frac{y\sqrt{x^2+y^2} - xy \, \frac{\frac{\partial}{\partial x}(x^2+y^2)}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{y\sqrt{x^2+y^2} - (xy) \cdot \frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\overset{\sqrt{x^2+y^2}}{y\sqrt{x^2+y^2}} - \frac{x^2 y}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{y(x^2+y^2) - x^2 y}{\sqrt{x^2+y^2}} \cdot \frac{1}{x^2+y^2} = \frac{y^3}{\sqrt{x^2+y^2}\,(x^2+y^2)}$$

obs. că $f(x,y) = f(y,x)$ (f sim)

trecem $\begin{matrix} x \to y \\ y \to x \end{matrix}$ în derivata calculată anterior

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{x^3}{\sqrt{y^2+x^2}} \cdot \frac{1}{y^2+x^2}$$

iii) $f(x,y,z) = x^y + y^z + z^x$

Dacă facem permutarea circulară $x \to y \to z \to x$.

$\Rightarrow f(y,z,x) = y^z + z^x + x^y = f(x,y,z) \Rightarrow f$ - invariantă la permutări ciclic

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^y + y^z + z^x) = \frac{\partial}{\partial x} x^y + \frac{\partial}{\partial x} y^z + \frac{\partial}{\partial x} z^x = yx^{y-1} + 0 + z^x \ln z$$

$$= yx^{y-1} + z^x \ln z$$

$x \to y \to z \to x \Rightarrow \frac{\partial f}{\partial y} = zy^{z-1} + x^z \ln x$

$y \to z \to x \to y \Rightarrow \frac{\partial f}{\partial z} = xz^{x-1} + y^x \ln y$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(yx^{y-1} + z^x \ln z) = y(y-1)x^{y-2} + z^x \ln^2 z$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(zy^{z-1} + x^z \ln x) = 0 + z \cdot x^{z-1} \ln x + x^{z-1}$$

$$= x^{z-1}(z \ln x + 1)$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2}{\partial x \partial y}(xz^{x-1} + y^x \ln y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(xz^{x-1} + y^x \ln y\right)\right)$$

$$= \frac{\partial}{\partial x}\left(0 + xy^{x-1} \cdot \ln y + \frac{y^x}{y}\right) = \frac{\partial}{\partial x}\left(y^{x-1}(\ln y + 1)\right)$$

$$= \frac{\partial}{\partial x}(y^{x-1})(\ln y + 1) + y^{x-1} \frac{\partial}{\partial x}(\ln y + 1)$$

$$= y^{x-1} \ln y (\ln y + 1) + y^{x-1} \cdot \ln y$$

$$= y^{x-1} \ln y (\ln y + 1) + y^{x-1} \ln y$$

$$\frac{\partial f}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2} + z^x \ln^2 z$$ $\Rightarrow$ facem permutare circulară $x \to y \to z \to x$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = z(z-1)^{z-2} y + x^y \ln^2 x$$

$$\Rightarrow \frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial^2}{\partial x^2}\left(z(z-1)y^{z-2} + x^y \ln^2 x\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(z(z-1)y^{z-2} + x^y \ln^2 x\right)\right)$$

$$= \frac{\partial}{\partial x}\left(y x^{y-1}\ln x + x^{y-1}\right) = y(y-1)x^{y-2}\ln x + y x^{y-1}\cdot\frac{1}{x} +$$

$$+ (y-1)x^{y-2}$$

$$= x^{y-2}\left(y(y-1)\ln x + y + y - 1\right)$$

N) $f(x,y,z) = x^{y^z}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^{y^z}\right) = y^z \cdot x^{y^z - 1}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^{y^z}\right) = x^{y^z} \ln x \cdot \frac{\partial}{\partial y}\left(y^z\right)$$

$$= \ln x^{y^z} \cdot z \cdot y^{z-1}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^{y^z}\right) = x^{y^z}\cdot \ln x \cdot \frac{\partial}{\partial z} y^z = \ln x \cdot x^{y^z} \cdot y^z \ln y$$

vi) $f(x,y) = x\sin(x^2+y^2)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\sin(x^2+y^2)) = \sin(x^2+y^2) + x\cdot\frac{\partial}{\partial x}\sin(x^2+y^2)$$

$$= \sin(x^2+y^2) + x\cos(x^2+y^2)\cdot 2x$$

$$= \sin(x^2+y^2) + 2x^2\cos(x^2+y^2)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\sin(x^2+y^2)) = x\frac{\partial}{\partial y}\sin(x^2+y^2) = x\cos(x^2+y^2)\cdot 2y$$

$$\frac{\partial^4 f}{\partial x^3\partial y} = \frac{\partial^3}{\partial x^3}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^3}{\partial x^3}(x\cos(x^2+y^2)2y)$$

$$= \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial x}((x\cos(x^2+y^2)\ 2y)\right)$$

$$= 2y\frac{\partial^2}{\partial x^2}(\cos(x^2+y^2) - x\sin(x^2+y^2)\cdot 2x)$$

$$= 2y\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(\cos(x^2+y^2) - 2x^3\sin(x^2+y^2))\right)$$

$$= 2y\frac{\partial}{\partial x}(-\sin(x^2+y^2)\cdot 2x - 6x^2\sin(x^2+y^2) - 2x^3\cos(x^2+y^2)2x)$$

$$= 2y\frac{\partial}{\partial x}(-2x\sin(x^2+y^2) - 6x^2\sin(x^2+y^2) - 4x^4\cos(x^2+y^2))$$

$$\cancel{= 2y(-2\sin(x^2+y^2) - 2x\cos(x^2+y^2)\cdot 2x - 16x^3\cos(x^2+y^2)}$$

$$= 2y(-2\sin(x^2+y^2) - 2x\cos(x^2+y^2)2x - 12x\sin(x^2+y^2) - 6x^2\cos(x^2+y^2)\cdot 2x$$
$$- 16x^3\cos(x^2+y^2) + 4x^4\sin(x^2+y^2))$$

$$= 2y(-2\sin(x^2+y^2) - 12x\sin(x^2+y^2) + 4x^4\sin(x^2+y^2) - 4x^2\cos(x^2+y^2) -$$
$$- 12x^3\cos(x^2+y^2) - 16x^3\cos(x^2+y^2).$$

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7. $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2yz$

$a = (1,1,0)$

$u = (1,0,-3)$

$\frac{\partial f}{\partial u}(a) = ?$

$$\lim_{t\to 0} \frac{f(a+tu)-f(a)}{t}$$

$$a+tu = (1,1,0) + t(1,0,-3)$$
$$= (1+t, 1, -3t)$$

$$f(a+tu) = f(1+t,1,-3t) = (1+t)^2(-3t) = (t^2+2t+1)(-3t)$$

$$\lim_{t\to 0} \frac{f(a+tu)-f(a)}{t} = \lim_{t\to 0} \frac{(t^2+2t+1)(-3\cancel{t})-0}{\cancel{t}} = -3$$

8. $f: \mathbb{R}^k \to \mathbb{R}$, $f(x) = \|x\|^2 = x_1^2 + \dots + x_k^2$, $\|x\|^2 = \langle x, x\rangle$

$$\frac{\partial f}{\partial u}(a) = \lim_{t\to 0} \frac{f(a+tu)-f(a)}{t} = \lim_{t\to 0} \frac{\langle a+tu, a+tu\rangle - \langle a,a\rangle}{t}$$

$$= \lim_{t\to 0} \frac{\cancel{\langle a,a\rangle} + \langle a, tu\rangle + \langle tu, a\rangle + \langle tu, tu\rangle - \cancel{\langle a,a\rangle}}{t}$$

$$= \lim_{t\to 0} \frac{t\langle a,u\rangle + t\langle u,a\rangle + t^2\langle u,u\rangle}{t}$$

$$= \lim_{t\to 0} 2\langle a,u\rangle + t\langle u,u\rangle = 2\langle a,u\rangle$$

Not. $g: \mathbb{R}^k \to \mathbb{R}$, $f(x) = \|x\|$

avem $f = g^2$

$$\frac{\partial f}{\partial u}(a) = \frac{\partial}{\partial u} g^2 = 2g \cdot \frac{\partial g(a)}{\partial u}$$

$$\Rightarrow \frac{\partial g}{\partial a}(a) = \frac{1}{2f(a)} \frac{\partial f}{\partial u}(a) \Rightarrow \frac{\partial g}{\partial u}(a) = \frac{1}{2\|a\|} \cdot 2\langle a,u\rangle = \frac{\langle a,u\rangle}{\|a\|}$$

9. $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

$v = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$

$a = (0,0)$

f are derivata in (0,0) după $\forall v = (\cos\theta, \sin\theta)$?

$$\frac{\partial f}{\partial v}(0,0) = \lim_{t\to 0} \frac{f(a+tv)-f(a)}{t} = \lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta) - 0}{t}$$

$$= \lim_{t\to 0} \frac{\frac{2t\cos\theta \, t\sin\theta}{t^2(\cos^2\theta+\sin^2\theta)}}{t} = \lim_{t\to 0} \frac{\sin 2\theta}{t} = 0$$

dacă $\sin 2\theta = 0 \Rightarrow \theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$

$\theta = 0 \Rightarrow v = (1,0)$

$\theta = \frac{\pi}{2} \Rightarrow v = (0,1)$

$\theta = \pi \Rightarrow v = (-1,0)$

$\theta = \frac{3\pi}{2} \Rightarrow v = (0,-1)$

10. Calc. $\frac{\partial T}{\partial u}(a)$ unde $T:\mathbb{R}^K \to \mathbb{R}$ este operator liniar, iar $a \in \mathbb{R}^K$, $u \in \mathbb{R}^k$, $u \neq 0$ sunt oarecare.

$$\frac{\partial T}{\partial u}(a) = \lim_{t\to 0} \frac{T(a+tu)-T(a)}{t} = \lim_{t\to 0} \frac{\cancel{T(a)} + T(tu) - \cancel{T(a)}}{t}$$

$$= \lim_{t\to 0} \frac{\cancel{t}\, T(u)}{\cancel{t}} = T(u)$$

11. $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{x^2y}{x^6+y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

Ar. că f nu e cont in (0,0).

$$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^3}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$f\left(\frac{1}{n}, \frac{1}{n^3}\right) = \frac{\frac{1}{n^5}}{\frac{1}{n^6} + \frac{1}{n^6}} = \frac{1}{\cancel{n^5}} \cdot \frac{\cancel{n^6}\, n}{2} = \frac{n}{2} \longrightarrow \infty \Rightarrow f \text{ nu e cont. în } (0,0)$$

$$\frac{\delta f}{\delta u}(0,0) = \lim_{t\to 0} \frac{f(tu) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^2 u_1^2 + u_2}{t^2 u_1^6 + t^2 u_2}}{t}$$

$u = (u_1, u_2)$

$$\cdot\ \frac{u_1^2}{u_2} \Rightarrow \exists \frac{\partial f}{\partial u}((0,0)), \ \forall u \neq (u_1, 0)$$

$\frac{\partial f}{\partial u}(0,0)$? pt $u = (u_1, 0)$

$$\frac{\partial f}{\partial u}(0,0) = \lim_{t\to 0} \frac{f((0,0) + t(u_1,0)) - f((0,0))}{t}$$

$$= \lim_{t\to 0} \frac{f(t(u_1,0))}{t} = \frac{0}{t} = 0 \Rightarrow \exists \frac{\partial f}{\partial u}((0,0)) \text{ pt. } u=(u_1,0).$$

12. $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{x^5}{(y-x^2)^2 + x^8}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

continuitatea în $(0,0)$

$$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^5}}{\left(\frac{1}{n^5} - \frac{1}{n^2}\right)^2 + \frac{1}{n^8}} = \lim_{n\to\infty} \frac{1}{n^5} \cdot n^8$$

$$= \lim_{n\to\infty} n^3 = \infty \neq f((0,0)) \Rightarrow f \text{ nu e cont. în } (0,0)$$

∘ Studiați derivabilitatea în $(0,0)$ a lui f după $v = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$.

$$\frac{\partial f}{\partial v}(0,0) = \lim_{t\to 0} \frac{f((0,0) + tv) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(tv)}{t}$$

$$= \lim_{t\to 0} \frac{t^5 \cos^5\theta \cdot \frac{1}{(t\sin\theta - t^2\cos^2\theta)^2 + t^8\cos^8\theta}}{t} \qquad \text{I. } \sin\theta \neq 0$$

$$= \lim_{t \to 0} \frac{t^2 \cos^5 \theta}{(\sin\theta - t\cos^2\theta)^2 + t^6 \cos^8 \theta}$$

$\underline{\text{II}}$. $\sin\theta = 0 \Rightarrow \lim_{t\to 0} \dfrac{t^2 \cos^5\theta}{t^2\cos^4\theta + t^6\cos^8\theta}$

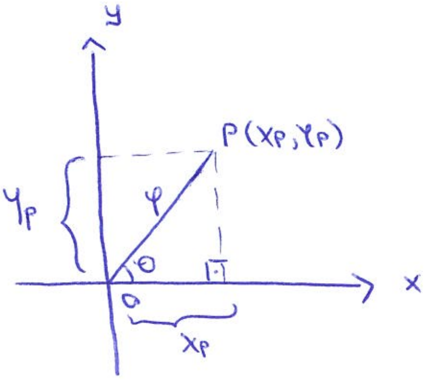
$$= \ldots = \cos\theta .$$

Seminar 19

Coordonate polare

- Fie funcția vectorială $F : \mathbb{R}_+ \times [0, 2\pi) \to \mathbb{R}^2$, $F(\varphi, \theta) = (\varphi\cos\theta, \varphi\sin\theta)$

