Contents

43 The Laplace Transform: Basic Definitions and Results

Laplace transform is yet another operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:
• The given "hard" problem is transformed into a "simple" equation.
• This simple equation is solved by purely algebraic manipulations.
• The solution of the simple equation is transformed back to obtain the solution of the given problem.
In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.
The above procedure can be summarized by Figure 43.1

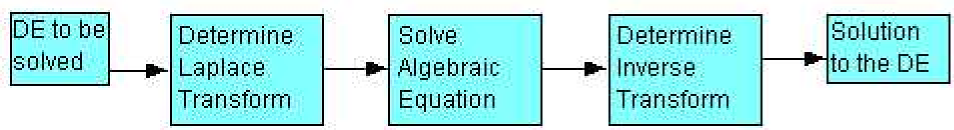


Figure 43.1

In this section we introduce the concept of Laplace transform and discuss some of its properties.
The Laplace transform is defined in the following way. Let $f(t)$ be defined for $t \geq 0$. Then the **Laplace transform** of f, which is denoted by $\mathcal{L}[f(t)]$ or by $F(s)$, is defined by the following equation

$$\mathcal{L}[f(t)] = F(s) = \lim_{T\to\infty} \int_0^T f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt$$

The integral which defined a Laplace transform is an improper integral. An improper integral may **converge** or **diverge**, depending on the integrand. When the improper integral in convergent then we say that the function $f(t)$ possesses a Laplace transform. So what types of functions possess Laplace transforms, that is, what type of functions guarantees a convergent improper integral.

Example 43.1
Find the Laplace transform, if it exists, of each of the following functions

$$(a)\ f(t) = e^{at} \quad (b)\ f(t) = 1 \quad (c)\ f(t) = t \quad (d)\ f(t) = e^{t^2}$$

Solution.
(a) Using the definition of Laplace transform we see that

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}dt = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}dt.$$

But

$$\int_0^T e^{-(s-a)t}dt = \begin{cases} T & \text{if } s=a \\ \frac{1-e^{-(s-a)T}}{s-a} & \text{if } s\neq a. \end{cases}$$

For the improper integral to converge we need $s > a$. In this case,

$$\mathcal{L}[e^{at}] = F(s) = \frac{1}{s-a}, \quad s > a.$$

(b) In a similar way to what was done in part (a), we find

$$\mathcal{L}[1] = \int_0^\infty e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-st}dt = \frac{1}{s}, \; s > 0.$$

(c) We have

$$\mathcal{L}[t] = \int_0^\infty te^{-st}dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}, \; s > 0.$$

(d) Again using the definition of Laplace transform we find

$$\mathcal{L}[e^{t^2}] = \int_0^\infty e^{t^2-st}dt.$$

If $s \leq 0$ then $t^2 - st \geq 0$ so that $e^{t^2-st} \geq 1$ and this implies that $\int_0^\infty e^{t^2-st}dt \geq \int_0^\infty$. Since the integral on the right is divergent, by the comparison theorem of improper integrals (see Theorem 43.1 below) the integral on the left is also divergent. Now, if $s > 0$ then $\int_0^\infty e^{t(t-s)}dt \geq \int_s^\infty dt$. By the same reasoning the integral on the left is divergent. This shows that the function $f(t) = e^{t^2}$ does not possess a Laplace transform ■

The above example raises the question of what class or classes of functions possess a Laplace transform. Looking closely at Example 43.1(a), we notice that for $s > a$ the integral $\int_0^\infty e^{-(s-a)t}dt$ is convergent and a critical component for this convergence is the type of the function $f(t)$. To be more specific, if $f(t)$ is a continuous function such that

$$|f(t)| \leq Me^{at}, \quad t \geq C \tag{1}$$

where $M \geq 0$ and a and C are constants, then this condition yields

$$\int_0^\infty f(t)e^{-st}dt \leq \int_0^C f(t)e^{-st}dt + M\int_C^\infty e^{-(s-a)t}dt.$$

Since $f(t)$ is continuous in $0 \leq t \leq C$, by letting $A = \max\{|f(t)| : 0 \leq t \leq C\}$ we have

$$\int_0^C f(t)e^{-st}dt \leq A\int_0^C e^{-st}dt = A\left(\frac{1}{s} - \frac{e^{-sC}}{s}\right) < \infty.$$

On the other hand, Now, by Example 43.1(a), the integral $\int_C^\infty e^{-(s-a)t}dt$ is convergent for $s > a$. By the comparison theorem of improper integrals (see Theorem 43.1 below) the integral on the left is also convergent. That is, $f(t)$ possesses a Laplace transform.
We call a function that satisfies condition (1) a function with an **exponential order at infinity.** Graphically, this means that the graph of $f(t)$ is contained in the region bounded by the graphs of $y = Me^{at}$ and $y = -Me^{at}$ for $t \geq C$. Note also that this type of functions controls the negative exponential in the transform integral so that to keep the integral from blowing up. If $C = 0$ then we say that the function is **exponentially bounded**.

Example 43.2
Show that any bounded function $f(t)$ for $t \geq 0$ is exponentially bounded.

Solution.
Since $f(t)$ is bounded for $t \geq 0$, there is a positive constant M such that $|f(t)| \leq M$ for all $t \geq 0$. But this is the same as (1) with $a = 0$ and $C = 0$. Thus, $f(t)$ has is exponentially bounded ■

Another question that comes to mind is whether it is possible to relax the condition of continuity on the function $f(t)$. Let's look at the following situation.

Example 43.3
Show that the square wave function whose graph is given in Figure 43.2 possesses a Laplace transform.

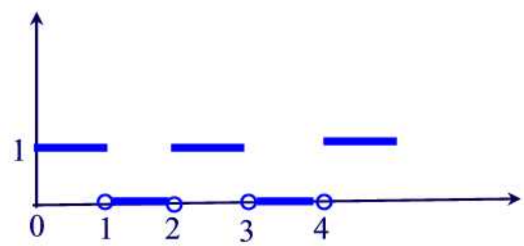


Figure 43.2

Note that the function is periodic of period 2.

Solution.
Since $f(t)e^{-st} \leq e^{-st}$, we have $\int_0^\infty f(t)e^{-st}dt \leq \int_0^\infty e^{-st}dt$. But the integral on the right is convergent for $s > 0$ so that the integral on the left is convergent as well. That is, $\mathcal{L}[f(t)]$ exists for $s > 0$ ■

The function of the above example belongs to a class of functions that we define next. A function is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (i.e. the subinterval without its endpoints) and has a finite limit at the endpoints (**jump discontinuities** and no vertical asymptotes) of each subinterval. Below is a sketch of a piecewise continuous function.

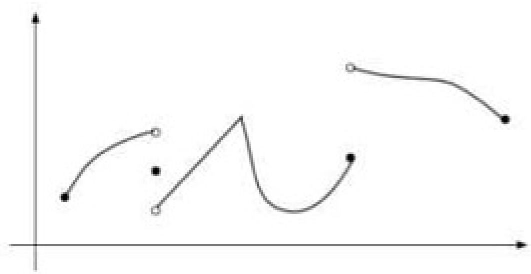


Figure 43.3

Note that a piecewise continuous function is a function that has a finite number of breaks in it and doesnt blow up to infinity anywhere. A function defined for $t \geq 0$ is said to be **piecewise continuous on the infinite interval** if it is piecewise continuous on $0 \leq t \leq T$ for all $T > 0$.

Example 43.4
Show that the following functions are piecewise continuous and of exponential order at infinity for $t \geq 0$

$$(a)\ f(t) = t^n \quad (b)\ f(t) = t^n \sin at$$

Solution.
(a) Since $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geq \frac{t^n}{n!}$, we have $t^n \leq n!e^t$. Hence, t^n is piecewise continuous and exponentially bounded.
(b) Since $|t^n \sin at| \leq n!e^t$, we have $t^n \sin at$ is piecewise continuous and exponentially bounded ■

Next, we would like to establish the existence of the Laplace transform for all functions that are piecewise continuous and have exponential order at infinity. For that purpose we need the following comparison theorem from calculus.

Theorem 43.1
Suppose that $f(t)$ and $g(t)$ are both integrable functions for all $t \geq t_0$ such that $|f(t)| \leq |g(t)$ for $t \geq t_0$. If $\int_{t_0}^{\infty} g(t)dt$ is convergent, then $\int_{t_0}^{\infty} f(t)dt$ is also convergent. If, on the other hand, $\int_{t_0}^{\infty} f(t)dt$ is divergent then $\int_{t_0}^{\infty} f(t)dt$ is also divergent.

Theorem 43.2 *(Existence)*
Suppose that $f(t)$ is piecewise continuous on $t \geq 0$ and has an exponential order at infinity with $|f(t)| \leq Me^{at}$ for $t \geq C$. Then the Laplace transform

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

exists as long as $s > a$. Note that the two conditions above are sufficient, but not necessary, for $F(s)$ to exist.

Proof.
The integral in the definition of $F(s)$ can be splitted into two integrals as follows

$$\int_0^{\infty} f(t)e^{-st}dt = \int_0^{C} f(t)e^{-st}dt + \int_C^{\infty} f(t)e^{-st}dt.$$

Since $f(t)$ is piecewise continuous in $0 \leq t \leq C$, it is bounded there. By letting $A = \max\{|f(t)| : 0 \leq t \leq C\}$ we have

$$\int_0^{C} f(t)e^{-st}dt \leq A\int_0^{C} e^{-st}dt = A\left(\frac{1}{s} - \frac{e^{-sC}}{s}\right) < \infty.$$

Now, by Example 43.1(a), the integral $\int_C^\infty f(t)e^{-st}dt$ is convergent for $s > a$. By Theorem 43.1 the integral on the left is also convergent. That is, $f(t)$ possesses a Laplace transform ■

In what follows, we will denote the class of all piecewise continuous functions with exponential order at infinity by $\mathcal{PE}$. The next theorem shows that any linear combination of functions in $\mathcal{PE}$ is also in $\mathcal{PE}$. The same is true for the product of two functions in $\mathcal{PE}$.

Theorem 43.3
Suppose that $f(t)$ and $g(t)$ are two elements of $\mathcal{PE}$ with

$$|f(t)| \le M_1 e^{a_1 t}, \quad t \ge C_1 \quad \text{and} \quad |g(t)| \le M_2 e^{a_1 t}, \quad t \ge C_2.$$

(i) For any constants α and β the function $\alpha f(t) + \beta g(t)$ is also a member of $\mathcal{PE}$. Moreover

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)].$$

(ii) The function $h(t) = f(t)g(t)$ is an element of $\mathcal{PE}$.