$\forall (\varphi, \theta) \in \mathbb{R}_+ \times [0, 2\pi)$



F exprimă legătura dintre ordonatele

$f_1(\varphi, \theta) = \varphi\cos\theta$

$f_2(\varphi, \theta) = \varphi\sin\theta$

$x_P = \varphi\cos\theta$

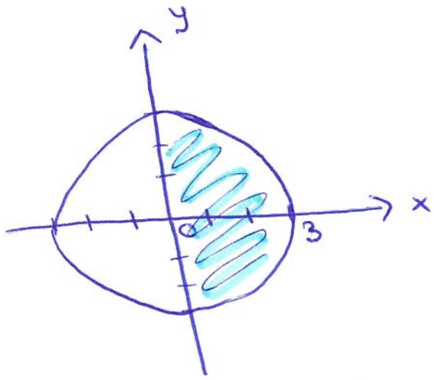
$y_P = \varphi\sin\theta$

Matricea Jacobiană

$$J_{F(\varphi,\theta)} = \begin{pmatrix} \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial \varphi} & \frac{\partial f_2}{\partial \theta} \end{pmatrix}$$

Jacobianul este $\det J_{F(\varphi,\theta)} = \begin{vmatrix} \cos\theta & -\varphi\sin\theta \\ \sin\theta & \varphi\cos\theta \end{vmatrix} = \varphi\cos^2\theta + \varphi\sin^2\theta = \varphi(\cos^2\theta + \sin^2\theta)$
$= \varphi$

$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9 \wedge x \geq 0\}$



$$\begin{cases} x = \varphi\cos\theta \\ y = \varphi\sin\theta \end{cases}$$

$\varphi^2\cos^2\theta + \varphi^2\sin^2\theta \leq 9$

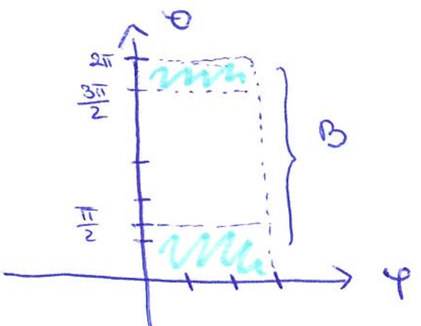
$\varphi^2(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) \leq 9$

$\varphi^2 \leq 9$

$\Rightarrow \varphi \in [0, 3]$

$$\begin{cases} \varphi\cos\theta \geq 0 \\ \cos\theta \geq 0 \end{cases}$$

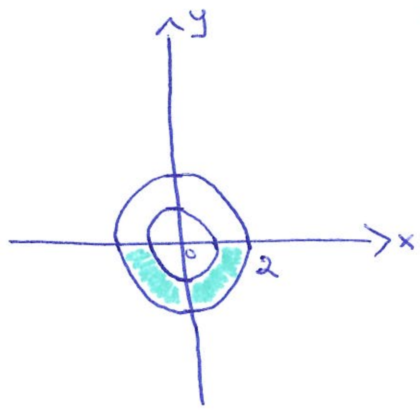
$\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$.



$$\iint_A f(x,y)\,dx\,dy = \iint_B f(\varphi\cos\theta, \varphi\sin\theta) \underline{J} \quad \frac{A}{B} = \lambda.$$

$J \to$ Jacobianul

$C=\{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 4 \wedge y\le 0\}$



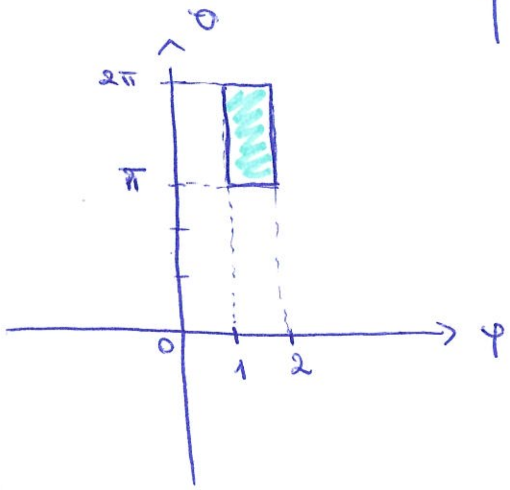
$1\le \varphi^2\le 4$

$\varphi\in[1,2]$

$\varphi\cos\theta \le 0$

$\sin\theta\le 0$

$\theta\in[\pi,2\pi]$.



Coordonatele cilindrice

Fie fct. $F:\mathbb{R}_+\times[0,2\pi)\times\mathbb{R}\to\mathbb{R}^3$, $F(x,y,z)=(\varphi\cos\theta,\varphi\sin\theta)$

$$\begin{cases} x=\varphi\cos\theta, & \varphi>0\\ y=\varphi\sin\theta, & \theta\in[0,2\pi]\\ z=z, & z\in\mathbb{R}\end{cases}$$

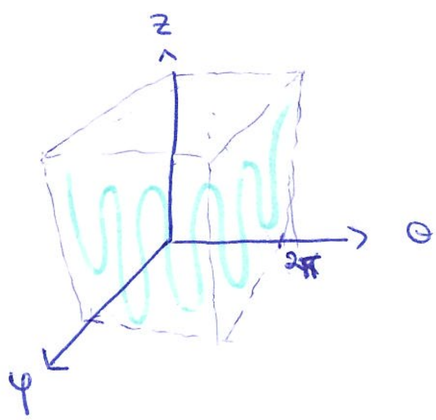
$$J_{F(\varphi,\theta,z)}=\begin{pmatrix}\frac{\partial f_1}{\partial\varphi} & \frac{\partial f_1}{\partial\theta} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial\varphi} & \frac{\partial f_2}{\partial\theta} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial\varphi} & \frac{\partial f_3}{\partial\theta} & \frac{\partial f_3}{\partial z}\end{pmatrix}=\begin{pmatrix}\cos\theta & -\varphi\sin\theta & 0\\ \sin\theta & \varphi\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

$\det J_F=\varphi$

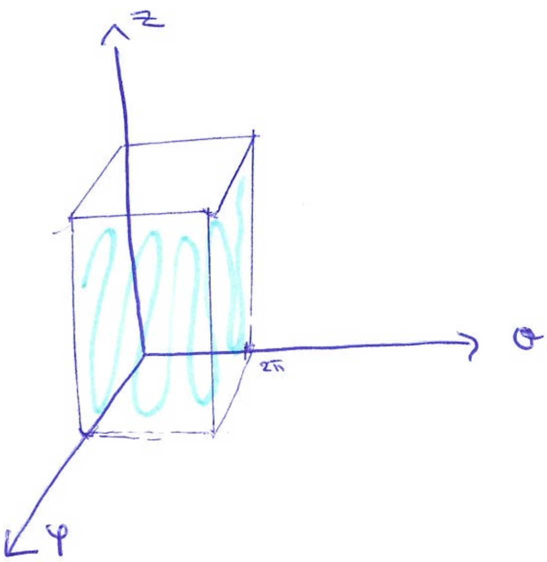
$\varphi^2\cos^2\theta+\varphi^2\sin^2\theta\le R^2$

$\varphi^2\le R^2 \Rightarrow \varphi\in(0,R]$

$(\varphi,\theta,z)\in(0,R]\times[0,2\pi)\times[0,h]$



$B = \{(x,y,z) \in \mathbb{R}^3 \mid 1 \leq x^2+y^2 \leq 4 \wedge z \in (0,h]\}$



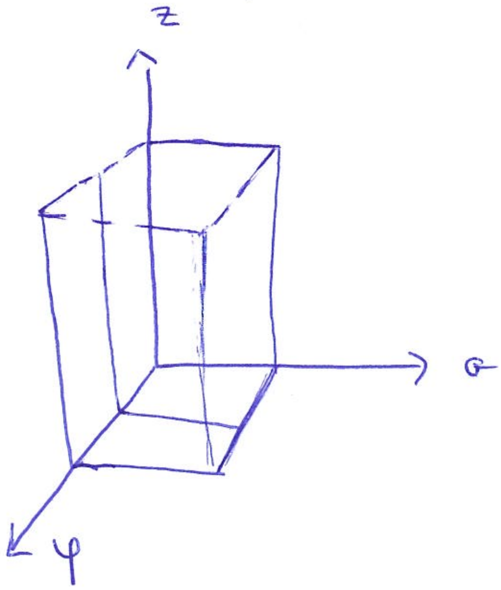
$1 \leq x^2+y^2 \leq 4$

$1 \leq \varphi^2\cos^2\theta + \varphi^2\sin^2\theta \leq 4$

$1 \leq \varphi \leq 2 \Rightarrow \varphi \in [1,2]$

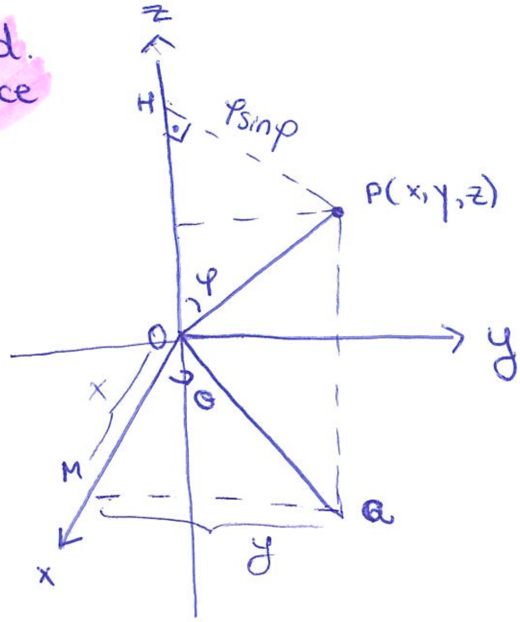
$\theta \in [0,2\pi]$

$z \in (0,h]$.



Seminar 20

Coord. sferice



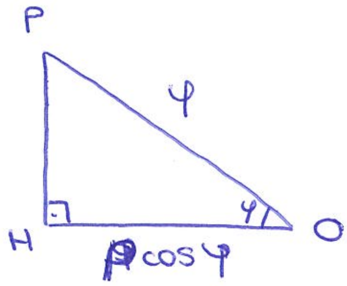
$\varphi = \measuredangle(\overrightarrow{OP}, \overrightarrow{Oz})$

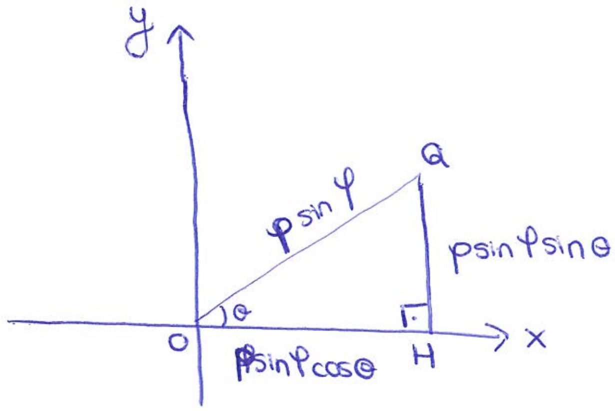
$\theta = \measuredangle(\overrightarrow{OQ}, Ox)$

$\rho = \sqrt{x^2+y^2+z^2}$

$Q = pr_{xOy}$

$\Rightarrow$





$$\begin{cases} x = OM = \rho \sin\varphi \cos\theta \\ y = MQ = \rho \sin\varphi \sin\theta \\ z = OH = \rho \cos\varphi \end{cases}$$

$\rho \in \mathbb{R}_+$

$\varphi \in [0, \pi]$

$\theta \in [0, 2\pi)$

$F: \mathbb{R}_+ \times [0,\pi] \times [0, 2\pi) \longrightarrow \mathbb{R}^3$

$F(\rho, \varphi, \theta) = (f_1, f_2, f_3) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi)$

$x^2+y^2+z^2 = \rho^2(\sin^2\varphi \cos^2\theta + \sin^2\varphi \sin^2\theta + \cos^2\varphi)$

$= \rho^2 (\sin^2\varphi \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + \cos^2\varphi) = \rho^2$

* Dacă $\rho = R$ - constant $\Rightarrow (x, y, z) \in Fr(S(0, R))$.

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial \rho} & \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial \rho} & \frac{\partial f_2}{\partial \varphi} & \frac{\partial f_2}{\partial \theta} \\ \frac{\partial f_3}{\partial \rho} & \frac{\partial f_3}{\partial \varphi} & \frac{\partial f_3}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\theta & \cancel{\theta}\ \rho\sin\varphi & \cancel{-\rho\sin\theta}\ 0 \end{pmatrix}$$

$$\cancel{\Rightarrow \det(J_F) = \rho^2\sin^2\theta\cos\varphi\sin\varphi\cos\theta + \rho^2\cos^3\theta\sin\varphi\cos\varphi}$$

$$\cancel{+\rho^2\sin\varphi\sin^2\theta\cos\theta\cos\varphi +}$$

$\det J_F = \rho^2 \sin\theta$

$A = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 9 ,\ y>0 \}$

$(x,y,z) \longmapsto (\rho, \varphi, \theta)$

$x = \rho\sin\varphi\cos\theta$

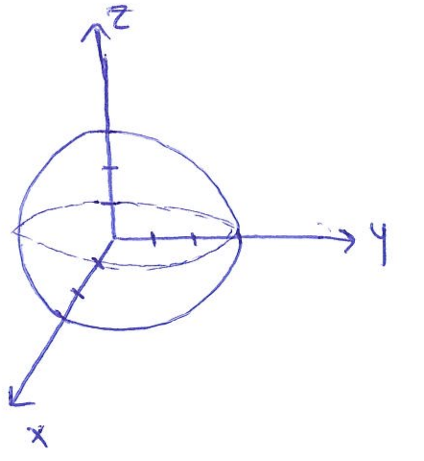
$y = \rho\sin\varphi\sin\theta$

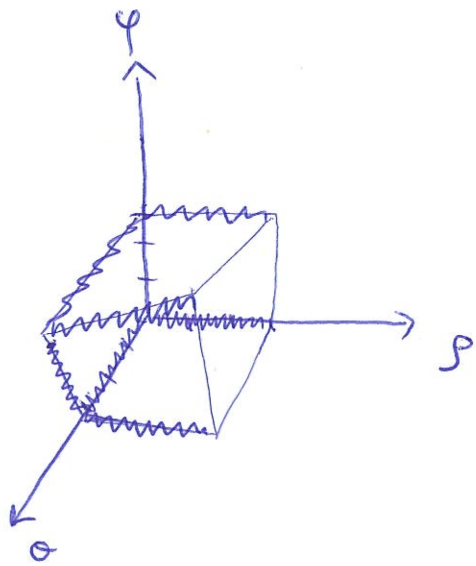
$z = \rho\cos\varphi$

$x^2+y^2+z^2 \leq 9 \Rightarrow \rho^2 \leq 9 \Rightarrow \rho \in [0,3]$;

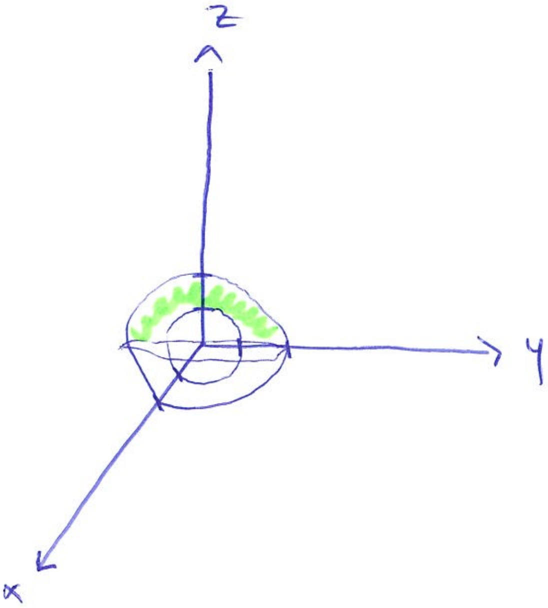
$$\left.\begin{array}{l} y>0,\ \rho\sin\theta\sin\varphi > 0 \Leftrightarrow \\ \varphi \in [0,\pi] \Rightarrow \rho\sin\theta\sin\varphi > 0 \Leftrightarrow \\ \varphi \in [0,\pi] \Rightarrow \sin\varphi \in [0,1] \end{array}\right\} \Rightarrow \sin\theta > 0 \Rightarrow \theta \in (0,\pi)$$

$(\rho, \theta, \varphi) \in (0,3] \times (0,\pi) \times (0,\pi)$





$B = \{(x,y,z) \in \mathbb{R}^3 \mid 1 \le x^2+y^2+z^2 \le 4,\ z \ge 0\}$



$$\begin{cases} \rho \in [1,2] \\ \varphi \in [0, \frac{\pi}{2}] \\ \theta \in [0, 2\pi). \end{cases}$$

~ Diferențiabilitate ~

Ex 1. (curs 11) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , (x,y) \ne (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

studiem continuitatea în $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

$$0 \le \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \le |x| \cdot \left| \frac{y}{\sqrt{x^2+y^2}} \right| \le |x|$$

mărginit, $0 \le \frac{y}{\sqrt{x^2+y^2}} \le 1$

0

$\Rightarrow f$ cont. pe $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = 0$$

$$T(h) = \frac{\partial f}{\partial x}(0,0) h_1 + \frac{\partial f}{\partial y}(0,0) h_2$$

$$T(h) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (T(x,y) - (0,0))}{\| (x,y) - (0,0) \|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$$h(x,y) = \frac{xy}{x^2+y^2}$$

$h\left(\frac{1}{n}, \frac{1}{n}\right) = \dfrac{\frac{1}{n^2}}{\frac{2}{n^2}} = \frac{1}{2} \neq 0 \Rightarrow$ nu e dif în $(0,0)$.

fie $(x_0, y_0) \in \mathbb{R}^2 \setminus \{0,0\}$

f este cont. în (x_0, y_0)

Studiem diferențiabilitatea

$$\frac{\partial f}{\partial x}(x,y) = \frac{\frac{\partial}{\partial x} xy \cdot \sqrt{x^2+y^2} \cdot xy \frac{\partial}{\partial x}\sqrt{x^2+y^2}}{x^2+y^2}$$

$$\left| \; \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{1}{2\sqrt{x^2+y^2}} \cdot \frac{\partial}{\partial x}(x^2+y^2) = \frac{2x}{\cancel{2}\sqrt{x^2+y^2}} \right.$$

$$= \frac{y\sqrt{x^2+y^2} - xy \cdot \frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x^2+y^2) - x^2 y}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{y^3}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^3}{(x^2+y^2)\sqrt{x^2+y^2}}$$ (simetrică)

$(x_0, y_0) \neq 0 \Rightarrow \exists V \in \mathcal{V}(x_0, y_0)$ a.î. $(0,0) \notin V$.

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} : V \to \mathbb{R}$ sunt continue pe $V \longrightarrow$ fct. f este de clasă C^1 pe V

$\Downarrow$

este diferențiabilă în (x_0, y_0).

$\Rightarrow f$ diferențiabilă pe $\mathbb{R}^2 \setminus \{0,0\}$.

ex. 2.

ex 3 v)

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Studiem continuitatea în $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{x|y|}{\sqrt{x^2+y^2}} = 0 \Rightarrow f(0,0) \Rightarrow f \text{ continuă pe } \mathbb{R}^2$$

$$T(h) = \frac{\partial f}{\partial x}(0,0) h_1 + \frac{\partial f}{\partial y}(0,0) h_2$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x-0} = 0$$

Analog $\frac{\partial f}{\partial y}(0,0) = 0$

$\Rightarrow T(h) = 0$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (T(x,y) - (0,0))}{\|(x,y) - (0,0)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x|y|}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \neq 0 \Rightarrow f \text{ nu e diferențiabilă în } (0,0)$$

Fie $(x_0, y_0) = (x_0, 0)$

$$\frac{\partial f}{\partial x}(x_0,0) = \lim_{x\to x_0} \frac{f(x,0)-f(x_0,0)}{x-x_0} = 0$$

$$\frac{\partial f}{\partial y}(x_0,0) = \lim_{y\to 0} \frac{f(x_0,y)-f(x_0,0)}{y-0} = \lim_{y\to 0} \frac{\frac{x_0|y|}{\sqrt{x_0^2+y^2}}}{y} = \frac{x_0}{|x_0|} \lim_{y\to 0} \frac{|y|}{y} \begin{matrix} \nearrow -1, \ y<0 \\ \searrow 1, \ y>0 \end{matrix}$$

$$\Rightarrow \nexists \lim_{y\to 0} \frac{f(x_0,y)-f(x_0,0)}{y-0}$$

$\Rightarrow \nexists$ ~~lim~~ $\frac{\partial f}{\partial y}(x_0,0) \Rightarrow$ nu există T - op. lin. candidat pt. diferențiala lui f.

$\Rightarrow f$ nu este diferențială în punctele de forma $(x_0, 0)$, $x_0 \in \mathbb{R}$.

Pentru puncte de forma (x_0, y_0), $y_0 \neq 0$.

$\exists V \in \mathcal{V}(x_0, y_0)$ a.î. $\forall (x,y) \in V \Rightarrow y = \operatorname{sgn} y_0$

$$\Rightarrow \forall (x,y) \in V, \ f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \cdot \operatorname{sgn}(y_0)$$

$\Rightarrow f$ este diferențiabilă în (x_0, y_0)

Seminar 21

ex. 2. (CURS 11) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

a) f este diferențiabilă

b) că f nu este de clasă C^1

a) Studiem continuitatea

caz I: $(x,y) \neq (0,0)$

f cont. (comp. de fct. elem)

caz II: $(x,y) = (0,0)$

$$\lim_{(x,y)\to(0,0)} \underbrace{(x^2+y^2)}_{\to 0} \underbrace{\sin\frac{1}{x^2+y^2}}_{\text{mărg.}} = 0 = f(0,0) \Rightarrow f \text{ este cont.}$$

Fie $(x_0, y_0) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left((x^2+y^2) + \sin\frac{1}{x^2+y^2}\right)$$

$$= \frac{\partial}{\partial x}(x^2+y^2)\sin\frac{1}{x^2+y^2} + (x^2+y^2)\cdot\frac{\partial}{\partial x}\left(\sin\frac{1}{x^2+y^2}\right)$$

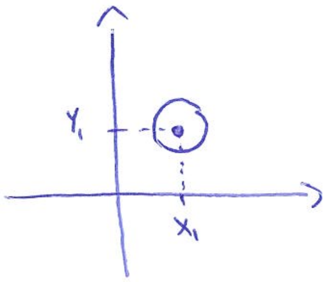
$$= 2x\sin\frac{1}{x^2+y^2} + (x^2+y^2)\cos\frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$= 2x\sin\frac{1}{x^2+y^2} + (x^2+y^2)\cos\frac{1}{x^2+y^2}\left(\frac{-2x}{(x^2+y^2)^2}\right)$$

$$\frac{\partial f}{\partial x} = 2x\sin\frac{1}{x^2+y^2} - \cos\frac{1}{x^2+y^2}\cdot\frac{2x}{x^2+y^2}$$

$$\left.\begin{matrix} x \to y \\ y \to x \end{matrix}\right\} \Rightarrow \frac{\partial f}{\partial y} = 2y\sin\frac{1}{x^2+y^2}\cos\frac{1}{x^2+y^2}\cdot\frac{2y}{x^2+y^2}$$

* obs. că $\exists V \in \mathcal{V}(x_0, y_0)$ a.î. $(0,0) \notin V$.



* Pe vecinătatea V derivatele parțiale $\frac{\partial f}{\partial x}$ și $\frac{\partial f}{\partial x}$ sunt bine definite.

Mai mult, acestea sunt continue în (x_0, y_0). $\xrightarrow[\text{(Crit. suf. de dif)}]{\text{T. cris}}$ f este dif. în (x_0, y_0)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{(x,y)\to(0,0)} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x^2}}{x}$$

$$= \lim_{x\to 0} \underbrace{x}_{0} \underbrace{\sin\frac{1}{x^2}}_{\text{mărg}} = 0$$

cum f simetrică $\Rightarrow \frac{\partial f}{\partial y}(0,0) = 0$

$$\frac{\partial f}{\partial x} = \begin{cases} 2x \sin\frac{1}{x^2+y^2} - \cos\frac{1}{x^2+y^2} \cdot \frac{2x}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) \neq (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} 2y \sin\frac{1}{x^2+y^2} - \cos\frac{1}{x^2+y^2} \cdot \frac{2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \left(\overbrace{2x}^{\to 0} \overbrace{\sin\frac{1}{x^2+y^2}}^{\text{mărg.}} - \cos\frac{1}{x^2+y^2} \cdot \frac{2x}{x^2+y^2} \right)$$

$$\lim_{(x,y)\to(0,0)} \left(-\cos\frac{1}{x^2+y^2} \frac{2x}{x^2+y^2} \right)$$

$$h(x_n, y_n) = h\left(\frac{1}{\sqrt{2n\pi}}, 0\right)$$

$$h\left(\frac{1}{\sqrt{2n\pi}}, 0\right) = -\cos\frac{1}{\frac{1}{2n\pi}} \cdot \frac{2 \cdot \frac{1}{\sqrt{2n\pi}}}{\frac{1}{2n\pi}}$$

$$= \underbrace{\cos 2n\pi}_{\to 1} \cdot \underbrace{2\sqrt{2n\pi}}_{\to\infty} \xrightarrow[n\to\infty]{} \infty \neq 0 \Rightarrow \frac{\partial f}{\partial x} \text{ nu e cont în } (0,0)$$

$$T(h_1,h_2)=\frac{\partial f}{\partial x}(0,0)\,h_1+\frac{\partial f}{\partial y}(0,0)\,h_2$$

$$=0$$

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-T\big((x,y)-(0,0)\big)}{\|(x,y)-(0,0)\|}$$

$$\lim_{(x,y)\to(0,0)}\frac{(x^2+y^2)\sin\frac{1}{x^2+y^2}}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\underbrace{\sqrt{x^2+y^2}}_{\to 0}\ \underbrace{\sin\frac{1}{x^2+y^2}}_{\text{mărg}}=0$$

f este diferențiala în $(0,0)$ și
diferențiala sa este $df(0,0)(h)=T(h)=0$

b)

$$C^1(\mathbb{R}^2)=\left\{ f:\mathbb{R}^2\to\mathbb{R}\ \middle|\ \exists\ \frac{\partial f}{\partial x} \text{ și } \frac{\partial f}{\partial y}(x,y),\ \forall (x,y)\in\mathbb{R}^2 \text{ și sunt cont pe } \mathbb{R}^2 \right.$$

$C^1(D)$
D- deschis.