Proof.
(i) It is easy to see that $\alpha f(t) + \beta g(t)$ is a piecewise continuous function. Now, let $C = C_1 + C_2,\ a = \max\{a_1, a_2\}$, and $M = |\alpha|M_1 + |\beta|M_2$. Then for $t \ge C$ we have

$$|\alpha f(t) + \beta g(t)| \le |\alpha||f(t)| + |\beta||g(t)| \le |\alpha|M_1 e^{a_1 t} + |\beta|M_2 e^{a_2 t} \le M e^{at}.$$

This shows that $\alpha f(t) + \beta g(t)$ is of exponential order at infinity. On the other hand,

$$\begin{aligned}
\mathcal{L}[\alpha f(t) + \beta g(t)] &= \lim_{T\to\infty} \int_0^T [\alpha f(t) + \beta g(t)]dt \\
&= \alpha \lim_{T\to\infty} \int_0^T f(t)dt + \beta \lim_{T\to\infty} \int_0^T g(t)dt \\
&= \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]
\end{aligned}$$

(ii) It is clear that $h(t) = f(t)g(t)$ is a piecewise continuous function. Now, letting $C = C_1 + C_2,\ M = M_1 M_2$, and $a = a_1 + a_2$ then we see that for $t \ge C$ we have

$$|h(t)| = |f(t)||g(t)| \le M_1 M_2 e^{(a_1+a_2)t} = M e^{at}.$$

Hence, $h(t)$ is of exponential order at infinity. By Theorem 43.2 , $\mathcal{L}[h(t)]$ exists for $s > a$ ■

We next discuss the problem of how to determine the function $f(t)$ if $F(s)$ is given. That is, how do we invert the transform. The following result on uniqueness provides a possible answer. This result establishes a one-to-one correspondence between the set $\mathcal{PE}$ and its Laplace transforms. Alternatively, the following theorem asserts that the Laplace transform of a member in $\mathcal{PE}$ is unique.

Theorem 43.4
Let $f(t)$ and $g(t)$ be two elements in $\mathcal{PE}$ with Laplace transforms $F(s)$ and $G(s)$ such that $F(s) = G(s)$ for some $s > a$. Then $f(t) = g(t)$ for all $t \geq 0$ where both functions are continuous.

The standard techniques used to prove this theorem(i.e., complex analysis, residue computations, and/or Fourier's integral inversion theorem) are generally beyond the scope of an introductory differential equations course. The interested reader can find a proof in the book "Operational Mathematics" by Ruel Vance Churchill or in D.V. Widder "The Laplace Transform".
With the above theorem, we can now officially define the inverse Laplace transform as follows: For a piecewise continuous function f of exponential order at infinity whose Laplace transform is F, we call f the **inverse Laplace transform** of F and write $f = \mathcal{L}^{-1}[F(s)]$. Symbolically

$$f(t) = \mathcal{L}^{-1}[F(s)] \Longleftrightarrow F(s) = \mathcal{L}[f(t)].$$

Example 43.5
Find $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right), \quad s > 1.$

Solution.
From Example 43.1(a), we have that $\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a$. In particular, for $a = 1$ we find that $\mathcal{L}[e^t] = \frac{1}{s-1}, \ s > 1$. Hence, $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t, \ t \geq 0$ ■.

The above theorem states that if $f(t)$ is continuous and has a Laplace transform $F(s)$, then there is no other function that has the same Laplace transform. To find $\mathcal{L}^{-1}[F(s)]$, we can inspect tables of Laplace transforms of known functions to find a particular $f(t)$ that yields the given $F(s)$.
When the function $f(t)$ is not continuous, the uniqueness of the inverse

Laplace transform is not assured. The following example addresses the uniqueness issue.

Example 43.6
Consider the two functions $f(t) = h(t)h(3-t)$ and $g(t) = h(t) - h(t-3)$.

(a) Are the two functions identical?
(b) Show that $\mathcal{L}[f(t)] = \mathcal{L}[g(t)$.

Solution.
(a) We have

$$f(t) = \begin{cases} 1, & 0 \le t \le 3 \\ 0, & t > 3 \end{cases}$$

and

$$g(t) = \begin{cases} 1, & 0 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

So the two functions are equal for all $t \neq 3$ and so they are not identical.
(b) We have

$$\mathcal{L}[f(t)] = \mathcal{L}[g(t)] = \int_0^3 e^{-st}dt = \frac{1-e^{-3s}}{s}, s > 0.$$

Thus, both functions $f(t)$ and $g(t)$ have the same Laplace transform even though they are not identical. However, they are equal on the interval(s) where they are both continuous ■

The inverse Laplace transform possesses a linear property as indicated in the following result.

Theorem 43.5
Given two Laplace transforms $F(s)$ and $G(s)$ then

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$$

for any constants a and b.

Proof.
Suppose that $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$. Since $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s)$ we have $\mathcal{L}^{-1}[aF(s) + bG(s)] = af(t) + bg(t) = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$ ■

Practice Problems

Problem 43.1
Determine whether the integral $\int_0^\infty \frac{1}{1+t^2}dt$ converges. If the integral converges, give its value.

Problem 43.2
Determine whether the integral $\int_0^\infty \frac{t}{1+t^2}dt$ converges. If the integral converges, give its value.

Problem 43.3
Determine whether the integral $\int_0^\infty e^{-t}\cos(e^{-t})dt$ converges. If the integral converges, give its value.

Problem 43.4
Using the definition, find $\mathcal{L}[e^{3t}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 43.5
Using the definition, find $\mathcal{L}[t-5]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 43.6
Using the definition, find $\mathcal{L}[e^{(t-1)^2}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 43.7
Using the definition, find $\mathcal{L}[(t-2)^2]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Problem 43.8
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & t \ge 1 \end{cases}$$

Problem 43.9
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$$

Problem 43.10
Let n be a positive integer. Using integration by parts establish the reduction formula

$$\int t^n e^{-st}dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s}\int t^{n-1}e^{-st}dt, \quad s > 0.$$

Problem 43.11
For $s > 0$ and n a positive integer evaluate the limits

$$\lim_{t\to 0} t^n e^{-st} \qquad \text{(b) } \lim_{t\to\infty} t^n e^{-st}$$

Problem 43.12
(a) Use the previous two problems to derive the reduction formula for the Laplace transform of $f(t) = t^n$,

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}], \quad s > 0.$$

(b) Calculate $\mathcal{L}[t^k]$, for $k = 1, 2, 3, 4, 5$.
(c) Formulate a conjecture as to the Laplace transform of $f(t), t^n$ with n a positive integer.

From a table of integrals,

$$\begin{aligned} \int e^{\alpha u}\sin\beta u du &= e^{\alpha u}\tfrac{\alpha\sin\beta u - \beta\sin\beta u}{\alpha^2+\beta^2} \\ \int e^{\alpha u}\cos\beta u du &= e^{\alpha u}\tfrac{\alpha\cos\beta u + \beta\sin\beta u}{\alpha^2+\beta^2} \end{aligned}$$

Problem 43.13
Use the above integrals to find the Laplace transform of $f(t) = \cos\omega t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Problem 43.14
Use the above integrals to find the Laplace transform of $f(t) = \sin\omega t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Problem 43.15
Use the above integrals to find the Laplace transform of $f(t) = \cos\omega(t-2)$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Problem 43.16
Use the above integrals to find the Laplace transform of $f(t) = e^{3t}\sin t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Problem 43.17
Use the linearity property of Laplace transform to find $\mathcal{L}[5e^{-7t} + t + 2e^{2t}]$. Find the domain of $F(s)$.

Problem 43.18
Consider the function $f(t) = \tan t$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

Problem 43.19
Consider the function $f(t) = t^2e^{-t}$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

Problem 43.20
Consider the function $f(t) = \frac{e^{t^2}}{e^{2t}+1}$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

Problem 43.21
Consider the floor function $f(t) = \lfloor t \rfloor$, where for any integer n we have $\lfloor t \rfloor = n$ for all $n \le t < n+1$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

Problem 43.22
Find $\mathcal{L}^{-1}\left(\frac{3}{s-2}\right)$.

Problem 43.23
Find $\mathcal{L}^{-1}\left(-\frac{2}{s^2}+\frac{1}{s+1}\right)$.

Problem 43.24
Find $\mathcal{L}^{-1}\left(\frac{2}{s+2}+\frac{2}{s-2}\right)$.

44 Further Studies of Laplace Transform

Properties of the Laplace transform enable us to find Laplace transforms without having to compute them directly from the definition. In this section, we establish properties of Laplace transform that will be useful for solving ODEs.

Laplace Transform of the Heaviside Step Function
The Heaviside step function is a piecewise continuous function defined by

$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Figure 44.1 displays the graph of $h(t)$.

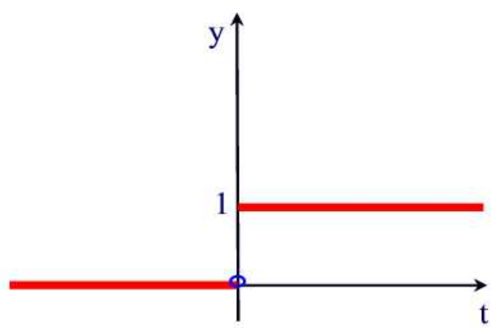


Figure 44.1

Taking the Laplace transform of $h(t)$ we find

$$\mathcal{L}[h(t)] = \int_0^\infty h(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}, \ s > 0.$$

A Heaviside function at $\alpha \geq 0$ is the shifted function $h(t-\alpha)$ (α units to the right). For this function, the Laplace transform is

$$\mathcal{L}[h(t-\alpha)] = \int_0^\infty h(t-\alpha)e^{-st}dt = \int_\alpha^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_\alpha^\infty = \frac{e^{-s\alpha}}{s}, \ s > 0.$$

Laplace Tranform of e^{at}
The Laplace transform for the function $f(t) = e^{at}$ is

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}, \ s > a.$$

Laplace Tranforms of $\sin at$ and $\cos at$
Using integration by parts twice we find

$$\begin{aligned}
\mathcal{L}[\sin at] &= \int_0^\infty e^{-st}\sin at dt \\
&= \left[-\frac{e^{-st}\sin at}{s} - \frac{ae^{-st}\cos at}{s^2}\right]_0^\infty - \frac{a^2}{s^2}\int_0^\infty e^{-st}\sin at dt \\
&= -\frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}[\sin at] \\
\left(\frac{s^2+a^2}{s^2}\right)\mathcal{L}[\sin at] &= \frac{a}{s^2} \\
\mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}, \quad s>0
\end{aligned}$$

A similar argument shows that

$$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}, \quad s>0.$$

Laplace Transforms of $\cosh at$ and $\sinh at$
Using the linear property of $\mathcal{L}$ we can write

$$\begin{aligned}
\mathcal{L}[\cosh at] &= \frac{1}{2}\left(\mathcal{L}[e^{at}] + \mathcal{L}[e^{-at}]\right) \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right), \quad s>|a| \\
&= \frac{s}{s^2-a^2}, \quad s>|a|
\end{aligned}$$

A similar argument shows that

$$\mathcal{L}[\sin at] = \frac{a}{s^2-a^2}, \quad s>|a|.$$

Laplace Transform of a Polynomial
Let n be a positive integer. Using integration by parts we can write

$$\int_0^\infty t^n e^{-st}dt = -\left[\frac{t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt.$$

By repeated use of L'Hôpital's rule we find $\lim_{t\to\infty} t^n e^{-st} = \lim_{t\to\infty}\frac{n!}{s^n e^{st}} = 0$ for $s>0$. Thus,

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}], \quad s>0.$$

Using induction on $n = 0, 1, 2, \cdots$ one can easily eastablish that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0.$$

Using the above result together with the linearity property of $\mathcal{L}$ one can find the Laplace transform of any polynomial.
The next two results are referred to as the first and second shift theorems. As with the linearity property, the shift theorems increase the number of functions for which we can easily find Laplace transforms.