în cazul nostru $\exists\ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ dar nu sunt continue în $(0,0)\Rightarrow f\in C^1(\mathbb{R}^2)$

$$f\notin C^1(\mathbb{R}^2)$$

ex. 12 (CURS 11)

i) $F(x,y)=(x^2+y^2, xy)$, $(x,y)\in\mathbb{R}^2$

$f_1=x^2+y^2$

$f_2=xy$

$$\frac{\partial f_1}{\partial x}=2x \qquad \frac{\partial f_1}{\partial y}=2y$$

$$\frac{\partial f_2}{\partial x}=y \qquad \frac{\partial f_2}{\partial y}=x$$

$$J(f_1,f_2)=\begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}$$

$$\Rightarrow J(f_1,f_2)=\begin{pmatrix}2x & 2y\\ y & x\end{pmatrix}$$

$$J(f_1,f_2)=\begin{vmatrix}2x & 2y\\ y & x\end{vmatrix}=2x^2-2y^2=2(x^2-y^2)$$

ii) $F(x,y) = (x + \cos y, y \sin x)$, $(x, y) \in \mathbb{R}^2$

$$J(f_1, f_2) = \begin{pmatrix} 1 & -\sin y \\ y\cos x & \sin x \end{pmatrix}$$

$$\Delta J(f_1, f_2) = \begin{vmatrix} 1 & -\sin y \\ y\cos x & \sin x \end{vmatrix}$$

$$= \sin x + y \sin y \cos x.$$

Ex 11. $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}\dfrac{x^3(x^2-y^2)}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Studiați dacă derivatele parțiale mix de ord. 2 în (0,0) coincid.

$$\frac{\partial f}{\partial x}\left(\frac{x^3(x^2-y^2)}{(x^2+y^2)^2}\right)=\frac{\partial f}{\partial x}\left(\frac{x^5-x^3y^2}{(x^2+y^2)^2}\right)=\cancel{\frac{\partial}{\partial x}(x^5-x^3y^2)(x^2+y^2)^2-(x^5-x^3y^2)\frac{\partial}{\partial x}}$$

$$=\frac{\frac{\partial}{\partial x}(x^5-x^3y^2)(x^2+y^2)^2-(x^5-x^3y^2)\frac{\partial}{\partial x}(x^2+y^2)^2}{(x^2+y^2)^2}$$

$$=\frac{(5x^4-3x^2y^2)(x^2+y^2)^2-(x^5-x^3y^2)\,2(x^2+y^2)2x}{(x^2+y^2)^4}$$

$$=\frac{5x^6+5x^4y^2-3x^4y^2-3x^2y^4-(x^5-x^3y^2)4x}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial x\partial y}(0,0)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0)=\lim_{x\to0}\frac{\frac{\partial f}{\partial y}(x,0)-\frac{\partial f}{\partial y}(0,0)}{x-0}$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to0}\frac{0-0}{y}=0$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to0}\frac{\frac{x^5}{x^4}}{x}=1$$

$$\lim_{x\to0}\frac{0-0}{x}=0$$

$$\frac{\partial^2 f}{\partial y\partial x}(0,0)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(0,0)\right)=\lim_{y\to0}\frac{\frac{\partial f}{\partial x}(0,y)-\frac{\partial f}{\partial x}(0,0)}{y-0}$$

$$\lim_{y\to0}\frac{0-1}{y}=\lim_{y\to0}\frac{-1}{y}=\pm\infty \Rightarrow \text{nu coincid}$$

caz I: $(x,y)\neq(0,0)$ } comp de fct. elem $\Rightarrow$ continuă

caz II: $(x,y)=(0,0)$

$$0 \le \left|\frac{x^3(x^2-y^2)}{(x^2+y^2)}\right| = \frac{|x^3||x^2-y^2|}{(x^2+y^2)^2} \le \frac{(x^2+y^2)^{\frac{3}{2}}(x^2+y^2)^{\frac{2}{2}}}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)^{\frac{5}{2}}}{(x^2+y^2)^2} = \sqrt{x^2+y^2}$$

$$0 \to 0 \leftarrow \sqrt{x^2+y^2}$$

$|x| \le \sqrt{x^2+y^2}\ \uparrow^3$

$|x^3| \le (x^2+y^2)^{\frac{3}{2}}$

$|x^2-y^2| \le x^2+y^2 \Rightarrow$ continuă

$$\frac{\partial f}{\partial x} = \frac{x^6+6x^4y^2-3x^2y^4}{(x^2+y^2)^3} \qquad \frac{\partial f}{\partial y} = \frac{-6x^5y+2x^3y^3}{(x^2+y^2)^3}$$

Fie $(x_0,y_0)\neq(0,0)$ $\exists$ V o vecinătate a lui (x_0,y_0) a.î. $(0,0)\notin V$

$\frac{\partial f}{\partial x}$ și $\frac{\partial f}{\partial y}$ bine definite, continue (x_0,y_0) } $\xrightarrow{\text{T. cond. suf. de dif.}}$ fct. f diferențiabil în (x_0,y_0)

Studiem în $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \lim_{(x,y)\to(0,0)} \frac{x^6+6x^4y^2-3x^2y^4}{(x^2+y^2)^3}$$

$\frac{\partial f}{\partial x}(0,0)=1$

$\frac{\partial f}{\partial x}\left(0,\frac{1}{n}\right)=0 \xrightarrow[n\to\infty]{} 0\neq 1 \Rightarrow \frac{\partial f}{\partial x}$ nu este cont. în $(0,0)$

$$T(h_1, h_2) = \frac{\partial f}{\partial x}(0,0) \cdot h_1 + \frac{\partial f}{\partial y}(0,0) h_2 = h_1$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (T(x,y) - (0,0))}{\|(x,y) - (0,0)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^3(x^2-y^2)}{(x^2+y^2)^2} - x}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^3(x^2-y^2)}{(x^2+y^2)^3\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}}$$

$$h\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{\frac{1}{n^3}\left(\frac{1}{n^2} - \frac{1}{n^2}\right)}{\left(\frac{1}{n^2} + \frac{1}{n^2}\right)^2 \sqrt{\frac{1}{n^2} + \frac{1}{n^2}}} - \frac{\frac{1}{n}}{\sqrt{\frac{1}{n^2} + \frac{1}{n^2}}} = -\frac{1}{\sqrt{2}} \xrightarrow[n\to\infty]{} -\frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow$ f nu e diferențiabilă în $(0,0)$.

Seminar 22

ex 5 (CURS 11)

Dacă $a \in \mathbb{R}^n$, iar $F: \mathbb{R}^n \to \mathbb{R}^m$ este pe rând:

a) o fct. constantă

b) un op. lin, ar. că F este diferențiabilă și calculați $dF(a)$.

a) Fie $F(x) = c$, unde $c \in \mathbb{R}^m$, c - constantă vectorială, $\forall x \in \mathbb{R}^n$

$T: \mathbb{R}^n \to \mathbb{R}^m$

$T(h) = 0$, $h \in \mathbb{R}^n$

$$\lim_{x \to a} \frac{F(x) - F(a) - T(x-a)}{\|x-a\|} = \lim_{x \to a} \frac{c - c - 0}{\|x-a\|} = 0 \Rightarrow$$

$\Rightarrow dF(a) = 0$, $\forall a \in \mathbb{R}^n$

b) $F: \mathbb{R}^n \to \mathbb{R}^m$ operator liniar

Alegem $T: \mathbb{R}^n \to \mathbb{R}^m$ să fie $T(h) = F(h)$, $\forall h \in \mathbb{R}^n$

$$\lim_{x \to a} \frac{F(x) - F(a) - T(x-a)}{\|x-a\|} = \lim_{x \to a} \frac{F(x-a) - F(x-a)}{\|x-a\|} = 0$$

$dF(a)(h) = F(h)$

ex. 6 (CURS 11)

Fie $F: \mathbb{R}^2 \to \mathbb{R}^3$

$F(x,y) = (x^2y, xy + y^2, x^3 - 2)$

Scrieți $J_{F(1,2)}$

$f_1(x,y) = x^2y$

$f_2(x,y) = xy + y^2$

$f_3(x,y) = x^3 - 2$

$$\frac{\partial f_1}{\partial x}(x,y) = 2xy \Rightarrow \frac{\partial f_1}{\partial x}(1,2) = 4$$

$$\frac{\partial f_1}{y}(x,y) = x^2 \Rightarrow \frac{\partial f_1}{\partial y}(1,2) = 1$$

$$\frac{\partial f_2}{\partial x}(x,y) = y \Rightarrow \frac{\partial f_2}{\partial x}(1,2) = 2$$

$$\frac{\partial f_2}{\partial y}(x,y) = x + 2y \Rightarrow \frac{\partial f_2}{\partial y}(1,2) = 5$$

$$\frac{\partial f_3}{\partial x}(x,y) = 2x^2 \Rightarrow \frac{\partial f_3}{\partial x}(1,2) = 3$$

$$\frac{\partial f_3}{\partial y}(x,y) = 0 \Rightarrow \frac{\partial f_3}{\partial y}(1,2) = 0$$

$$J_{F(1,2)} = \begin{pmatrix} 4 & 1 \\ 2 & 5 \\ 3 & 0 \end{pmatrix}$$

ex 7 (CURS 11)

Arătați că funcția $f: S((0,0),r) \subset \mathbb{R}^2 \to \mathbb{R}$ îndeplinind condiția $\underbrace{|f(x,y)| \le x^2+y^2, \forall (x,y) \in S((0,0),r)}_{(*)}$ este diferențiabilă în $(0,0)$.

În $(*)$ luăm $(x,y) = (0,0)$

$|f(0,0)| \le 0^2 + 0^2 \Rightarrow f(0,0) = 0$

$0 \le |f(x,y)| \le x^2+y^2$

0

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$$

$$T(h_1,h_2) = \frac{\partial f}{\partial x}(0,0)h_1 + \frac{\partial f}{\partial y}(0,0)h_2$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0}\frac{f(x,0)}{x}$$

$$0 \leq \left|\frac{f(x,0)}{x}\right| = \frac{|f(x,0)|}{|x|} \overset{(*)\, y=0}{\leq} \frac{x^2}{|x|} \quad \Rightarrow \frac{\partial f}{\partial x}(0,0)=0$$

$$\downarrow 0$$

Analog $\frac{\partial f}{\partial y}(0,0)=0$

$$\left.\begin{array}{l} T(h_1,h_2) = 0\cdot h_1 + 0\cdot h_2 = 0. \\ f(0,0)=0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{f(x,y)-0-0}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}}$$

$$0 \leq \left|\frac{f(x,y)}{\sqrt{x^2+y^2}}\right| = \frac{|f(x,y)|}{|\sqrt{x^2+y^2}|} = \frac{|f(x,y)|}{\sqrt{x^2+y^2}} \overset{(*)}{\leq} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

$$\downarrow 0$$

$\exists\, dF(0,0)=0.$

ex. 3 (CURS II)

Studiați diferențiabilitatea

a) $f(x,y) = \sqrt{|xy|}$

$f: \mathbb{R}^2 \to \mathbb{R}$

f e cont pe $\mathbb{R}^2$ (compunere de fct. elem)

$f: D_1 \to \mathbb{R}$, $f: D_2 \to \mathbb{R}$

$D_1 = \{(x,y) \mid xy > 0\}$

$D_2 = \{(x,y) \mid xy < 0\}$

$D_3 = \{(x,y) \mid xy = 0\}$

Pe D_1

$f_1(x,y) = \sqrt{xy}$

$$\frac{\partial f}{\partial x} = \frac{y}{2\sqrt{xy}}$$

$$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{xy}}$$

Derivatele parțiale sunt continue pe D_1 (combinații fct. elem) $\Big\}$ f clasă C^1 pe deschisul D_1

D_1 - mulț. deschisă

Ar. că D_1 - mulț. deschisă

Fie $h(x,y) = xy$

$D_1 = \{(x,y) \mid xy > 0\} = \{(x,y) \mid h(x,y) > 0\} = \{(x,y) \mid h(x,y) \in (0;+\infty)\}$

$= h^{-1}(0;+\infty)$ (deschisă) $\Big\} \Rightarrow D_1$ - deschis

h cont

· f e clasă C^1 pe $D_1 \Rightarrow$ f e dif. pe D_1

$f_2(x,y) = \sqrt{-xy}$

$$\frac{\partial f_2}{\partial x} = \frac{-y}{2\sqrt{-xy}}$$

$$\frac{\partial f_2}{\partial y} = \frac{-x}{2\sqrt{-xy}}$$

D_2 - deschis

cont. pe D_2 $\Big\} \Rightarrow$ f e de clasă C^1 pe $D_2 \Rightarrow$ f diferențiabilă

$D_3 = \{(x,y) \mid xy = 0\} = \{(x,y) \mid (x=0) \vee (y=0)\}$

Studiem diferențiabilitatea în $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \frac{0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \frac{0}{y} = 0$$

$T(h_1,h_2) = 0$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-(T(x,y)-(0,0))}{\|(x,y)-(0,0)\|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \underbrace{\sqrt{\frac{|xy|}{x^2+y^2}}}_{\overset{not}{=}\, g} \neq 0 \Rightarrow f \text{ nu e diferențiabilă în } (0,0).$$

$$\lim_{n\to\infty} g\left(\frac{1}{n},\frac{1}{n}\right) = \lim_{n\to\infty} \sqrt{\frac{\frac{1}{n^2}}{\frac{2}{n^2}}} = \frac{1}{\sqrt{2}} \neq 0$$

- Studiem dif. în pct. de forma $(a,0)$, $a \neq 0$

$$\frac{\partial f}{\partial y}(a,0) = \lim_{y\to 0} \frac{f(a,y)-f(a,0)}{y} = \lim_{y\to 0} \frac{\sqrt{|ay|}}{y}$$

$$= \lim_{y\to 0} \frac{\sqrt{|a|}\cdot\sqrt{|y|}}{y} = \sqrt{|a|} \lim_{y\to 0} \frac{\sqrt{|y|}}{y}$$

$$\lim_{\substack{y\to 0\\ y<0}} \frac{\sqrt{|y|}}{y} = \lim_{\substack{y\to 0\\ y<0}} \frac{\sqrt{-y}}{y} = \lim_{\substack{y\to 0\\ y<0}} \frac{\sqrt{-y}}{-(\sqrt{-y})^2} = \lim_{\substack{y\to 0\\ y<0}} \frac{1}{\sqrt{-y}} = -\frac{1}{0_+} = -\infty$$

$\Rightarrow$ nu există $\frac{\partial f}{\partial y}$ în pct $(a,0)$

$\Rightarrow$ nu există $T \Rightarrow f$ nu e diferențiabilă în $(a,0)$.