Theorem 44.1 *(First Shifting Theorem)*
If $f(t)$ is a piecewise continuous function for $t \geq 0$ and has exponential order at infinity with $|f(t)| \leq Me^{at}, \quad t \geq C$, then for any real number α we have

$$\mathcal{L}[e^{\alpha t} f(t)] = F(s - \alpha), \quad s > a + \alpha$$

where $\mathcal{L}[f(t)] = F(s)$.

Proof.
From the definition of the Laplace transform we have

$$\mathcal{L}[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt.$$

Using the change of variable $\beta = s - a$ the previous equation reduces to

$$\mathcal{L}[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t) dt = \int_0^\infty e^{-\beta t} f(t) dt = F(\beta) = F(s - a), \;\; s > a + \alpha \;\blacksquare$$

Theorem 44.2 *(Second Shifting Theorem)*
If $f(t)$ is a piecewise continuous function for $t \geq 0$ and has exponential order at infinity with $|f(t)| \leq Me^{at}, \quad t \geq C$, then for any real number $\alpha \geq 0$ we have

$$\mathcal{L}[f(t - \alpha) h(t - \alpha)] = e^{-\alpha s} F(s), \quad s > a$$

where $\mathcal{L}[f(t)] = F(s)$ and $h(t)$ is the Heaviside step function.

Proof.
From the definition of the Laplace transform we have

$$\mathcal{L}[f(t - \alpha) h(t - \alpha)] = \int_0^\infty f(t - \alpha) h(s - \alpha) e^{-st} dt = \int_\alpha^\infty f(t - \alpha) e^{-st} dt.$$

Using the change of variable $\beta = t - \alpha$ the previous equation reduces to

$$\begin{aligned}\mathcal{L}[f(t-\alpha)h(t-\alpha)] &= \int_0^\infty f(\beta)e^{-s(\beta+\alpha)}d\beta \\ &= e^{-s\alpha}\int_0^\infty f(\beta)e^{-s\beta}d\beta = e^{-s\alpha}F(s), \quad s > a \blacksquare\end{aligned}$$

Example 44.1
Find

(a) $\mathcal{L}[e^{2t}t^2]$ (b) $\mathcal{L}[e^{3t}\cos 2t]$ (c) $\mathcal{L}^{-1}[e^{-2t}s^2]$

Solution.
(a) By Theorem 44.1, we have $\mathcal{L}[e^{2t}t^2] = F(s-2)$ where $\mathcal{L}[t^2] = \frac{2!}{s^3} = F(s),\ s > 0$. Thus, $\mathcal{L}[e^{2t}t^2] = \frac{2}{(s-2)^3},\ s > 2$.
(b) As in part (a), we have $\mathcal{L}[e^{3t}\cos 2t] = F(s-3)$ where $\mathcal{L}[\cos 2t] = F(s-3)$. But $\mathcal{L}[\cos 2t] = \frac{s}{s^2+4},\ s > 0$. Thus,

$$\mathcal{L}[e^{3t}\cos 2t] = \frac{s-3}{(s-3)^2+4}, \quad s > 3$$

(c) Since $\mathcal{L}[t] = \frac{1}{s^2}$, by Theorem 44.2, we have

$$\frac{e^{-2t}}{s^2} = \mathcal{L}[(t-2)h(t-2)].$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{e^{-2t}}{s^2}\right] = (t-2)h(t-2) = \begin{cases} 0, & 0 \le t < 2 \\ t-2, & t \ge 2 \end{cases} \blacksquare$$

The following result relates the Laplace transform of derivatives and integrals to the Laplace transform of the function itself.

Theorem 44.3
Suppose that $f(t)$ is continuous for $t \ge 0$ and $f'(t)$ is piecewise continuous of exponential order at infinity with $|f'(t)| \le Me^{at},\ t \ge C$ Then

(a) $f(t)$ is of exponential order at infinity.
(b) $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) = sF(s) - f(0),\ s > \max\{a, 0\} + 1$.
(c) $\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0) = s^2F(s) - sf(0) - f(0),\ s > \max\{a, 0\} + 1$.
(d) $\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{\mathcal{L}[f(t)]}{s} = \frac{F(s)}{s}, \quad s > \max\{a, 0\} + 1$.

Proof.

(a) By the Fundamental Theorem of Calculus we have $f(t) = f(0) - \int_0^t f'(u)du$. Also, since f' is piecewise continuous, $|f'(t)| \leq T$ for some $T > 0$ and all $0 \leq t \leq C$. Thus,

$$\begin{aligned} |f(t)| &= \left| f(0) - \int_0^t f'(u)du \right| = |f(0) - \int_0^C f'(u)du - \int_C^t f'(u)du| \\ &\leq |f(0)| + TC + M\int_C^t e^{au}du \end{aligned}$$

Note that if $a > 0$ then

$$\int_C^t e^{au}du = \frac{1}{a}(e^{at} - e^{aC}) \leq \frac{e^{at}}{a}$$

and so

$$|f(t)| \leq [|f(0)| + TC + \frac{M}{a}]e^{at}.$$

If $a = 0$ then

$$\int_C^t e^{au}du = t - C$$

and therefore

$$|f(t)| \leq |f(0)| + TC + M(t - C) \leq (|f(0)| + TC + M)e^t.$$

Now, if $a < 0$ then

$$\int_C^t e^{au}du = \frac{1}{a}(e^{at} - e^{aC}) \leq \frac{1}{|a|}$$

so that

$$|f(t)| \leq (|f(0)| + TC + \frac{M}{|a|})e^t$$

It follows that

$$|f(t)| \leq Ne^{bt}, \quad t \geq 0$$

where $b = \max\{a, 0\} + 1$.

(b) From the definition of Laplace transform we can write

$$\mathcal{L}[f'(t)] = \lim_{A \to \infty} \int_0^A f'(t)e^{-st}dt.$$

Since $f'(t)$ may have jump discontinuities at $t_1, t_2, \cdots, t_N$ in the interval $0 \le t \le A$, we can write

$$\int_0^A f'(t)e^{-st}dt = \int_0^{t_1} f'(t)e^{-st}dt + \int_{t_1}^{t_2} f'(t)e^{-st}dt + \cdots + \int_{t_N}^{A} f'(t)e^{-st}dt.$$

Integrating each term on the RHS by parts and using the continuity of $f(t)$ to obtain

$$\begin{array}{rcl} \int_0^{t_1} f'(t)e^{-st}dt & = & f(t_1)e^{-st_1} - f(0) + s\int_0^{t_1} f(t)e^{-st}dt \\ \int_{t_1}^{t_2} f'(t)e^{-st}dt & = & f(t_2)e^{-st_2} - f(t_1)e^{-st_1} + s\int_{t_1}^{t_2} f(t)e^{-st}dt \\ \vdots & & \\ \int_{t_{N-1}}^{t_N} f'(t)e^{-st}dt & = & f(t_N)e^{-st_N} - f(t_{N-1})e^{-st_{N-1}} + s\int_{t_{N-1}}^{t_N} f(t)e^{-st}dt \\ \int_{t_N}^{A} f'(t)e^{-st}dt & = & f(A)e^{-sA} - f(t_N)e^{-st_N} + s\int_{t_N}^{A} f(t)e^{-st}dt \end{array}$$

Also, by the continuity of $f(t)$ we can write

$$\int_0^A f(t)e^{-st}dt = \int_0^{t_1} f(t)e^{-st}dt + \int_{t_1}^{t_2} f(t)e^{-st}dt + \cdots + \int_{t_N}^{A} f(t)e^{-st}dt.$$

Hence,

$$\int_0^A f'(t)e^{-st}dt = f(A)e^{-sA} - f(0) + s\int_0^A f(t)e^{-st}dt.$$

Since $f(t)$ has exponential order at infinity,$\lim_{A\to\infty} f(A)e^{-sA} = 0$. Hence,

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

(c) Using part (b) we find

$$\begin{array}{rcl} \mathcal{L}[f''(t)] & = & s\mathcal{L}[f'(t)] - f'(0) \\ & = & s(sF(s) - f(0)) - f'(0) \\ & = & s^2F(s) - sf(0) - f'(0), \quad s > \max\{a, 0\} + 1 \end{array}$$

(d) Since $\frac{d}{dt}\left(\int_0^t f(u)du\right) = f(t)$, by part (b) we have

$$F(s) = \mathcal{L}[f(t)] = s\mathcal{L}\left\{\int_0^t f(u)du\right\}$$

and therefore

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{\mathcal{L}[f(t)]}{s} = \frac{F(s)}{s}, \quad s > \max\{a, 0\} + 1 \blacksquare$$

The argument establishing part (b) of the previous theorem can be extended to higher order derivatives.

Theorem 44.4
Let $f(t), f'(t), \cdots, f^{(n-1)}(t)$ be continuous and $f^{(n)}(t)$ be piecewise continuous of exponential order at infinity with $|f^{(n)}(t)| \leq Me^{at}, \quad t \geq C$. Then

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0), \;\; s > \max\{a, 0\}+1.$$

We next illustrate the use of the previous theorem in solving initial value problems.

Example 44.2
Solve the initial value problem

$$y'' - 4y' + 9y = t, \quad y(0) = 0, \; y'(0) = 1.$$

Solution.
We apply Theorem 44.4 that gives the Laplace transform of a derivative. By the linearity property of the Laplace transform we can write

$$\mathcal{L}[y''] - 4\mathcal{L}[y'] + 9\mathcal{L}[y] = \mathcal{L}[t].$$

Now since

$$\begin{array}{rcl} \mathcal{L}[y''] &=& s^2\mathcal{L}[y] - sy(0) - y'(0) = s^2Y(s) - 1 \\ \mathcal{L}[y'] &=& sY(s) - y(0) = sY(s) \\ \mathcal{L}[t] &=& \frac{1}{s^2} \end{array}$$

where $\mathcal{L}[y] = Y(s)$, we obtain

$$s^2Y(s) - 1 - 4sY(s) + 9Y(s) = \frac{1}{s^2}.$$

Rearranging gives

$$(s^2 - 4s + 9)Y(s) = \frac{s^2+1}{s^2}.$$

Thus,

$$Y(s) = \frac{s^2+1}{s^2(s^2-4s+9)}$$

and

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2+1}{s^2(s^2-4s+9)}\right] \blacksquare$$

In the next section we will discuss a method for finding the inverse Laplace transform of the above expression.