- În pct de forma $(0,b)$, $b\neq 0$ $\nexists \frac{\partial f}{\partial x} \Rightarrow f$ nu e diferențiabilă în $(0,b)$.

$\Rightarrow f$ este diferențiabilă pe $\mathbb{R}^2 / D_3$.

*b) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=x|y|$

$D_1=\{(x,y)\mid y>0\}$

$D_2=\{(x,y)\mid y<0\}$

$D_3=\{(x,y)\mid y=0\}$

$h(x,y)=y$

$D_1=\{(x,y)\mid y>0\}$

$=\{(x,y)\mid h(x,y)>0\}$

$=h^{-1}(0,\infty)\Rightarrow D_1$ deschis

Pe D_1 $f(x,y)=xy$

$\left.\begin{array}{l}\frac{\partial f}{\partial x}=y;\ \frac{\partial f}{\partial y}=x\\ D_1 \text{ deschis}\end{array}\right\}\Rightarrow f$ fct. de clasă C^1 pe D_1

$\Downarrow$

f e diferențiabilă

Analog pt. $D_2\Rightarrow f$ e diferențiabilă.

Studiem pe $D_3=\{(x,0)\mid x\in\mathbb{R}\}$

Studiem pe $D_3\setminus\{(0,0)\}$, $(x_0,0)$, $x_0\neq 0$.

$$\frac{\partial f}{\partial x}(x_0,0)=\lim_{x\to x_0}\frac{f(x,0)-f(x_0,0)}{x-x_0}=\lim_{x\to x_0}\frac{0-0}{x-x_0}=0$$

$$\frac{\partial f}{\partial y}(x_0,0)=\lim_{y\to 0}\frac{f(x_0,y)-f(x_0,0)}{y}=\lim_{y\to 0}\frac{x_0|y|}{y}$$

$$=x_0\lim_{y\to 0}\frac{|y|}{y}=x_0\cdot(\pm\infty)$$

$\Rightarrow \nexists\ \frac{\partial f}{\partial y}(x_0,0)\Rightarrow \nexists\ T\Rightarrow f$ nu e diferențiabilă pe $D_3\setminus\{0,0\}$.

Studiem diferențiabilitatea în $(0,0)$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=0$$

$T(h_1,h_2)=0$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-T((x,y)-(0,0))}{\|(x,y)-(0,0)\|}$$

$$= \lim_{(x,y)\to(0,0)} \underbrace{x}_{\to 0} \underbrace{\frac{|y|}{\sqrt{x^2+y^2}}}_{\text{mărg}} = 0$$

$\Rightarrow$ f diferențiabilă în $(0,0)$

$df(0)=0$

c) $h: \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) = \begin{cases} \dfrac{x|y|}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$\frac{\partial h}{\partial x}(0,0) = \lim_{x\to 0} \frac{h(x,0)-h(0,0)}{x} = 0$$

$$\frac{\partial h}{\partial y}(0,0) = \lim_{y\to 0} \frac{h(0,y)-h(0,0)}{y} = 0$$

$T=0$

$$\lim_{(x,y)\to(0,0)} \frac{h(x,y)-h(0,0)-T((x,y)-(0,0))}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{x|y|}{x^2+y^2}}_{\overset{not}{=}\, g}$$

$$\lim_{n\to\infty} g\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \frac{1}{2} \neq 0 \Rightarrow h \text{ nu e diferențiabilă în } (0,0).$$

SEMINAR 23

ex 3 | (CURS12)

Ar. că funcția $u = \frac{1}{\sqrt{y}} f(\underbrace{4x + \frac{z^2}{y}}_{t})$ verifică relația:

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$t(x,y) = 4x + \frac{z^2}{y}$$

$u = u(x,y,z)$

Derivăm după y

$$\frac{1}{\sqrt{y}} = y^{-\frac{1}{2}} \quad (y^{-\frac{1}{2}})' = -\frac{1}{2} y^{-\frac{3}{2}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y}} \cdot f(t)\right)$$

$$= \frac{\partial}{\partial y} \cdot \frac{1}{\sqrt{y}} \cdot f(t) + \frac{1}{\sqrt{y}} \cdot \frac{\partial}{\partial y} f(t)$$

$$\boxed{\frac{\partial}{\partial y} f(t) = f'(t) \cdot \frac{\partial}{\partial y}}$$

$$= -\frac{1}{2} y^{-\frac{3}{2}} f(t) + y^{-\frac{1}{2}} \cdot f'(t)\left(-\frac{z^2}{y^2}\right)$$

$$= -\frac{1}{2} y^{-\frac{3}{2}} f(t) - z^2 y^{-\frac{5}{2}} f'(t)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{2} y^{-\frac{3}{2}} f(t) - z^2 y^{-\frac{5}{2}} f'(t)\right)$$

$$= -\frac{1}{2} y^{-\frac{3}{2}} \frac{\partial}{\partial x} f(t) - z^2 y^{-\frac{5}{2}} \frac{\partial}{\partial x} f'(t)$$

$$= -\frac{1}{2} y^{-\frac{3}{2}}\left(f'(t) \cdot \frac{\partial}{\partial x} t\right) - z^2 y^{-\frac{5}{2}}\left(f''(t) \cdot \frac{\partial}{\partial x} t\right)$$

$$= -\frac{1}{2} y^{-\frac{3}{2}} \cdot f'(t) \cdot 4 - z^2 y^{-\frac{5}{2}} f''(t) \cdot 4$$

$$= -2y^{-\frac{3}{2}} f'(t) - 4z^2 y^{-\frac{5}{2}} f''(t) \quad (*)$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{\sqrt{y}} \cdot f(t)\right) = y^{-\frac{1}{2}} \frac{\partial}{\partial z} f(t) = y^{-\frac{1}{2}} \frac{2z}{y} f'(t)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(y^{-\frac{1}{2}} \cdot \frac{2z}{y} \cdot f'(t)\right)$$

$$= y^{-\frac{3}{2}} \cdot 2 \cdot \frac{\partial}{\partial z}\left(f'(t)\right) z$$

$$= y^{-\frac{3}{2}} \cdot 2 \cdot \left[f''(t) \cdot \frac{\partial}{\partial z} t \cdot z + f'(t) \cdot 1\right]$$

$$= 2y^{-\frac{3}{2}}\left[f''(t) \cdot \frac{2z}{y} \cdot z + f'(t)\right] \qquad (**)$$

$(*) + (**) = 0.$

ex 41 | CURS 12

Arătați că funcția $f(x,y,z) = f(xy, x^2+y^2-z^2)$

verifică relația: $\quad xz \frac{\partial f}{\partial x} - yz \frac{\partial f}{\partial y} + (x^2-y^2) \frac{\partial f}{\partial z} = 0$

$f = f(x,y,z) \qquad u(x,y,z) = xy$

$h = h(u,v) \qquad v(x,y,z) = x^2+y^2-z^2$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(h\left(u(x,y,z), v(x,y,z)\right)\right) = \frac{\partial h}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial h}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial h}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial h}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial u} \cdot \frac{\partial}{\partial x}(xy) + \frac{\partial h}{\partial v} \cdot \frac{\partial}{\partial x}(x^2+y^2-z^2)$$

$$= \frac{\partial h}{\partial u} y + \frac{\partial h}{\partial v} \cdot 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial h}{\partial u} \cdot \frac{\partial}{\partial y}(xy) + \frac{\partial h}{\partial v} \cdot \frac{\partial}{\partial y}(x^2+y^2-z^2)$$

$$= \frac{\partial h}{\partial u} x + \frac{\partial h}{\partial v} \cdot 2y$$

$$\frac{\partial f}{\partial z} = \frac{\partial h}{\partial u} \cdot \frac{\partial}{\partial z}(xy) + \frac{\partial h}{\partial v} \cdot \frac{\partial}{\partial z}(x^2+y^2-z^2) = \frac{\partial h}{\partial v}(-2z)$$

$$xz\frac{\partial f}{\partial x} - yz\frac{\partial f}{\partial y} + (x^2-y^2)\frac{\partial f}{\partial z} =$$

$$= xz\left(\frac{\partial h}{\partial u} \cdot y + \frac{\partial h}{\partial v} \cdot 2x\right) - yz\left(\frac{\partial h}{\partial u} x + \frac{\partial h}{\partial v} 2y\right) + (x^2-y^2)\left(\frac{\partial h}{\partial v}(-2z)\right)$$

$$= \frac{\partial h}{\partial u}(xzy - yzx) + \frac{\partial h}{\partial v}(2x^2z - 2y^2z - 2zx^2 + 2y^2z) = 0$$

ex 1 | CURS 12

Ar. că urm. fct. verif. ecuațiile

i) $f(x,y) = h\left(\frac{y}{x}\right) \cdot x f'(x + y f'y = 0$

Not. $u = \frac{y}{x} \Rightarrow f(x,y) = h(u(x,y))$, $h: \mathbb{R} \to \mathbb{R}$

$$f'(x) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[h\left(\frac{y}{x}\right)\right] = h'(u) \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = h'(u)\left(-\frac{y}{x^2}\right)$$

$$f'(x) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[h\left(\frac{y}{x}\right)\right] = h'(u) \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = h'(u)\frac{1}{x}$$

$$\Rightarrow x f'(x) + y f'y = x \cdot h' \cdot \cancel{u} \cdot \left(-\frac{y}{x^2}\right) + y \cdot h'(u) \cdot \frac{1}{x} = -\frac{y}{x}h'(u) + \frac{y}{x}h'(u) = 0$$

ii) $f(x,y) = xyh(x^2-y^2)$: $xy^2 f'x + x^2 y f'y = (x^2+y^2) f$

Not. $u(x,y) = x^2 - y^2 \Rightarrow f(x,y) = xyh(u(x,y))$

$$f'x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[xyh(u(x,y))] = \frac{\partial}{\partial x}(xy) \cdot h(u(x,y)) + xy \cdot \frac{\partial(h(u(x,y)}{\partial x}$$

$$= y \cdot h(u(x,y)) + xy \cdot h'(u) \cdot \frac{\partial u}{\partial x}$$

$$= y \cdot h(u(x,y)) + 2x^2 y h'(u)$$

$$f'y = \frac{\partial}{\partial y}[xyh(u(x,y))] = \frac{\partial}{\partial y}(xy) \cdot h(u(x,y)) + xy \cdot h'(u) \cdot \frac{\partial u}{\partial y}$$

$$= xh(u(x,y)) - 2xy^2 h'(u).$$

$xy^2(yh(u(x,y)) + 2x^2yh'(u)) + x^2y(xh(u(x,y)) - 2xy^2h'(u))$

$= h(u(x,y))(xy^3 + x^3y) + h'(u(x,y))(2x^3y^3 - 2x^3y^3)$

$= xy \cdot h(u(x,y))(y^2 + x^2) = f(x^2+y^2)$

iii) $f(x,y) = xy + xh\left(\frac{y}{x}\right)$: $xf'_x + yf'_y = xy + f$

Not. $u(x,y) = \frac{y}{x} \Rightarrow f(x,y) = xy + x \cdot h(u(x,y))$

$f'_x = \frac{\partial}{\partial x}[xy + xh(u(x,y))] = \frac{\partial}{\partial x}(xy) + x\frac{\partial}{\partial x}[h(u(x,y))]$.