Example 44.3
Consider the mass-spring oscillator without friction: $y'' + y = 0$. Suppose we add a force which corresponds to a push (to the left) of the mass as it oscillates. We will suppose the push is described by the function

$$f(t) = -h(t-2\pi) + u(t-(2\pi+a))$$

for some $a > 2\pi$ which we are allowed to vary. (A small a will correspond to a short duration push and a large a to a long duration push.) We are interested in solving the initial value problem

$$y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0.$$

Solution.
To begin, determine the Laplace transform of both sides of the DE:

$$\mathcal{L}[y'' + y] = \mathcal{L}[f(t)]$$

or

$$s^2Y - sy(0) - y'(0) + Y(s) = -\frac{1}{s}e^{-2\pi s} + \frac{1}{s}e^{-(2\pi+a)s}.$$

Thus,

$$Y(s) = \frac{e^{-(2\pi+a)s}}{s(s^2+1)} - \frac{e^{-2\pi s}}{s(s^2+1)} + \frac{s}{s^2+1}.$$

Now since $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$ we see that

$$Y(s) = e^{-(2\pi+a)s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] - e^{-2\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{s}{s^2+1}$$

and therefore

$$\begin{array}{rcl} y(t) & = & h(t-(2\pi+a))\left[\mathcal{L}^{-1}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)\right](t-(2\pi+a)) \\ & - & h(t-2\pi)\left[\mathcal{L}^{-1}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)\right](t-2\pi)+\cos t \\ & = & h(t-(2\pi+a))[1-\cos(t-(2\pi+a))]-u(t-2\pi)[1-\cos(t-2\pi)] \\ & + & \cos t \ \blacksquare \end{array}$$

We conclude this section with the following table of Laplace transform pairs.

f(t)	F(s)
$h(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$	$\frac{1}{s},\ s>0$
$t^n,\ n=1,2,\cdots$	$\frac{n!}{s^{n+1}},\ s>0$
$e^{\alpha t}$	$\frac{s}{s-\alpha},\ s>\alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2},\ s>0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2},\ s>0$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2},\ s>\lvert\omega\rvert$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2},\ s>\lvert\omega\rvert$
$e^{\alpha t}f(t),\ with\ \lvert f(t)\rvert\leq Me^{at}$	$F(s-\alpha),\ s>\alpha+a$
$e^{\alpha t}h(t)$	$\frac{1}{s-\alpha},\ s>\alpha$
$e^{\alpha t}t^n,\ n=1,2,\cdots$	$\frac{n!}{(s-\alpha)^{n+1}},\ s>\alpha$
$e^{\alpha t}\sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2},\ s>\alpha$
$e^{\alpha t}\cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2},\ s>\alpha$
$f(t-\alpha)h(t-\alpha),\ \alpha\geq 0$ $with\ \lvert f(t)\rvert\leq Me^{at}$	$e^{-\alpha s}F(s),\ s>a$

f(t)	F(s) (continued)
$h(t-\alpha),\ \alpha \geq 0$	$\frac{e^{-\alpha s}}{s},\ s > 0$
$tf(t)$	-$F'(s)$
$\frac{t}{2\omega}\sin \omega t$	$\frac{s}{(s^2+\omega^2)^2},\ s > 0$
$\frac{1}{2\omega^3}[\sin \omega t - \omega t \cos \omega t]$	$\frac{1}{(s^2+\omega^2)^2},\ s > 0$
$f'(t),\ \textit{with } f(t) \textit{ continuous}$ $\textit{and } \lvert f'(t)\rvert \leq Me^{at}$	$sF(s) - f(0)$ $s > \max\{a, 0\} + 1$
$f''(t),\ \textit{with } f'(t) \textit{ continuous}$ $\textit{and } \lvert f''(t)\rvert \leq Me^{at}$	$s^2F(s) - sf(0) - f'(0)$ $s > \max\{a, 0\} + 1$
$f^{(n)}(t),\ \textit{with } f^{(n-1)}(t) \textit{ continuous}$ $\textit{and } \lvert f^{(n)}(t)\rvert \leq Me^{at}$	$s^nF(s) - s^{n-1}f(0) - \cdots$ -$sf^{(n-2)}(0) - f^{(n-1)}(0)$ $s > \max\{a, 0\} + 1$
$\int_0^t f(u)du,\ \textit{with } \lvert f(t)\rvert \leq Me^{at}$	$\frac{F(s)}{s},\ \ s > \max\{a, 0\} + 1$

Table $\mathcal{L}$

Practice Problems

Problem 44.1
Use Table $\mathcal{L}$ to find $\mathcal{L}[2e^t + 5]$.

Problem 44.2
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{3t-3}h(t-1)]$.

Problem 44.3
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin^2 \omega t]$.

Problem 44.4
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin 3t \cos 3t]$.

Problem 44.5
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{2t} \cos 3t]$.

Problem 44.6
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{4t}(t^2 + 3t + 5)]$.

Problem 44.7
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{10}{s^2+25} + \frac{4}{s-3}]$.

Problem 44.8
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{5}{(s-3)^4}]$.

Problem 44.9
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}]$.

Problem 44.10
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-3s}(2s+7)}{s^2+16}]$.

Problem 44.11
Graph the function $f(t) = h(t-1) + h(t-3)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Problem 44.12
Graph the function $f(t) = t[h(t-1) - h(t-3)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

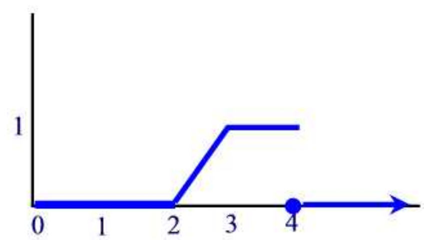
Problem 44.13
Graph the function $f(t) = 3[h(t-1) - h(t-4)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Problem 44.14
Graph the function $f(t) = |2-t|[h(t-1) - h(t-3)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

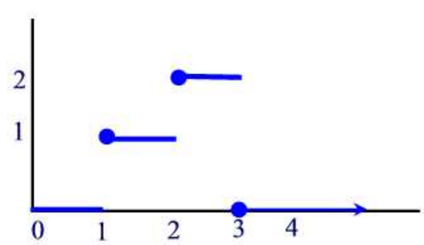
Problem 44.15
Graph the function $f(t) = h(2-t)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Problem 44.16
Graph the function $f(t) = h(t-1) + h(4-t)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Problem 44.17
The graph of $f(t)$ is given below. Represent $f(t)$ as a combination of Heaviside step functions, and use Table $\mathcal{L}$ to calculate the Laplace transform of $f(t)$.



Problem 44.18
The graph of $f(t)$ is given below. Represent $f(t)$ as a combination of Heaviside step functions, and use Table $\mathcal{L}$ to calculate the Laplace transform of $f(t)$.



Problem 44.19
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{12}{(s-3)(s+1)}\right]$.

Problem 44.20
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{24e^{-5s}}{s^2-9}\right]$.

Problem 44.21
Use Laplace transform technique to solve the initial value problem

$$y' + 4y = g(t), \quad y(0) = 2$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 1 \\ 12, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

Problem 44.22
Use Laplace transform technique to solve the initial value problem

$$y'' - 4y = e^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Problem 44.23
Obtain the Laplace transform of the function $\int_2 tf(\lambda)d\lambda$ in terms of $\mathcal{L}[f(t)] = F(s)$ given that $\int_0^2 f(\lambda)d\lambda = 3$.

45 The Laplace Transform and the Method of Partial Fractions

In the last example of the previous section we encountered the equation

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2+1}{s^2(s^2-4s+9)}\right].$$

We would like to find an explicit expression for $y(t)$. This can be done using the method of partial fractions which is the topic of this section. According to this method, finding $\mathcal{L}^{-1}\left(\frac{N(s)}{D(s)}\right)$, where $N(s)$ and $D(s)$ are polynomials, require decomposing the rational function into a sum of simpler expressions whose inverse Laplace transform can be recognized from a table of Laplace transform pairs.
The method of integration by partial fractions is a technique for integrating rational functions, i.e. functions of the form

$$R(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials.
The idea consists of writing the rational function as a sum of simpler fractions called **partial fractions**. This can be done in the following way:

Step 1. Use long division to find two polynomials $r(s)$ and $q(s)$ such that

$$\frac{N(s)}{D(s)} = q(s) + \frac{r(s)}{D(s)}.$$

Note that if the degree of $N(s)$ is smaller than that of $D(s)$ then $q(s) = 0$ and $r(s) = N(s)$.

Step 2. Write $D(s)$ as a product of factors of the form $(as+b)^n$ or $(as^2+bs+c)^n$ where as^2+bs+c is irreducible, i.e. $as^2+bs+c = 0$ has no real zeros.

Step 3. Decompose $\frac{r(s)}{D(s)}$ into a sum of partial fractions in the following way:
(1) For each factor of the form $(s-\alpha)^k$ write

$$\frac{A_1}{s-\alpha} + \frac{A_2}{(s-\alpha)^2} + \cdots + \frac{A_k}{(s-\alpha)^k},$$

where the numbers $A_1, A_2, \cdots, A_k$ are to be determined.
(2) For each factor of the form $(as^2+bs+c)^k$ write

$$\frac{B_1s+C_1}{as^2+bs+c}+\frac{B_2s+C_2}{(as^2+bs+c)^2}+\cdots+\frac{B_ks+C_k}{(as^2+bs+c)^k},$$

where the numbers $B_1, B_2, \cdots, B_k$ and $C_1, C_2, \cdots, C_k$ are to be determined.

Step 4. Multiply both sides by $D(s)$ and simplify. This leads to an expression of the form

$$r(s) = \text{a polynomial whose coefficients are combinations of } \mathrm{A_i}, \mathrm{B_i}, \text{ and } \mathrm{C_i}.$$

Finally, we find the constants, A_i, B_i, and C_i by equating the coefficients of like powers of s on both sides of the last equation.

Example 45.1
Decompose into partial fractions $R(s) = \frac{s^3+s^2+2}{s^2-1}$.

Solution.
Step 1. $\frac{s^3+s^2+2}{s^2-1} = s+1+\frac{s+3}{s^2-1}$.
Step 2. $s^2-1 = (s-1)(s+1)$.
Step 3. $\frac{s+3}{(s+1)(s-1)} = \frac{A}{s+1}+\frac{B}{s-1}$.
Step 4. Multiply both sides of the last equation by $(s-1)(s+1)$ to obtain

$$s+3 = A(s-1)+B(s+1).$$

Expand the right hand side, collect terms with the same power of s, and identify coefficients of the polynomials obtained on both sides:

$$s+3 = (A+B)s+(B-A).$$

Hence, $A+B=1$ and $B-A=3$. Adding these two equations gives $B=2$. Thus, $A=-1$ and so

$$\frac{s^3+s^2+2}{s^2-1} = s+1-\frac{1}{s+1}+\frac{2}{s-1}.\blacksquare$$

Now, after decomposing the rational function into a sum of partial fractions all we need to do is to find the Laplace transform of expressions of the form $\frac{A}{(s-\alpha)^n}$ or $\frac{Bs+C}{(as^2+bs+c)^n}$.

Example 45.2
Find $\mathcal{L}^{-1}\left[\frac{1}{s(s-3)}\right]$.

Solution.
We write
$$\frac{1}{s(s-3)} = \frac{A}{s} + \frac{B}{s-3}.$$
Multiply both sides by $s(s-3)$ and simplify to obtain
$$1 = A(s-3) + Bs$$
or
$$1 = (A+B)s - 3A.$$
Now equating the coefficients of like powers of s to obtain $-3A = 1$ and $A + B = 0$. Solving for A and B we find $A = -\frac{1}{3}$ and $B = \frac{1}{3}$. Thus,
$$\begin{aligned}\mathcal{L}^{-1}\left[\tfrac{1}{s(s-3)}\right] &= -\tfrac{1}{3}\mathcal{L}^{-1}\left[\tfrac{1}{s}\right] + \tfrac{1}{3}\mathcal{L}^{-1}\left[\tfrac{1}{s-3}\right] \\ &= -\tfrac{1}{3}h(t) + \tfrac{1}{3}e^{3t},\ t \geq 0\end{aligned}$$
where $h(t)$ is the Heaviside unit step function ■

Example 45.3
Find $\mathcal{L}^{-1}\left[\frac{3s+6}{s^2+3s}\right]$.

Solution.
We factor the denominator and split the integrand into partial fractions:
$$\frac{3s+6}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}.$$
Multiplying both sides by $s(s+3)$ to obtain
$$\begin{aligned}3s+6 &= A(s+3) + Bs \\ &= (A+B)s + 3A\end{aligned}$$
Equating the coefficients of like powers of x to obtain $3A = 6$ and $A+B = 3$. Thus, $A = 2$ and $B = 1$. Finally,
$$\mathcal{L}^{-1}\left[\frac{3s+6}{s^2+3s}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = 2h(t) + e^{-3t},\ t \geq 0. \blacksquare$$

Example 45.4
Find $\mathcal{L}^{-1}\left[\frac{s^2+1}{s(s+1)^2}\right]$.

Solution.
We factor the denominator and split the rational function into partial fractions:

$$\frac{s^2+1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}.$$

Multiplying both sides by $s(s+1)^2$ and simplifying to obtain

$$\begin{array}{rcl} s^2+1 & = & A(s+1)^2 + Bs(s+1) + Cs \\ & = & (A+B)s^2 + (2A+B+C)s + A. \end{array}$$

Equating coefficients of like powers of s we find $A = 1, 2A+B+C = 0$ and $A+B = 1$. Thus, $B = 0$ and $C = -2$. Now finding the inverse Laplace transform to obtain

$$\mathcal{L}^{-1}\left[\frac{s^2+1}{s(s+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = h(t) - 2te^{-t},\ t \geq 0. \blacksquare$$

Example 45.5
Use Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = e^{-t}, \quad y(0) = y'(0) = 0.$$

Solution.
By the linearity property of the Laplace transform we can write

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[e^{-t}].$$

Now since

$$\begin{array}{rcl} \mathcal{L}[y''] & = & s^2\mathcal{L}[y] - sy(0) - y'(0) = s^2Y(s) \\ \mathcal{L}[y'] & = & sY(s) - y(0) = sY(s) \\ \mathcal{L}[e^{-t}] & = & \frac{1}{s+1} \end{array}$$

where $\mathcal{L}[y] = Y(s)$, we obtain

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s+1}.$$

Rearranging gives

$$(s^2 + 3s + 2)Y(s) = \frac{1}{s+1}.$$

Thus,

$$Y(s) = \frac{1}{(s+1)(s^2 + 3s + 2)}.$$

and

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2 + 3s + 2)}\right].$$

Using the method of partial fractions we can write

$$\frac{1}{(s+1)(s^2 + 3s + 2)} = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}.$$

Thus,

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-2t} - e^{-t} + te^{-t}, \;\; t \geq 0 \blacksquare$$

Practice Problems

In Problems 45.1 - 45.4, give the form of the partial fraction expansion for $F(s)$. You need not evaluate the constants in the expansion. However, if the denominator has an irreducible quadratic expression then use the completing the square process to write it as the sum/difference of two squares.

Problem 45.1

$$F(s) = \frac{s^3 + 3s + 1}{(s-1)^3(s-2)^2}.$$

Problem 45.2

$$F(s) = \frac{s^2 + 5s - 3}{(s^2 + 16)(s-2)}.$$

Problem 45.3

$$F(s) = \frac{s^3 - 1}{(s^2 + 1)^2(s+4)^2}.$$

Problem 45.4

$$F(s) = \frac{s^4 + 5s^2 + 2s - 9}{(s^2 + 8s + 17)(s-2)^2}.$$

Problem 45.5
Find $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3}\right]$.

Problem 45.6
Find $\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-3s+2}\right]$.

Problem 45.7
Find $\mathcal{L}^{-1}\left[\frac{4s^2+s+1}{s^3+s}\right]$.

Problem 45.8
Find $\mathcal{L}^{-1}\left[\frac{s^2+6s+8}{s^4+8s^2+16}\right]$.

Problem 45.9
Use Laplace transform to solve the initial value problem

$$y' + 2y = 26\sin 3t,\ y(0) = 3.$$

Problem 45.10
Use Laplace transform to solve the initial value problem

$$y' + 2y = 4t,\ y(0) = 3.$$

Problem 45.11
Use Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = 6e^{-t},\ y(0) = 1,\ y'(0) = 2.$$

Problem 45.12
Use Laplace transform to solve the initial value problem

$$y'' + 4y = \cos 2t,\ y(0) = 1,\ y'(0) = 1.$$

Problem 45.13
Use Laplace transform to solve the initial value problem

$$y'' - 2y' + y = e^{2t},\ y(0) = 0,\ y'(0) = 0.$$

Problem 45.14
Use Laplace transform to solve the initial value problem

$$y'' + 9y = g(t),\ y(0) = 1,\ y'(0) = 0$$

where

$$g(t) = \begin{cases} 6, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$$

Problem 45.15
Determine the constants α, β, y_0, and y_0' so that $Y(s) = \frac{2s-1}{s^2+s+2}$ is the Laplace transform of the solution to the initial value problem

$$y'' + \alpha y' + \beta y = 0,\ y(0) = y_0,\ y'(0) = y_0'.$$

Problem 45.16
Determine the constants α, β, y_0, and y_0' so that $Y(s) = \frac{s}{(s+1)^2}$ is the Laplace transform of the solution to the initial value problem

$$y'' + \alpha y' + \beta y = 0,\ y(0) = y_0,\ y'(0) = y_0'.$$

46 Laplace Transforms of Periodic Functions

In many applications, the nonhomogeneous term in a linear differential equation is a periodic function. In this section, we derive a formula for the Laplace transform of such periodic functions.
Recall that a function $f(t)$ is said to be $T-$**periodic** if we have $f(t+T) = f(t)$ whenever t and $t+T$ are in the domain of $f(t)$. For example, the sine and cosine functions are $2\pi-$periodic whereas the tangent and cotangent functions are $\pi-$periodic.
If $f(t)$ is $T-$periodic for $t \geq 0$ then we define the function

$$f_T(t) = \begin{cases} f(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

The Laplace transform of this function is then

$$\mathcal{L}[f_T(t)] = \int_0^\infty f_T(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt.$$

The Laplace transform of a $T-$periodic function is given next.

Theorem 46.1
If $f(t)$ is a $T-$periodic, piecewise continuous fucntion for $t \geq 0$ then

$$\mathcal{L}[f(t)] = \frac{\mathcal{L}[f_T(t)]}{1-e^{-sT}}, \quad s > 0.$$

Proof.
Since $f(t)$ is piecewise continuous, it is bounded on the interval $0 \leq t \leq T$. By periodicity, $f(t)$ is bounded for $t \geq 0$. Hence, it has an exponential order at infinity. By Theorem 43.2, $\mathcal{L}[f(t)]$ exists for $s > 0$. Thus,

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = \sum_{n=0}^\infty \int_0^T f_T(t-nT)h(t-nT)e^{-st}dt,$$

where the last sum is the result of decomposing the improper integral into a sum of integrals over the constituent periods.
By the Second Shifting Theorem (i.e. Theorem 44.2) we have

$$\mathcal{L}[f_T(t-nT)h(t-nT)] = e^{-nTs}\mathcal{L}[f_T(t)],\ s > 0$$

Hence,

$$\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs}\mathcal{L}[f_T(t)] = \mathcal{L}[f_T(t)]\left(\sum_{n=0}^{\infty} e^{-nTs}\right).$$

Since $s > 0$, it follows that $0 < e^{-nTs} < 1$ so that the series $\sum_{n=0}^{\infty} e^{-nTs}$ is a convergent geometric series with limit $\frac{1}{1-e^{-sT}}$. Therefore,

$$\mathcal{L}[f(t)] = \frac{\mathcal{L}[f_T(t)]}{1-e^{-sT}}, \quad s > 0 \blacksquare$$

Example 46.1
Determine the Laplace transform of the function

$$f(t) = \begin{cases} 1, & 0 \le t \le \frac{T}{2} \\ 0, & \frac{T}{2} < t < T \end{cases} \qquad f(t+T) = f(t),\ t \ge 0.$$

Solution.
The graph of $f(t)$ is shown in Figure 46.1.

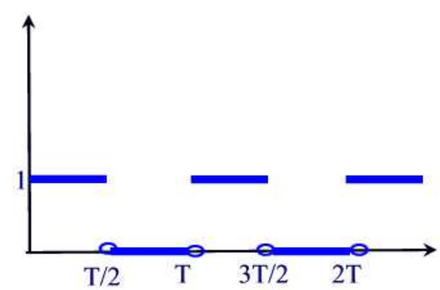


Figure 46.1

By Theorem 46.1,

$$\mathcal{L}[f(t)] = \frac{\int_0^{\frac{T}{2}} e^{-st}dt}{1-e^{-sT}}, \quad s > 0.$$

Evaluating this last integral, we find

$$\mathcal{L}[f(t)] = \frac{\frac{1-e^{-\frac{sT}{2}}}{s}}{1-e^{-sT}} = \frac{1}{s(1+e^{-\frac{sT}{2}})}, \quad s > 0 \blacksquare$$

Example 46.2
Find the Laplace transform of the sawtooth curve shown in Figure 46.2