$= y + \frac{\partial}{\partial x}(x) \cdot h(u(x,y)) + x \cdot \frac{\partial}{\partial x}(h(u(x,y)))$

$= y + h(u(x,y)) + x \cdot h'(u(x,y)) \cdot \left(-\frac{y}{x^2}\right) = y + h(u(x,y)) - \frac{y}{x}h'(u(x,y))$

$f'_y = \frac{\partial}{\partial y}[xy + xh(u(x,y))] = x + \frac{\partial}{\partial y}(x) \cdot h(u(x,y)) + x \cdot \frac{\partial}{\partial y}[h(u(x,y))]$

$= x + xh'(u(x,y)) \cdot \frac{1}{x} = x + h'(u(x,y))$

$x = f'_x + yf'_y = x(y + h(u(x,y)) - \frac{y}{x}h'(u(x,y)) + y(x + h'(u(x,y)))$

$= 2xy + xh(u(x,y)) = xy + f$

ex 2 | CURS 12

Ar. că funcția $f(x,y) = h(x-ay) + g(x+ay)$, unde funcțiile h, g admit derivate de ord II, satisface ec:

$f''_{y^2} = a^2 f''_{x^2}$

$u(x,y) = x - ay$

$v = (x+ay)$

$f'_x = \frac{\partial}{\partial x}[h(u) + g(v)] = \frac{\partial}{\partial x}(h(u)) + \frac{\partial}{\partial x}(g(v))$

$\Rightarrow f'_x = h'(u) \cdot 1 + g'(v) \cdot 1 = h'(u) + g'(v)$

$$f_y' = \frac{\partial}{\partial y}[h(u) + g(v)] = h\,(u)\cdot(-a) + g'(v)\,a$$

$$f''_{x^2} = \frac{\partial}{\partial x}[h'(u) + g'(v)] = \frac{\partial}{\partial x}(h'(u)) + \frac{\partial}{\partial x}(g'(v))$$

$$= h''(u)\cdot 1 + g''(v)\cdot 1 = h''(u) + g''(v)$$

$$f''_{y^2} = \frac{\partial}{\partial y}[h'(u)(-a) + g'(v)a] = \frac{\partial}{\partial y}[h'(u)(-a)] + \frac{\partial}{\partial y}[g'(v)\cdot a]$$

$$= h''(u)\cdot a^2 + g''(v)a^2 = a^2\cdot f''_{x^2}$$

Seminar 24

ex 5 (CURS 12)

Calculați $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ pentru:

i) $f(u,v) = \ln(u^2+v)$

$u = u(x,y) = e^{x+y^2}$

$v = v(x,y) = x^2+y$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial u} = \frac{\partial}{\partial u}(\ln(u^2+v)) = \frac{1}{u^2+v} \cdot 2u = \frac{2u}{u^2+v}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^{x+y^2}) = 1 \cdot e^{x+y^2} = e^{x+y^2}$$

$$\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}(\ln u^2+v) = \frac{1}{u^2+v}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x^2+y) = 2x$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{2u}{u^2+v} \cdot e^{x+y^2} + \frac{1}{u^2+v} \cdot 2x$$

$$\frac{\partial f}{\partial x} = \frac{2u \cdot e^{x+y^2} + 2x}{u^2+v} = \frac{2e^{x+y^2} \cdot e^{x+y^2} + 2x}{(e^{x+y^2})^2 + x^2 + y}$$

$$\frac{\partial f}{\partial x} = \frac{2(e^{2x+2y^2}+x)}{e^{2x+2y^2}+x^2+y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial u} = \frac{2u}{u^2+v}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^{x+y^2}) = 2y \cdot e^{x+y^2}$$

$$\frac{\partial f}{\partial v} = \frac{1}{u^2+v}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(x^2+y) = 1$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{2u}{u^2+v} \cdot 2y \cdot e^{x+y^2} + \frac{1}{u^2+v} \cdot 1$$

$$= \frac{4y\, e^{2x+2y^2}+1}{e^{2x+2y^2}+x^2+y}$$

ii) $f(u,v) = \operatorname{arctg} \frac{u}{v}$

$u = u(x,y) = x \sin y$

$v = v(x,y) = x \cos y$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial u} = \frac{\partial}{\partial u}\left(\operatorname{arctg} \frac{u}{v}\right) = \frac{1}{1+\left(\frac{u}{v}\right)^2} \cdot \left(\frac{u}{v}\right)' = \frac{1}{1+\left(\frac{u}{v}\right)^2} \cdot \frac{v}{v^2}$$

$$= \frac{1}{\frac{u^2+v^2}{v^2}} \cdot \frac{1}{v} = \frac{v^2}{u^2+v^2} \cdot \frac{1}{v} = \frac{v}{u^2+v^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x \sin y) = \sin y$$

$$\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\left(\operatorname{arctg}\frac{u}{v}\right) = \frac{1}{1+\left(\frac{u}{v}\right)^2}\cdot\left(\frac{u}{v}\right)' = \frac{1}{1+\left(\frac{u}{v}\right)^2}\cdot\frac{-u}{v^2}$$

$$= \frac{\cancel{v^2}}{u^2+v^2}\cdot\frac{-u}{\cancel{v^2}} = \frac{-u}{u^2+v^2}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x\cos y) = \cos y$$

$$\rightarrow \frac{\partial f}{\partial x} = \frac{v}{u^2+v^2}\sin y - \frac{u}{u^2+v^2}\cos y$$

$$= \frac{x\cos y\sin y - x\sin y\cos y}{x^2\underbrace{(\sin^2 y+\cos^2 y)}_{=1}} = \frac{x(\overbrace{\cos y\sin y-\cos y\sin y}^{=0})}{x^2} = 0$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial u} = \frac{v}{u^2+v^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x\sin y) = x\cos y$$

$$\frac{\partial f}{\partial v} = \frac{-u}{u^2+v^2}$$

$$\frac{\partial v}{\partial y} = \frac{v}{u^2+v^2}\cdot x\cos y + \frac{u}{u^2+v^2}\,x\sin y$$

$$= \frac{x^2\cos^2 y + x^2\sin^2 y}{x^2(\cos^2 y+\sin^2 y)} = \frac{x^2(\cos^2 y+\sin^2 y)}{x^2(\cos^2 y+\sin^2 y)} = 1.$$

ex 6 | CURS 12

$f: \mathbb{R}^2 \to \mathbb{R}$ diferențiabilă, at. $w = f(x+y, x-y)$ are deriv. parțiale care verifică relația:

$$\frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} = \left(\frac{\partial f}{\partial u}\right)^2 - \left(\frac{\partial f}{\partial v}\right)^2$$

$u = u(x, y) = x + y$

$v = v(x, y) = x - y$

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}(x+y) = 1 \\ \frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}(x-y) = 1 \end{aligned}\right\} \Rightarrow \frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \quad (*)$$

$$\left.\begin{aligned} \frac{\partial w}{\partial y} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(x+y) = 1 \\ \frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}(x-y) = -1 \end{aligned}\right\} \Rightarrow \frac{\partial w}{\partial y} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} \quad (**)$$

din $(*) + (**) \Rightarrow \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} = \left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)\left(\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}\right)$

$$= \left(\frac{\partial f}{\partial u}\right)^2 - \left(\frac{\partial f}{\partial v}\right)^2$$

ex 7 | CURS 12

Calculați : $h'(x)$ dacă $h(x) = f(u(x), v(x))$ pentru:

(i) $f(u,v) = u + u \cdot v$, $u = u(x) = \cos x$ și $v = v(x) = \sin x$

(ii) $f(u,v) = e^{u-2v}$, $u = u(x) = x^2$ și $v = v(x) = x^2 - 2$

(i) $h'(x) = \frac{\partial f}{\partial u} \cdot u'(x) + \frac{\partial f}{\partial v} \cdot v'(x)$

$$\frac{\partial f}{\partial u} = \frac{\partial}{\partial u} u + \frac{\partial}{\partial v} (uv) = 1 + v$$

$$\frac{\partial f}{\partial v} = \frac{\partial}{\partial v} u + \frac{\partial}{\partial v} (uv) = u$$

$u'(x) = -\sin x$

$v'(x) = \cos x$

$h'(x) = (1+v)(-\sin x) + u \cdot \cos x$

$h'(x) = (1+\sin x)(-\sin x) + \cos^2 x$

$h'(x) = -\sin x - \sin^2 x + \cos^2 x$

$h'(x) = \cos 2x - \sin x$

(ii) $f(u,v) = e^{u-2v}$

$$\frac{\partial f}{\partial u} e^{u-2v} = e^{u-2v} \frac{\partial}{\partial u} (u - 2v)$$

$$= e^{u-2v} \cdot 1 = e^{u-2v}$$

$$\frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left(e^{u-2v} \right) = e^{u-2v} \cdot \frac{\partial}{\partial v} (u - 2v)$$

$$= e^{u-2v} \cdot (-2) = -2e^{u-2v}$$

$u'(x) = 2x$

$v'(x) = 2x$

$h'(x) = e^{u-2v} \cdot 2x - 2e^{u-2v} \cdot 2x = e^{u-2v}(2x - 4x) = -2xe^{u-2v}$

$u - 2v = x^2 - 2(x^2-2) = 4 - x^2$

$h'(x) = -2x \cdot e^{4-x^2}$

ex 8 | CURS 12

Cercetați dacă $h(x,y) = e^y f(ye^{\frac{x^2}{2y^2}})$ verifică

$$(x^2-y^2)\frac{\partial h}{\partial x} + xy\frac{\partial h}{\partial y} = xyh$$

$h = h(x,y)$

$f = f(x)$

notăm : $t(x,y) = ye^{\frac{x^2}{2y^2}}$

$h(x,y) = e^y f(t(x,y))$

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\left(e^y(f(t(x,y))\right) = e^y \cdot f'(t)\frac{\partial v}{\partial x}$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial x}\left(ye^{\frac{x^2}{2y^2}}\right) = ye^{\frac{x^2}{2y^2}}\frac{\partial}{\partial x}\left(\frac{x^2}{2y^2}\right)$$

$$= y \cdot e^{\frac{x^2}{2y^2}} \cdot \frac{x}{y^2}$$

$$\frac{\partial h}{\partial x} = e^y f'(t) \cdot \frac{x}{y}e^{\frac{x^2}{2y^2}}, \quad h(x,y) = e^y f(ye^{\frac{x^2}{2y^2}})$$

$t(x,y) = ye^{\frac{x^2}{2y^2}}$

$$\frac{\partial h}{\partial y} = e^y \cdot f(t) + e^y \cdot f'(t)\frac{\partial t}{\partial y}$$

$$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y} \cdot y \cdot e^{\frac{x^2}{2y^2}} + y \cdot \frac{\partial}{\partial y} \cdot e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{x^2}{2y^2}\right) = e^{\frac{x^2}{2y^2}} \cdot \underbrace{\frac{\partial}{\partial y}\left(\frac{x^2}{2y^2}\right)}_{-\frac{x^2}{y^3}} = e^{\frac{x^2}{2y^2}} - y \cdot \frac{x^2}{y^3}$$

$$= e^{\frac{x^2}{2y^2}}\left(1 - \frac{x^2}{y^2}\right)$$

$$(x^2-y^2)\frac{\partial h}{\partial x} + xy\frac{\partial h}{\partial y} = xyh \Leftrightarrow (x^2-y^2)\cdot e^y \varphi'(t)\frac{x}{y} e^{\frac{x^2}{2y^2}}$$

$$+xy\left(e^y\varphi(t)+e^y\varphi'(t)\cdot e^{\frac{x^2}{2y^2}}\left(1-\frac{x^2}{y^2}\right)\right)$$

$$= \varphi\left[(x^2-y^2)e^y\frac{x}{y}e^{\frac{x^2}{2y^2}} + xy\cdot e^y\cdot e^{\frac{x^2}{2y^2}}\cdot\frac{y^2-x^2}{y^2}\right] + xyh = xyh$$

ex 10 / CURS 12

$$f(x,y)=\begin{cases} y^2\ln\left(1+\frac{x^2}{y^2}\right), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$\frac{\partial^2 f}{\partial x\partial y}$ și $\frac{\partial^2 f}{\partial y\partial x}$ nu sunt continue în $(0,0)$.

$$\frac{\partial^2 f}{\partial x\partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{x\to 0}\frac{\frac{\partial f}{\partial y}(x,0)-\frac{\partial f}{\partial y}(0,0)}{x}$$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{y\to 0}\frac{f(x,y)-f(x,0)}{y} = \lim_{y\to 0}\frac{y^2\ln\left(1+\frac{x^2}{y^2}\right)}{y}$$

$$= \lim_{y\to 0} y\ln\left(1+\frac{x^2}{y^2}\right) = \lim_{y\to 0} y\cdot\left(\ln\frac{x^2+y^2}{y^2}\right)$$

$$= \lim_{y\to 0} y\cdot \ln(x^2+y^2) - 2\ln y^2$$

$$= \lim_{y\to 0} \underbrace{y\ln(x^2+y^2)}_{=0} - \lim_{y\to 0}\underbrace{2y\ln y}_{=0} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0}\frac{0}{y} = 0.$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0} \frac{0-0}{x} = 0$$

analog $\frac{\partial^2 f}{\partial y \partial x} = 0$.

Studiem continuitatea

$$f(x,y) = y^2 \ln\left(1 + \frac{x^2}{y^2}\right) = y^2 \ln(x^2+y^2) - 2y^2 \ln y$$

$$\frac{\partial f}{\partial x} = \cancel{\frac{\partial}{\partial x}}\ y^2 \cdot \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial x}(x^2+y^2) - 0$$

$$\frac{\partial f}{\partial x} = \frac{2y^2 \cdot x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x \cdot y^2}{x^2+y^2}\right) = 2x \cdot \frac{2y(x^2+y^2) - y^2 \cdot 2y}{(x^2+y^2)^2} = \frac{4yx^3}{(x^2+y^2)^2}$$

$D_1 = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$

$D_2 = \mathbb{R}^2 \setminus D_1 = 0$

complementara lui $D_1 = D_2$

$$\left.\begin{aligned} &cD_1 = D_2 = \{(x,y) \mid y = 0\} - \{(x,y) \mid h(x,y) = 0\} \\ &h(x,y) = y \end{aligned}\right\} \Rightarrow \quad = h^{-1}\{0\}$$

= închis (mulț. finită)

$\Rightarrow D_1$ deschis.

$\Rightarrow$ fct. f este de clasă $C_2 \Rightarrow \frac{\partial^2 f}{\partial x \partial y}(a) = \frac{\partial^2 f}{\partial y \partial x}(a), \forall a \in D_1$.

$$\lim_{n \to \infty} g\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \to \infty} \frac{4 \cdot \frac{1}{n^4}}{\frac{4}{n^4}} = 1 \neq 0 \Rightarrow \lim_{n \to \infty} g\left(\frac{1}{n}, \frac{1}{n}\right) \neq g(0,0)$$

$\Rightarrow g$ nu e cont. în $(0,0)$.