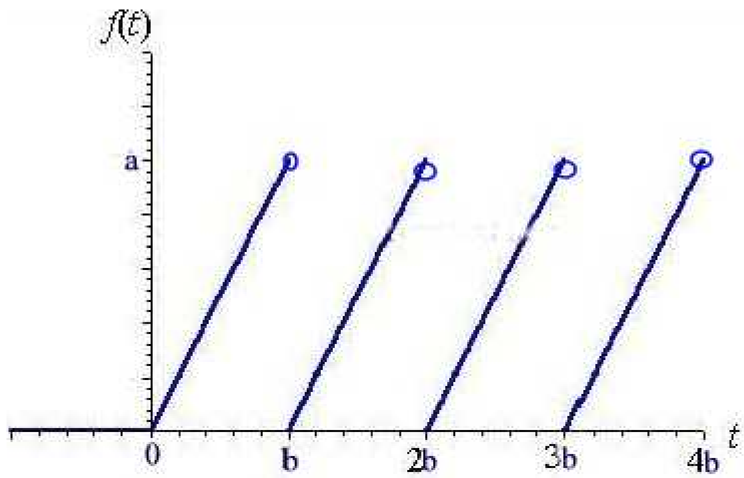


Figure 46.2

Solution.
The given function is periodic of period b. For the first period the function is defined by

$$f_b(t) = \frac{a}{b}t[h(t) - h(t-b)].$$

So we have

$$\begin{array}{rcl}\mathcal{L}[f_b(t)] &=& \mathcal{L}[\frac{a}{b}t(h(t)-h(t-b))] \\ &=& -\frac{a}{b}\frac{d}{ds}\mathcal{L}[h(t)-h(t-b)]\end{array}$$

But

$$\begin{array}{rcl}\mathcal{L}[h(t)-h(t-b)] &=& \mathcal{L}[h(t)] - \mathcal{L}[h(t-b)] \\ &=& \frac{1}{s} - \frac{e^{-bs}}{s}, \quad s > 0\end{array}$$

Hence,

$$\mathcal{L}[f_b(t)] = \frac{a}{b}\left(\frac{1}{s^2} - \frac{bse^{-bs}+e^{-bs}}{s^2}\right).$$

Finally,

$$\mathcal{L}[f(t)] = \frac{\mathcal{L}[f_b(t)]}{1-e^{-bs}} = \frac{a}{b}\left[\frac{1-e^{-bs}-bse^{-bs}}{s^2(1-e^{-bs})}\right] \blacksquare$$

Example 46.3
Find $\mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}\right]$.

Solution.
Note first that

$$\frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})} = \frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-s})}.$$

According to the previous example with $a = 1$ and $b = 1$ we find that $\mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}\right]$ is the sawtooth function shown in Figure 46.2 ■

Linear Time Invariant Systems and the Transfer Function

The Laplace transform is a powerful technique for analyzing linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems, to name just a few. A mathematical model described by a linear differential equation with constant coefficients of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 u' + b_0 u$$

is called a **linear time invariant system.** The function $y(t)$ denotes the system output and the function $u(t)$ denotes the system input. The system is called time-invariant because the parameters of the system are not changing over time and an input now will give the same result as the same input later. Applying the Laplace transform on the linear differential equation with null initial conditions we obtain

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \cdots + a_0 Y(s) = b_m s^m U(s) + b_{m-1} s^{m-1} U(s) + \cdots + b_0 U(s).$$

The function

$$\Phi(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

is called the **system transfer function.** That is, the transfer function of a linear time-invariant system is the ratio of the Laplace transform of its output to the Laplace transform of its input.

Example 46.4

Consider the mathematical model described by the initial value problem

$$my'' + \gamma y' + ky = f(t), \quad y(0) = 0, \; y'(0) = 0.$$

The coefficients m, γ, and k describe the properties of some physical system, and $f(t)$ is the input to the system. The solution y is the output at time t. Find the system transfer function.

Solution.
By taking the Laplace transform and using the initial conditions we obtain

$$(ms^2 + \gamma s + k)Y(s) = F(s).$$

Thus,

$$\mathbf{\Phi}(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + \gamma s + k} \blacksquare \tag{2}$$

Parameter Identification
One of the most useful applications of system transfer functions is for system or parameter identification.

Example 46.5
Consider a spring-mass system governed by

$$my'' + \gamma y' + ky = f(t), \quad y(0) = 0, \; y'(0) = 0. \tag{3}$$

Suppose we apply a unit step force $f(t) = h(t)$ to the mass, initially at equilibrium, and you observe the system respond as

$$y(t) = -\frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t + \frac{1}{2}.$$

What are the physical parameters m, γ, and k?

Solution.
Start with the model (3)) with $f(t) = h(t)$ and take the Laplace transform of both sides, then solve to find $Y(s) = \frac{1}{s(ms^2+\gamma s+k)}$. Since $f(t) = h(t)$, $F(s) = \frac{1}{s}$. Hence

$$\Phi(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + \gamma s + k}.$$

On the other hand, for the input $f(t) = h(t)$ the corresponding observed output is

$$y(t) = -\frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t + \frac{1}{2}.$$

Hence,

$$\begin{aligned} Y(s) &= \mathcal{L}[-\tfrac{1}{2}e^{-t}\cos t - \tfrac{1}{2}e^{-t}\sin t + \tfrac{1}{2}] \\ &= -\tfrac{1}{2}\tfrac{s+1}{(s+1)^2+1} - \tfrac{1}{2}\tfrac{1}{(s+1)^2+1} + \tfrac{1}{2s} \\ &= \tfrac{1}{s(s^2+2s+2)} \end{aligned}$$

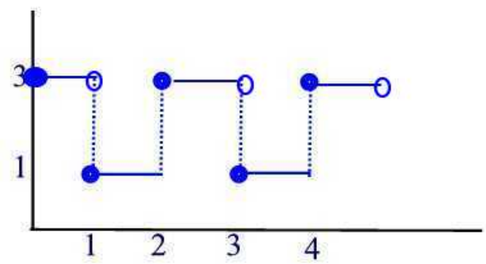
Thus,

$$\Phi(s) = \frac{Y(s)}{F(s)} = \frac{1}{s^2 + 2s + 2}.$$

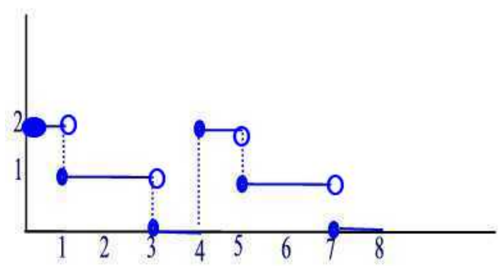
By comparison we conclude that $m = 1,\ \gamma = 2$, and $k = 2$ ■

Practice Problems

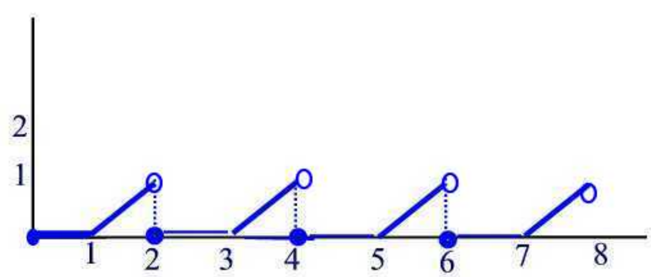
Problem 46.1
Find the Laplace transform of the periodic function whose graph is shown.



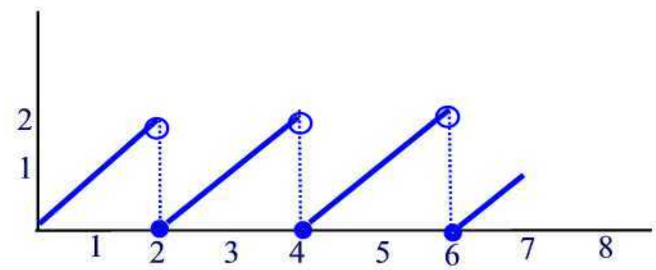
Problem 46.2
Find the Laplace transform of the periodic function whose graph is shown.



Problem 46.3
Find the Laplace transform of the periodic function whose graph is shown.



Problem 46.4
Find the Laplace transform of the periodic function whose graph is shown.



Problem 46.5
State the period of the function $f(t)$ and find its Laplace transform where

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases} \qquad f(t+2\pi) = f(t),\ t \ge 0.$$

Problem 46.6
State the period of the function $f(t) = 1 - e^{-t},\ 0 \le t < 2,\quad f(t+2) = f(t)$, and find its Laplace transform.

Problem 46.7
Using Example 44.3 find

$$\mathcal{L}^{-1}\left[\frac{s^2 - s}{s^3} + \frac{e^{-s}}{s(1-e^{-s})}\right].$$

Problem 46.8
An object having mass m is initially at rest on a frictionless horizontal surface. At time $t = 0$, a periodic force is applied horizontally to the object, causing it to move in the positive x-direction. The force, in newtons, is given by

$$f(t) = \begin{cases} f_0, & 0 \le t \le \frac{T}{2} \\ 0, & \frac{T}{2} < t < T \end{cases} \qquad f(t+T) = f(t),\ t \ge 0.$$

The initial value problem for the horizontal position, $x(t)$, of the object is

$$mx''(t) = f(t), \quad x(0) = x'(0) = 0.$$

(a) Use Laplace transforms to determine the velocity, $v(t) = x'(t)$, and the position, $x(t)$, of the object.
(b) Let $m = 1\ kg,\ f_0 = 1\ N$, and $T = 1\ sec$. What is the velocity, v, and position, x, of the object at $t = 1.25$ sec?

Problem 46.9
Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y'(0) = 0, \quad t > 0$$

Suppose that the transfer function of this system is given by $\Phi(s) = \frac{1}{2s^2+5s+2}$.
(a) What are the constants $a,\ b$, and c?
(b) If $f(t) = e^{-t}$, determine $F(s),\ Y(s)$, and $y(t)$.

Problem 46.10
Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y'(0) = 0, \quad t > 0$$

Suppose that an input $f(t) = t$, when applied to the above system produces the output $y(t) = 2(e^{-t} - 1) + t(e^{-t} + 1), \quad t \geq 0$.
(a) What is the system transfer function?
(b) What will be the output if the Heaviside unit step function $f(t) = h(t)$ is applied to the system?

Problem 46.11
Consider the initial value problem

$$y'' + y' + y = f(t), \quad y(0) = y'(0) = 0,$$

where

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ -1, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$$

(a) Determine the system transfer function $\Phi(s)$.
(b) Determine $Y(s)$.

Problem 46.12
Consider the initial value problem

$$y''' - 4y = e^t + t, \quad y(0) = y'(0) = y''(0) = 0.$$

(a) Determine the system transfer function $\Phi(s)$.
(b) Determine $Y(s)$.

Problem 46.13
Consider the initial value problem

$$y'' + by' + cy = h(t), \quad y(0) = y_0,\ y'(0) = y_0',\ t > 0.$$

Suppose that $\mathcal{L}[y(t)] = Y(s) = \frac{s^2+2s+1}{s^3+3s^2+2s}$. Determine the constants $b,\ c,\ y_0,$ and y_0'.

47 Convolution Integrals

We start this section with the following problem.

Example 47.1
A spring-mass system with a forcing function $f(t)$ is modeled by the following initial-value problem

$$mx'' + kx = f(t), \quad x(0) = x_0, \ x'(0) = x'_0.$$

Find solution to this initial value problem using the Laplace transform method.

Solution.
Apply Laplace transform to both sides of the equation to obtain

$$ms^2X(s) - msx_0 - mx'_0 + kX(s) = F(s).$$

Solving the above algebraic equation for $X(s)$ we find

$$\begin{aligned} X(s) &= \frac{F(s)}{ms^2+k} + \frac{msx_0}{ms^2+k} + \frac{mx'_0}{ms^2+k} \\ &= \frac{1}{m}\frac{F(s)}{s^2+\frac{k}{m}} + \frac{sx_0}{s^2+\frac{k}{m}} + \frac{x'_0}{s^2+\frac{k}{m}} \end{aligned}$$

Apply the inverse Laplace transform to obtain

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X(s)] \\ &= \frac{1}{m}\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+\frac{k}{m}}\right\} + x_0\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{k}{m}}\right\} + x'_0\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{k}{m}}\right\} \\ &= \frac{1}{m}\mathcal{L}^{-1}\left\{F(s)\cdot\frac{1}{s^2+\frac{k}{m}}\right\} + x_0\cos\left(\sqrt{\frac{k}{m}}\right)t + x'_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\right)t \end{aligned}$$

Finding $\mathcal{L}^{-1}\left\{F(s)\cdot\frac{1}{s^2+\frac{k}{m}}\right\}$,i.e., the inverse Laplace transform of a product, requires the use of the concept of convolution, a topic we discuss in this section ■
Convolution integrals are useful when finding the inverse Laplace transform of products $H(s) = F(s)G(s)$. They are defined as follows: The **convolution** of two scalar piecewise continuous functions $f(t)$ and $g(t)$ defined for $t \geq 0$ is the integral

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds.$$

Example 47.2
Find $f * g$ where $f(t) = e^{-t}$ and $g(t) = \sin t$.

Solution.
Using integration by parts twice we arrive at

$$\begin{aligned}(f * g)(t) &= \int_0^t e^{-(t-s)} \sin s ds \\ &= \frac{1}{2}\left[e^{-(t-s)}(\sin s - \cos s)\right]_0^t \\ &= \frac{e^{-t}}{2} + \frac{1}{2}(\sin t - \cos t) \blacksquare\end{aligned}$$

Graphical Interpretation of Convolution Operation
For the convolution

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds$$

we perform the following:
Step 1. Given the graphs of $f(s)$ and $g(s)$.(Figure 47.1(a) and (b))
Step 2. Time reverse $f(-s)$. (See Figure 47.1(c))
Step 3. Shift $f(-s)$ right by an amount t to get $f(t-s)$. (See Figure 47.1(d))
Step 4. Determine the product $f(t-s)g(s)$. (See Figure 47.1(e))
Step 5. Determine the area under the graph of $f(t-s)g(s)$ between 0 and t. (See Figure 47.1(e))

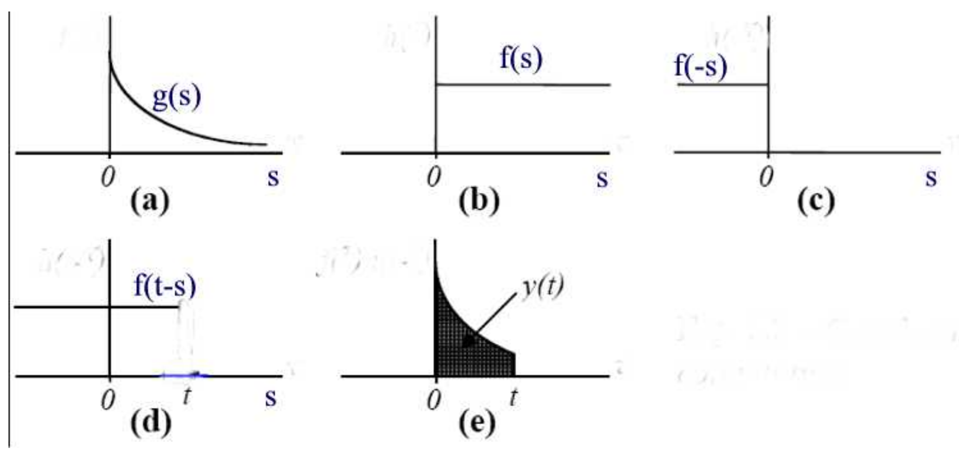


Figure 47.1

Next, we state several properties of convolution product, which resemble those of ordinary product.

Theorem 47.1
Let $f(t), g(t)$, and $k(t)$ be three piecewise continuous scalar functions defined for $t \geq 0$ and c_1 and c_2 are arbitrary constants. Then
(i) $f * g = g * f$ (Commutative Law)
(ii) $(f * g) * k = f * (g * k)$ (Associative Law)
(iii) $f * (c_1 g + c_2 k) = c_1 f * g + c_2 f * k$ (Distributive Law)

Proof.
(i) Using the change of variables $\tau = t - s$ we find

$$\begin{aligned} (f * g)(t) &= \int_0^t f(t-s)g(s)ds \\ &= -\int_t^0 f(\tau)g(t-\tau)d\tau \\ &= \int_0^t g(t-\tau)f(\tau)d\tau = (g * f)(t) \end{aligned}$$

(ii) By definition, we have

$$\begin{aligned} [(f * g) * k)](t) &= \int_0^t (f * g)(t-u)k(u)du \\ &= \int_0^t \left[\int_0^{t-u} f(t-u-w)g(w)k(u)dw\right] du \end{aligned}$$

For the integral in the bracket, make change of variable $w = s - u$. We have

$$[(f * g) * k)](t) = \int_0^t \left[\int_u^t f(t-s)g(s-u)k(u)ds\right] du.$$

This multiple integral is carried over the region

$$\{(s, u) : 0 \leq u \leq s \leq t\}$$

as depicted by shaded region in the following graph.

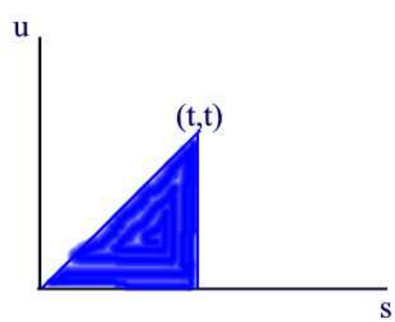


Figure 47.2

Changing the order of integration, we have

$$\begin{array}{rcl}[(f*g)*k](t) & = & \int_0^t \left[\int_0^s f(t-s)g(s-u)k(u)du\right] ds \\ & = & \int_0^t f(t-s)(g*k)(s)ds \\ & = & [f*(g*k)](t)\end{array}$$

(iii) We have

$$\begin{array}{rcl}(f*(c_1g+c_2k))(t) & = & \int_0^t f(t-s)(c_1g(s)+c_2k(s))ds \\ & = & c_1\int_0^t f(t-s)g(s)ds + c_2\int_0^t f(t-s)k(s)ds \\ & = & c_1(f*g)(t)+c_2(f*k)(t) \blacksquare\end{array}$$

Example 47.3
Express the solution to the initial value problem $y' + \alpha y = g(t),\ y(0) = y_0$ in terms of a convolution integral.

Solution.
Solving this initial value problem by the method of integrating factor we find

$$y(t) = e^{-\alpha t}y_0 + \int_0^t e^{-\alpha(t-s)}g(s)ds = e^{-\alpha t}y_0 + e^{-\alpha t} * g(t) \blacksquare$$

Example 47.4
If $\mathbf{f}(t)$ is an $m \times n$ matrix function and $\mathbf{g}$(t) is an $n \times p$ matrix function then we define

$$(\mathbf{f} * \mathbf{g})(t) = \int_0^t \mathbf{f}(t-s)\mathbf{g}(s)ds, \quad t \geq 0.$$

Express the solution to the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(t),\ \mathbf{y}(0) = \mathbf{y}_0$ in terms of a convolution integral.

Solution.
The unique solution is given by

$$\mathbf{y}(t) = e^{t\mathbf{A}}\mathbf{y}_0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)ds = e^{t\mathbf{A}}\mathbf{y}_0 + e^{t\mathbf{A}} * \mathbf{g}(t) \blacksquare$$

The following theorem, known as the Convolution Theorem, provides a way for finding the Laplace transform of a convolution integral and also finding the inverse Laplace transform of a product.

Theorem 47.2
If $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$, and of exponential order at infinity then

$$\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = F(s)G(s).$$

Thus, $(f*g)(t) = \mathcal{L}^{-1}[F(s)G(s)]$.

Proof.
First we show that $f*g$ has a Laplace transform. From the hypotheses we have that $|f(t)| \leq M_1 e^{a_1 t}$ for $t \geq C_1$ and $|g(t)| \leq M_2 e^{a_2 t}$ for $t \geq C_2$. Let $M = M_1 M_2$ and $C = C_1 + C_2$. Then for $t \geq C$ we have

$$\begin{array}{lll} |(f*g)(t)| & = & \left|\int_0^t f(t-s)g(s)ds\right| \leq \int_0^t |f(t-s)||g(s)|ds \\ & \leq & M_1 M_2 \int_0^t e^{a_1(t-s)} e^{a_2 s} ds \\ & = & \left\{ \begin{array}{cc} Mte^{a_1 t}, & a_1 = a_2 \\ M\frac{e^{a_2 t} - e^{a_1 t}}{a_2 - a_1}, & a_1 \neq a_2 \end{array} \right. \end{array}$$

This shows that $f*g$ is of exponential order at infinity. Since f and g are piecewise continuous, the first fundamental theorem of calculus implies that $f*g$ is also piecewise continuous. Hence, $f*g$ has a Laplace transform.
Next, we have

$$\begin{array}{lll} \mathcal{L}[(f*g)(t)] & = & \int_0^\infty e^{-st} \left(\int_0^t f(t-\tau)g(\tau)d\tau\right) dt \\ & = & \int_{t=0}^\infty \int_{\tau=0}^t e^{-st} f(t-\tau)g(\tau)d\tau dt \end{array}$$

Note that the region of integration is an infinite triangular region and the integration is done vertically in that region. Integration horizontally we find

$$\mathcal{L}[(f*g)(t)] = \int_{\tau=0}^\infty \int_{t=\tau}^\infty e^{-st} f(t-\tau)g(\tau)dt d\tau.$$

We next introduce the change of variables $\beta = t - \tau$. The region of integration becomes $\tau \geq 0, t \geq 0$. In this case, we have

$$\begin{array}{lll} \mathcal{L}[(f*g)(t)] & = & \int_{\tau=0}^\infty \int_{\beta=0}^\infty e^{-s(\beta+\tau)} f(\beta)g(\tau)d\tau d\beta \\ & = & \left(\int_{\tau=0}^\infty e^{-s\tau} g(\tau)d\tau\right)\left(\int_{\beta=0}^\infty e^{-s\beta} f(\beta)d\beta\right) \\ & = & G(s)F(s) = F(s)G(s) \blacksquare \end{array}$$

Example 47.5
Use the convolution theorem to find the inverse Laplace transform of

$$H(s) = \frac{1}{(s^2+a^2)^2}.$$

Solution.
Note that

$$H(s) = \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+a^2}\right).$$

So, in this case we have, $F(s) = G(s) = \frac{1}{s^2+a^2}$ so that $f(t) = g(t) = \frac{1}{a}\sin{(at)}$. Thus,

$$(f * g)(t) = \frac{1}{a^2}\int_0^t \sin{(at - as)}\sin{(as)}ds = \frac{1}{2a^3}(\sin{(at)} - at\cos{(at)}) \blacksquare$$

Convolution integrals are useful in solving initial value problems with forcing functions.