Seminar 25

ex 9 | curs 12

Cercetați dacă funcțiile urm. verifică ec. indicate:

$\varphi, \psi : \mathbb{R} \to$

i) $f(x,y) = x\varphi(x+y) + y\psi(x-y)$: $\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} = 0$

Not. $u = u(x+y) = x+y$
$v = v(x,y) = x-y$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x+y) = 1$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x+y) = 1$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\varphi(u) + y\psi(v)) = \frac{\partial}{\partial x}x \cdot \varphi(u) + x \cdot \varphi'(u) \cdot \frac{\partial u}{\partial x} + y \cdot \psi'(v) \cdot \frac{\partial v}{\partial x}$$

$$= 1 \cdot \varphi(u) + x \cdot \varphi'(u) \cdot 1 + y \cdot \psi'(v) \cdot 1$$

$$= \varphi(u) + x\varphi'(u) + y \cdot \psi'(v)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\varphi(u) + y \cdot \psi(v))$$

$$= x \cdot \varphi'(u) \cdot 1 + 1 \cdot \psi(v) + y \cdot \psi'(v) \cdot (-1)$$

$$= x \cdot \varphi'(u) + \psi(v) - y \cdot \psi'(v)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\varphi(u) + x\varphi'(u) + y\psi'(v))$$

$$= \varphi'(u) \cdot 1 + 1 \cdot \varphi'(u) + x \cdot \varphi''(u) \cdot 1 + y \cdot \psi''(v) \cdot 1$$

$$= 2\varphi'(u) + x\cdot\varphi''(u) + y\cdot\psi''(v)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(x\cdot\varphi'(u) + \psi(v) - y\cdot\psi'(v)\right)$$

$$= x\cdot\varphi''(u)\cdot 1 + \psi''(v)\cdot(-1) - (1\cdot\psi'(v) + y\cdot\psi''(v)(-1))$$

$$= x\cdot\varphi''(u) - \psi''(v) - \psi'(v) + y\cdot\psi''(v)$$

$$= x\cdot\varphi''(u) - 2\psi'(v) + y\cdot\psi''(v)$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\,\varphi'(u) + \psi(v) - y\cdot\psi(v)\right)$$

$$= 1\cdot\varphi'(u) + x\cdot\varphi''(u)\cdot 1 + \psi'(v)\cdot 1 - y\cdot\psi''(v)\cdot 1$$

$$= \varphi'(u) + x\cdot\varphi''(u) + \psi'(v) - y\cdot\psi''(v)$$

$$d\,f(x,y)(h_1,h_2) = \frac{\partial f}{\partial x}(x,y)h_1 + \frac{\partial f}{\partial y}(x,y)h_2$$

$$= \left[\varphi(u) + x\,\varphi'(u) + y\psi'(v)\right]h_1 + \left[x\cdot\varphi'(u) + \psi(v)\cdot y\psi'(v)\right]h_2$$

$$\Rightarrow df(x,y)(1,-1) = \varphi(u) + \cancel{x\,\varphi'(u)} + y\psi'(v) - \cancel{x\cdot\varphi'(u)} - \psi(v) + y\psi'(v)$$

$$\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial^2 f}{\partial y^2} = d^2 f(x,y)(1,-1).$$

$$d^2 f(x,y)(h_1,h_2) = \frac{\partial^2 f}{\partial x^2}(x,y)\cdot h_1^2 + 2\frac{\partial^2 f}{\partial x\partial y}(x,y)h_1h_2 + \frac{\partial^2 f}{\partial y^2}(x,y)h_2^2$$

$$d^2 f(x,y)(1,-1) = 0,\ \forall (x,y) \subset \mathbb{R}^2$$

$\Rightarrow df(x,y)(1,-1) =$ constantă (în general nu)

ii) $f(x,y) = \varphi(x\cdot y) + \sqrt{xy}\,\psi\left(\frac{y}{x}\right)$: $x^2\cdot\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} = 0$

$$f(x,y) = \varphi(x\cdot y) + x^{\frac{1}{2}}\cdot y^{\frac{1}{2}}\cdot\psi(y\cdot x^{-1})$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\varphi(x\cdot y) + x^{\frac{1}{2}}y^{\frac{1}{2}}\psi(y\cdot x^{-1})\right)$$

$= \varphi'(xy)\cdot y + \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}\cdot\psi(y\cdot x^{-1}) + x^{\frac{1}{2}}y^{\frac{1}{2}}\cdot\psi'(y\cdot x^{-1})\cdot(-y\cdot x^{-2})$

$= \varphi'(xy)\,y + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{1}{2}}\cdot\psi(y\cdot x^{-1}) - x^{-\frac{3}{2}}\cdot y^{\frac{3}{2}}\psi'(y\cdot x^{-1})$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\varphi(xy) + x^{\frac{1}{2}}y^{\frac{1}{2}}\psi(yx^{-1})\right)$

$= \varphi'(xy)\cdot x + \frac{1}{2}x^{\frac{1}{2}}y^{\frac{1}{2}}\cdot\psi(yx^{-1}) + x^{\frac{1}{2}}y^{\frac{1}{2}}\cdot\psi'(y\cdot x^{-1})\,x^{-1}$

$= \varphi'(xy)\cdot x + \frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}\cdot\psi(yx^{-1}) + x^{-\frac{1}{2}}y^{\frac{1}{2}}\cdot\psi'(y\cdot x^{-1})$

~~$\varphi'(xy)\cdot x$~~

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(y\cdot\varphi'(xy) + \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}\psi(yx^{-1}) - x^{-\frac{3}{2}}y^{\frac{3}{2}}\cdot\psi'(yx^{-1})\right)$

$= y\cdot\varphi''(xy)\cdot y - \frac{1}{4}x^{-\frac{3}{2}}y^{\frac{1}{2}}\psi(yx^{-1}) + \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}\cdot\psi'(yx^{-1})\cdot(-yx^{-2}) -$

$-\left(-\frac{3}{2}x^{-\frac{5}{2}}\cdot y^{\frac{3}{2}}\cdot\psi'(yx^{-1})\right) + x^{-\frac{3}{2}}y^{\frac{3}{2}}\psi'(yx^{-1})(-yx^{-2})$

$\frac{\partial^2 f}{\partial y^2} = x^2\varphi''(xy) - \frac{1}{4}x^{\frac{1}{2}}y^{-\frac{3}{2}}\psi(yx^{-1}) + x^{-\frac{1}{2}}y^{-\frac{1}{2}}\psi'(yx^{-1}) +$
$+ x^{-\frac{3}{2}}y^{\frac{1}{2}}\psi''(yx^{-1})$

$x^2\cdot\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} = \varphi''(x^2y^2 - x^2y^2) + \psi''(\cancel{x^2}\cdot x^{-\frac{3}{2}}\cdot y^{\frac{3}{2}}\cdot(-\cancel{yx^{-2}}) -$
$- y^2\cdot x^{-\frac{3}{2}}y^{\frac{1}{2}})$

$= 0.$

ex 11 | CURS 12

Fie $f:\mathbb{R}^3\to\mathbb{R}$, $f(x,y,z) = x\cos(y\sin z)$

Motivați diferențiabilitatea funcției f și, în caz afirmativ, calculați:

$df(0,\frac{\pi}{2},\pi)$

f e diferențiabilă, fiind comp. de funcții elementare

$df(a)(h_1,h_2,h_3) = \frac{\partial f}{\partial x}(a)h_1 + \frac{\partial f}{\partial y}(a)h_2 + \frac{\partial f}{\partial z}(a)h_3$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y\sin z) = \cos(y\sin z)$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos(y\sin z)) = x\frac{\partial}{\partial y}\cos(y\sin z) = x(-\sin(y\sin z)\sin z$$

$$= -x\sin(y\sin z)\sin z$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x\cos(y\sin z)) = x\frac{\partial}{\partial x}\cos(y\sin z) = x(-\sin(y\sin z)\,y\cos z$$

$$= -xy\sin(y\sin z)\cos z$$

$$df\left(0,\tfrac{\pi}{2},\pi\right)(h_1,h_2,h_3) = \frac{\partial f}{\partial x}\left(0,\tfrac{\pi}{2},\pi\right)h_1 + \frac{\partial f}{\partial y}\left(0,\tfrac{\pi}{2},\pi\right)h_2 + \frac{\partial f}{\partial z}\left(0,\tfrac{\pi}{2},\pi\right)h_3$$

$$df\left(0,\tfrac{\pi}{2},\pi\right)(h_1,h_2,h_3) = \cos\left(\tfrac{\pi}{2}\sin\pi\right)h_1 + (-0\sin\left(\tfrac{\pi}{2}\sin\pi\right)\sin\pi\, h_2 -$$
$$- 0\,\tfrac{\pi}{2}\sin(0\sin\pi)\cos\pi\, h_3 = h_1$$

ex 14 | curs 12

Scrieți formulele lui Taylor cu rest Lagrange de ordin 2 pentru $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=e^x\sin y$ în $(0,0)$

$$f(x,y) = f(a) + \frac{1}{1!}df(a)((x,y)a) + \frac{1}{2!}(f(a)((x,y)-a)$$

$$df(a)(h_1,h_2) = \frac{\partial f}{\partial x}(a)h_1 + \frac{\partial f}{\partial y}(a)h_2$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}e^x\sin y = e^x\sin y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}e^x\sin y = e^x\cos y$$

$a=(0,0)$

$$df(0,0)(h_1,h_2) = \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0)h_2 = 0\cdot h_1 + 1\,h_2 = h_2$$

$$df(0,0)\,((x,y)-(0,0)) = y$$

$$d^2f(a)(h_1,h_2) = \frac{\partial^2 f}{\partial x^2}(a)h_1^2 + 2\frac{\partial^2 f}{\partial x\partial y}(a)h_1h_2 + \frac{\partial^2 f}{\partial y^2}(a)h_2^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(e^x\sin y) = e^x\sin y$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(e^x\cos y) = e^x\cos y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y$$

$$\Rightarrow d^2 f(0,0)(h_1,h_2) = \frac{\partial^2 f}{\partial x^2}(0,0)\, h_1^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0) h_1 h_2 + \frac{\partial^2 f}{\partial y^2}(0,0) h_2^2$$

$$= 0 h_1^2 + 2 \cdot 1 h_1 h_2 - 0 h_2^2 = 2 h_1 h_2$$

$$\Rightarrow d^2 f(0,0)\big((x,y)-(0,0)\big) = 2xy.$$

$a = (0,0)$

$$d^3 f(a)(h_1,h_2) = \frac{\partial f}{\partial x^2}(a) h_1^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(a) h_1^2 h_2 + 3\frac{\partial^3 f}{\partial x \partial y^2}(a) h_1 h_2^3 + \frac{\partial^3 f}{\partial y^3}(a) h_2^3$$

$$\frac{\partial^3 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x^2}\right) = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x^2}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x^2}\right) = \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial x}(-e^x \sin y) = -e^x \sin y$$

$$\frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial y}(-e^x \sin y) = e^x \cos y$$

$$f(x,y) = 0 + 1 \cdot y + \frac{1}{2!} 2xy + \frac{1}{3!}(e^{\xi_1} \sin \xi_2)\, x^3 + 3e^{\xi_1} \cos \xi_2 \,(x^2 y) - 3e^{\xi_1} \sin \xi_2 (xy$$
$$- e^{\xi_1} \cos \xi_2\, y^3)$$

Vrem să calculăm aprox. valoarea $e^{0,01} \sin(0,02)$.

$e^x \sin y \simeq y + xy$

obs! (x,y) în vecinătatea lui $(0,0)$

$x = 0,01$

$y = 0,02$

$$\Rightarrow e^{0,01} \sin(0,02) \simeq 0,02 + 0,01 \cdot 0,02$$

$$= 0,02 + 0,0002$$

$$= 0,0202.$$

$x = 0$

$y = \frac{\pi}{2}$

$$\Rightarrow e^0 \sin\frac{\pi}{2} \simeq \frac{\pi}{2} + 0\,\frac{\pi}{2} \Rightarrow 1 \simeq \frac{\pi}{2} \Rightarrow 1 \simeq \frac{3,14}{2}$$

$$\Rightarrow 1 \simeq 1,6$$

$(0,\frac{\pi}{2})$ departe de $(0,0)$

Seminar 26

ex 1 | CURS 13

Det. punctele de extrem local ale funcțiilor $f:\mathbb{R}\to\mathbb{R}$ definit prin:

i) $f(x,y) = x^4+y^4+2x^2y^2-8x+8y$, $a\in\mathbb{R}^2$ este pct. de extrem dacă $\nearrow f(x,y)\geq f(a), \forall(x,y)\in V$; $\searrow f(x,y)\leq f(a), \forall(x,y)\in V$

I. Det. pct. critice:

$$\begin{cases} \dfrac{\partial f}{\partial x}=0 \\ \dfrac{\partial f}{\partial y}=0 \end{cases} \quad (1)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^4+y^4+2x^2y^2-8x+8y) = 4x^3+4xy^2-8 = 4x(x^2+y^2)-8$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^4+y^4+2x^2y^2-8x+8y) = 4y^3+4x^2y+8 = 4y(y^2+x^2)+8$$

(1) devine: $\begin{cases} 4x(x^2+y^2)-8=0 \\ 4y(x^2+y^2)+8=0 \end{cases} \Leftrightarrow \begin{cases} 4x(x^2+y^2)=8 \\ 4y(x^2+y^2)=-8 \end{cases}$ $\left(\frac{a}{b}\right)$

$$\frac{4x}{4y}\cdot\frac{x^2+y^2}{x^2+y^2} = \frac{8}{-8} \Leftrightarrow \frac{x}{y}=-1$$

$\Rightarrow x=-y$

În prima ecuație: $x=-y \Rightarrow -y((-y^2)+y^2)=2$

$-y(2y^2)=2 \Rightarrow -2y^3=2 \Rightarrow y^3=-1 \Rightarrow \begin{cases} y=-1 \\ x=1 \end{cases}$

$\Rightarrow$ pct. critic $M(1,-1)$

II. Det. semnul Hessianei (H):

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}(4x^3+4xy^2-8) = 12x^2+4y^2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}(4x^3+4x^2y+8) = 12y^2+4x^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}(4y^3+4x^2+8) = 8xy$$

$$H_f(x,y) = \begin{pmatrix} 12x^2+4y^2 & 8xy \\ 8xy & 12y^2+4x^2 \end{pmatrix}$$

$$Hf(1,-1) = \begin{pmatrix} 16 & -8 \\ -8 & 16 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 16 > 0 \\ \Delta_2 = 192 > 0 \end{array}\right\} \Rightarrow M(1,-1) \text{ min. local}$$

$$f(1,-1) = 1^4 + (-1)^4 + 2\cdot 1^2 \cdot (-1)^2 - 8\cdot 1 + 8(-1)$$
$$= 1+1+2-8-8 = -12$$

$$f(x,y) \geq f(1,-1) \Leftrightarrow x^4+y^4+2x^2y^2-8x+8y \geq -12 \quad \forall (x,y) \in V(1,-1)$$

ii) $f(x,y) = xy + \frac{50}{x} + \frac{20}{y}$, $x>0$, $y>0$

I. $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xy + \frac{50}{x} + \frac{20}{y}\right) = y - \frac{50}{x^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy + \frac{50}{x} + \frac{20}{y}\right) = x - \frac{20}{y^2}$$

$$y - \frac{50}{x^2} = 0 \Rightarrow y = \frac{50}{x^2}$$
$$x - \frac{20}{y^2} = 0 \Rightarrow x = \frac{20}{y^2}$$

$$\Rightarrow \begin{cases} y = \frac{50}{x^2} \\ x - \frac{20}{\left(\frac{50}{x^2}\right)^2} = 0 \Rightarrow \end{cases}$$

$$x - 20 \cdot \frac{1}{\frac{50^2}{x^4}} = 0 \Rightarrow x - 20 \cdot \frac{x^4}{50^2} = 0$$

$$x - 2 \cdot \frac{x^4}{250} = 0 \Rightarrow x - \frac{x^4}{125} = 0$$

$$125x - x^4 = 0 \Rightarrow x(125 - x^3) = 0 \Rightarrow x = 5$$

$$y = 2$$

$M(5,2)$

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(y - \frac{50}{x^2}\right) = \frac{100}{x^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x - \frac{20}{y^2}\right) = 1$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x - \frac{20}{y^2}\right) = \frac{40}{y^3}$$

$$H_f(x,y) = \begin{pmatrix} \frac{100}{x^3} & 1 \\ 1 & \frac{40}{y^3} \end{pmatrix}$$

$$H_f(5,2) = \begin{pmatrix} \frac{100}{125} & 1 \\ 1 & \frac{40}{8} \end{pmatrix} = \begin{pmatrix} 4/5 & 1 \\ 1 & 5 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = \frac{4}{5} > 0 \\ \Delta_2 = 4 - 1 = 3 > 0 \end{array}\right\} \Rightarrow M(5,2) \text{ pct. minim Local}$$

$$f(5,2) = 5 \cdot 2 + \frac{50}{5} + \frac{20}{2} = 10 + 10 + 10 = 30$$

$$f(x,y) \geq 30 \Leftrightarrow xy + \frac{50}{x} + \frac{20}{y} \geq 30, \forall (x,y) \in V(5,2)$$

$$\begin{cases} x = 1 \\ y = -200 \end{cases}$$

$f(1,-200) = -100 + 50 - \frac{1}{10} \geq 30$ (F) $\Rightarrow M$ nu e pct. min. global

ex 2 | CURS 13

Det. pct. de ext. local ale fct. $f: \mathbb{R}^3 \to \mathbb{R}$

(i) $f(x,y,z) = x^2+y^2+z^2+2x+4y-6z$

I. pct. critice

$$\frac{\partial f}{\partial x} = 0 \Leftrightarrow \frac{\partial f}{\partial x}(x^2+y^2+z^2+2x+4y-6z) = 2x+2$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2+z^2+2x+4y-6z) = 2y+4$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+z^2+2x+4y-6z) = 2z-6$$

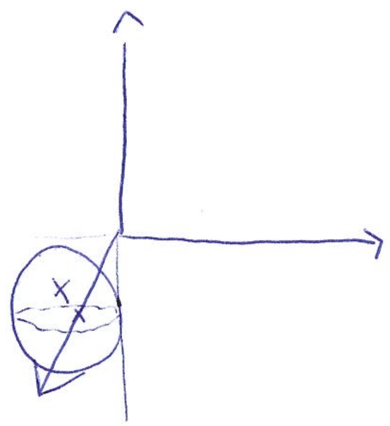
$$\begin{cases} 2x+2=0 \Rightarrow x=-1 \\ 2y+4=0 \Rightarrow y=-2 \\ 2z-6=0 \Rightarrow z=3 \end{cases} \Rightarrow M(-1,-2,3)$$

II. $$Hf(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$H(-1,-2,3) = \begin{pmatrix} \boxed{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Delta_3$$

$$\begin{rcases} \Delta_1 = 2 > 0 \\ \Delta_2 = 4 > 0 \\ \Delta_3 = 8 > 0 \end{rcases} \Rightarrow M \text{ - pct. min local}$$

$$f(-1,-2,3) = 1+4+9-2-4\cdot 2-6(3)$$
$$= -14$$



$f(x,y,z) \geq -14, \forall (x,y,z) \in V(-1,-2,3)$

$x^2+y^2+z^2+2x+4y-6z \geq -14 \Leftrightarrow (x+1)^2+(y+2)^2+(z-3)^2 \geq 0.$

$\Rightarrow$ „A" $\forall (x,y,z) \in \mathbb{R}^3 \Rightarrow M(-1,-2,3)$ minim global

ii) $f(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z} = 1 - \frac{y^2}{4x^2}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}\right) = 1 - \frac{y^2}{4x^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}\right) = \frac{y}{2x} - \frac{z^2}{y^2}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}\right) = \frac{2z}{y} - \frac{2}{z^2}$$

$$\begin{cases} 1 - \frac{y^2}{4x^2} = 0 \\ \frac{y}{2x} - \frac{z^2}{y^2} = 0 \\ \frac{2z}{y} - \frac{2}{z^2} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ \frac{2x}{2x} - \frac{z^2}{4x^2} = 0 \\ \frac{2z}{2x} - \frac{2}{z^2} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 2x \\ z = 2x \\ x = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} x = 1/2 \\ y = 1 \\ z = 1 \end{cases}$$

$\Rightarrow M\left(\frac{1}{2}, 1, 1\right)$ pct. critic.

$$Hf(x,y,z) = \begin{pmatrix} \frac{y^2}{2x^3} & -\frac{y}{2x^2} & 0 \\ -\frac{y}{2x^2} & \frac{1}{2x} + \frac{2z^2}{y^3} & -\frac{2z}{y^2} \\ 0 & -\frac{2z}{y^2} & \frac{2}{y} + \frac{4}{z^3} \end{pmatrix} \Rightarrow H\left(\frac{1}{2},1,1\right) = \begin{pmatrix} \boxed{4}_{\Delta_1} & -2 & 0 \\ -2 & 3_{\Delta_2} & -2 \\ 0 & 2 & 6 \end{pmatrix}_{\Delta_3}$$

$$\Downarrow$$

$$\left.\begin{array}{l} \Delta_1 = 4 > 0 \\ \Delta_2 = 8 > 0 \\ \Delta_3 = 32 > 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow M\left(\frac{1}{2}, 1, 1\right)$ pct. de min local

$$f\left(\frac{1}{2}, 1, 1\right) = \frac{1}{2} + \frac{1}{2} + 1 + 2 = 4$$

$$x + y^2/4x + z^2/y + 2/z > 4, \ \forall (x,y,z) \in V\left(\frac{1}{2}, 1, 1\right)$$

$$x - 2 \cdot \frac{\sqrt{x} \cdot y}{2\sqrt{x}} + \frac{y^2}{4x} + y - 2 \cdot \frac{z}{\sqrt{y}} \cdot \sqrt{y} + \frac{z^2}{y} + 2z - 2 \cdot \frac{\sqrt{2}}{\sqrt{z}} \cdot \sqrt{2} \cdot \sqrt{z} \cdot \frac{2}{z} \geq 0$$

$$\Rightarrow \left(\sqrt{x} - \frac{y}{2\sqrt{x}}\right)^2 + \left(\sqrt{y} - \frac{z}{\sqrt{y}}\right)^2 + \left(\sqrt{2z} - \frac{\sqrt{2}}{\sqrt{z}}\right) \geq 0, \ \forall (x,y,z) > 0.$$

Seminar 27

ex 2 | CURS 13

iii) $f(x,y,z) = \sin x + \sin y + \sin z - \sin(x+y+z)$, $\forall x,y,z \in (0,\pi)$

I. Det. pct. critice:

$$\frac{\partial f}{\partial x} = \cos x - \cos(x+y+z)$$

$$\frac{\partial f}{\partial y} = \cos y - \cos(x+y+z)$$

$$\frac{\partial f}{\partial z} = \cos z - \cos(x+y+z)$$

$$\begin{cases} \cos x - \cos(x+y+z) = 0 \\ \cos y - \cos(x+y+z) = 0 \\ \cos z - \cos(x+y+z) = 0 \end{cases} \Leftrightarrow \begin{cases} \cos x = \cos(x+y+z) \\ \cos y = \cos(x+y+z) \\ \cos z = \cos(x+y+z) \end{cases}$$

$$\left.\begin{array}{l} \Rightarrow \cos x = \cos y = \cos z = \cos(x+y+z) \\ (\cos x \text{ pe } (0,\pi) \to \text{injectivă}) \end{array}\right\} \Rightarrow x+y+z \in (0,3\pi)$$

$\Downarrow$

$x = y = z$.

$\cos x = \cos 3x$

$\cos x - \cos 3x = 0 \Rightarrow \cos x - (4\cos^3 x - 3\cos x) = 0$

$\cos x - 4\cos^3 x + 3\cos x = 0 \Leftrightarrow$

$4\cos x - 4\cos^3 x = 0$

$\Leftrightarrow \cos x(1-\cos^2 x) = 0 \Rightarrow \cos x = 0$ sau $\cos = \pm 1$

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}$

$\cos = \pm 1 \Rightarrow x = 0 \notin (0,\pi)$, $x = \pi \notin (0,\pi)$

$\Rightarrow M\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$ pct. critic

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos x - \cos(x+y+z)) = -\sin x + \sin(x+y+z)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(\cos y - \cos(x+y+z)) = -\sin y + \sin(x+y+z)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(\cos z - (\cos x+y+z\right) = -\sin z + \sin(x+y+z)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\cos y - \cos(x+y+z)\right)$$

$$= \sin(x+y+z)$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\cos z - \cos(x+y+z)\right) = \sin(x+y+z)$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\cos x - \cos(x+y+z)\right) = \sin(x+y+z)$$

$$H_f(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

$$= \begin{pmatrix} -\sin x + \sin(x+y+z) & \sin(x+y+z) & \sin(x+y+z) \\ \sin(x+y+z) & -\sin y + \sin(x+y+z) & \sin(x+y+z) \\ \sin(x+y+z) & \sin(x+y+z) & -\sin z + \sin(x+y+z) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} \Rightarrow \begin{array}{l} \Delta_1 = -2 < 0 \\ \Delta_2 = 3 > 0 \\ \Delta_3 = -4 < 0 \end{array}$$

$- + - \Rightarrow M\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$ pct. de max local

Valoarea maximă:

$f\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = 4$

$f(x,y,z) \leq 4 \ \forall (x,y,z)$ în jurul lui $M\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$

$f(x,y,z) \leq 4, \ \forall (x,y,z) \in (0,\pi) \times (0,\pi) \times (0,\pi)$?

$\sin x + \sin y + \sin z - \sin(x+y+z) \leq 4$ "A" $\Rightarrow M\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$ pct. de max. global

iv) $f(x,y,z) = 2(xy+z) - x^2 - y^2 - z^2$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2xy + 2z - x^2 - y^2 - z^2) = 2y - 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2xy + 2z - x^2 - y^2 - z^2) = 2x - 2y$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(2xy + 2z - x^2 - y^2 - z^2) = 2 - 2z$$

$$\begin{cases} 2y - 2x = 0 \\ 2x - 2y = 0 \\ 2 - 2z = 0 \end{cases} \Rightarrow \begin{cases} 2y = 2x \\ x = 2y \\ z = 1 \end{cases} \Leftrightarrow \begin{cases} y = x \\ z = 1 \end{cases} \Rightarrow M(x,x,1)\ \forall x \in \mathbb{R}$$

sunt critice

$$\frac{\partial^2 f}{\partial x^2} = -2 \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(2x - 2y) = 2$$

$$\frac{\partial^2 f}{\partial y^2} = -2 \qquad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}(2 - 2z) = 0$$

$$\frac{\partial^2 f}{\partial z^2} = -2 \qquad \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}(2y - 2x) = 0$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}(2 - 2z) = 0$$

$$H_f(x,y,z) = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\left.\begin{matrix} \Delta_1 = -2 \\ \Delta_2 = 0 \\ \Delta_3 = 0 \end{matrix}\right\} \text{caz de dubiu}$$

$f(x,x,1) = 2x^2 + 2 - x^2 - x^2 - 1 = 1$

$f(x,y,z) = 2(xy+z) - x^2 - y^2 - z^2$

$= 2xy + 2z - x^2 - y^2 - z^2 \leq 1 = f(x,x,1)$

$(x+y)^2 + (z+1)^2 \geq f(x,x,1)$

$\Rightarrow (x,x,1)$ pct. de maxim local.