Example 47.6
Solve the initial value problem

$$4y'' + y = g(t), \quad y(0) = 3, \ y'(0) = -7$$

Solution.
Take the Laplace transform of all the terms and plug in the initial conditions to obtain

$$4(s^2Y(s) - 3s + 7) + Y(s) = G(s)$$

or

$$(4s^2+1)Y(s) - 12s + 28 = G(s).$$

Solving for $Y(s)$ we find

$$\begin{array}{rcl} Y(s) & = & \frac{12s-28}{4\left(s^2+\frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2+\frac{1}{4}\right)} \\ & = & \frac{3s}{s^2+(\left(\frac{1}{2}\right)^2} - 7\frac{\left(\frac{1}{2}\right)^2}{s^2+\left(\frac{1}{2}\right)^2} + \frac{1}{4}G(s)\frac{\left(\frac{1}{2}\right)^2}{s^2+\left(\frac{1}{2}\right)^2} \end{array}$$

Hence,

$$y(t) = 3\cos\left(\frac{t}{2}\right) - 7\sin\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^t \sin\left(\frac{s}{2}\right)g(t-s)ds.$$

So, once we decide on a $g(t)$ all we need to do is to evaluate the integral and we'll have the solution $\blacksquare$

Practice Problems

Problem 47.1
Consider the functions $f(t) = g(t) = h(t),\ t \geq 0$ where $h(t)$ is the Heaviside unit step function. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Problem 47.2
Consider the functions $f(t) = e^t$ and $g(t) = e^{-2t},\ t \geq 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Problem 47.3
Consider the functions $f(t) = \sin t$ and $g(t) = \cos t,\ t \geq 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

Problem 47.4
Use Laplace transform to comput the convolution $\mathbf{P} * \mathbf{y}$, where $|bfP(t) = \left[\begin{array}{cc} h(t) & e^t \\ 0 & t \end{array}\right]$ and $\mathbf{y}(t) = \left[\begin{array}{c} h(t) \\ e^{-t} \end{array}\right]$.

Problem 47.5
Compute and graph $f * g$ where $f(t) = h(t)$ and $g(t) = t[h(t) - h(t-2)]$.

Problem 47.6
Compute and graph $f * g$ where $f(t) = h(t) - h(t-1)$ and $g(t) = h(t-1) - 2h(t-2)]$.

Problem 47.7
Compute $t * t * t$.

Problem 47.8
Compute $h(t) * e^{-t} * e^{-2t}$.

Problem 47.9
Compute $t * e^{-t} * e^t$.

Problem 47.10

Suppose it is known that $\overbrace{h(t) * h(t) * \cdots * h(t)}^{n\ functions} = Ct^8$. Determine the constants C and the poisitive integer n.

Problem 47.11
Use Laplace transform to solve for $y(t)$:

$$\int_0^t \sin(t-\lambda)y(\lambda)d\lambda = t^2.$$

Problem 47.12
Use Laplace transform to solve for $y(t)$:

$$y(t) - \int_0^t e^{(t-\lambda)}y(\lambda)d\lambda = t.$$

Problem 47.13
Use Laplace transform to solve for $y(t)$:

$$t * y(t) = t^2(1 - e^{-t}).$$

Problem 47.14
Use Laplace transform to solve for $y(t)$:

$$\mathbf{y}' = h(t) * \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Problem 47.15
Solve the following initial value problem.

$$y' - y = \int_0^t (t-\lambda)e^{\lambda}d\lambda, \quad y(0) = -1.$$

48 The Dirac Delta Function and Impulse Response

In applications, we are often encountered with linear systems, originally at rest, excited by a sudden large force (such as a large applied voltage to an electrical network) over a very short time frame. In this case, the output corresponding to this sudden force is referred to as the "impulse response". Mathematically, an impulse can be modeled by an initial value problem with a special type of function known as the **Dirac delta function** as the external force, i.e., the nonhomogeneous term. To solve such IVP requires finding the Laplace transform of the delta function which is the main topic of this section.

An Example of Impulse Response
Consider a spring-mass system with a time-dependent force $f(t)$ applied to the mass. The situation is modeled by the second-order differential equation

$$my'' + \gamma y' + ky = f(t) \tag{4}$$

where t is time and $y(t)$ is the displacement of the mass from equilibrium. Now suppose that for $t \leq 0$ the mass is at rest in its equilibrium position, so $y(0) = y'(0) = 0$. Hence, the situation is modeled by the initial value problem

$$my'' + \gamma y' + ky = f(t), \quad y(0) = 0, \quad y'(0) = 0. \tag{5}$$

Solving this equation by the method of variation of parameters one finds the unique solution

$$y(t) = \int_0^t \phi(t-s)f(s)ds \tag{6}$$

where

$$\phi(t) = \frac{e^{(-\gamma/2m)t}\sin\left(t\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}\right)}{m\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}}.$$

Next, we consider the problem of strucking the mass by an "instantaneous" hammer blow at $t = 0$. This situation actually occurs frequently in practice-a system sustains a forceful, almost-instantaneous input. Our goal is to model the situation mathematically and determine how the system will respond.

In the above situation we might describe f(t) as a large constant force applied on a very small time interval. Such a model leads to the forcing function

$$f_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & 0 \le t \le \epsilon \\ 0, & \text{otherwise} \end{cases}$$

where ϵ is a small positive real number. When ϵ is close to zero the applied force is very large during the time interval $0 \le t \le \epsilon$ and zero afterwards. A possible graph of $f_\epsilon(t)$ is given in Figure 48.1

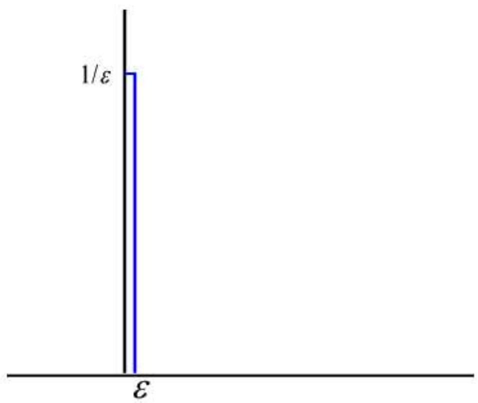


Figure 48.1

In this case it's easy to see that for any choice of ϵ we have

$$\int_{-\infty}^{\infty} f_\epsilon dt = 1$$

and

$$\lim_{\epsilon \to 0^+} f_\epsilon(t) = 0,\ t \neq 0, \quad \lim_{\epsilon \to 0^+} f_\epsilon(0) = \infty. \tag{7}$$

Our ultimate interest is the behavior of the solution to equation (4) with forcing function $f_\epsilon(t)$ in the limit $\epsilon \to 0^+$. That is, what happens to the system output as we make the applied force progressively "sharper" and "stronger?".

Let $y_\epsilon(t)$ be the solution to equation (4) with $f(t) = f_\epsilon(t)$. Then the unique solution is given by

$$y_\epsilon(t) = \int_0^t \phi(t-s) f_\epsilon(s) ds.$$

For $t \geq \epsilon$ the last equation becomes

$$y_\epsilon(t) = \frac{1}{\epsilon}\int_0^\epsilon \phi(t-s)ds.$$

Since $\phi(t)$ is continuous for all $t \geq 0$ we can apply the mean value theorem for integrals and write

$$y_\epsilon(t) = \phi(t-\psi)$$

for some $0 \leq \psi \leq \epsilon$. Letting $\epsilon \to 0^+$ and using the continuity of ϕ we find

$$y(t) = \lim_{\epsilon \to 0^+} y_\epsilon(t) = \phi(t).$$

We call $y(t)$ the **impulse response** of the linear system.

The Dirac Delta Function

The problem with the integral

$$\int_0^t \phi(t-s)f_\epsilon(s)ds$$

is that $\lim_{\epsilon\to 0^+} f_\epsilon(0)$ is undefined. So it makes sense to ask the question of whether we can find a function $\delta(t)$ such that

$$\begin{array}{rcl} \lim_{\epsilon\to 0^+} y_\epsilon(t) & = & \lim_{\epsilon\to 0^+} \int_0^t \phi(t-s)f_\epsilon(s)ds \\ & = & \int_0^t \phi(t-s)\delta(s)ds \\ & = & \phi(t) \end{array}$$

where the role of $\delta(t)$ would be to evaluate the integrand at $s = 0$. Note that because of Fig 48.1 and (7), we cannot interchange the opeartions of limit and integration in the above limit process. Such a function δ exist in the theory of distributions and can be defined as follows:
If $f(t)$ is continuous in $a \leq t \leq b$ then we define the function $\delta(t)$ by the integral equation

$$\int_a^b f(t)\delta(t-t_0)dt = \lim_{\epsilon\to 0^+}\int_a^b f(t)f_\epsilon(t-t_0)dt.$$

The object $\delta(t)$ on the left is called the **Dirac Delta function,** or just the **delta function** for short.

Finding the Impulse Function Using Laplace Transform
For $\epsilon > 0$ we can solve the initial value problem (5) using Laplace transforms. To do this we need to compute the Laplace transform of $f_\epsilon(t)$, given by the integral

$$\mathcal{L}[f_\epsilon(t)] = \int_0^\infty f_\epsilon(t)e^{-st}dt = \frac{1}{\epsilon}\int_0^\epsilon e^{-st}dt = \frac{1-e^{-\epsilon s}}{\epsilon s}.$$

Note that by using L'Hôpital's rule we can write

$$\lim_{\epsilon\to 0^+} \mathcal{L}[f_\epsilon(t)] = \lim_{\epsilon\to 0^+} \frac{1-e^{-\epsilon s}}{\epsilon s} = 1, \quad s > 0.$$

Now, to find $y_\epsilon(t)$, we apply the Laplace transform to both sides of equation (4) and using the initial conditions we obtain

$$ms^2 Y_\epsilon(s) + \gamma s Y_\epsilon(s) + kY_\epsilon(s) = \frac{1-e^{-\epsilon s}}{\epsilon s}.$$

Solving for $Y_\epsilon(s)$ we find

$$Y_\epsilon(s) = \frac{1}{ms^2+\gamma s+k}\frac{1-e^{-\epsilon s}}{\epsilon s}.$$

Letting $\epsilon \to 0^+$ we find

$$Y(s) = \frac{1}{ms^2+\gamma s+k}$$

which is the transfer function of the system. Now inverse transform $Y(s)$ to find the solution to the initial value problem. That is,

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{ms^2+\gamma s+k}\right) = \phi(t).$$

Now, impulse inputs are usually modeled in terms of delta functions. Thus, knowing the Laplace transform of such functions is important when solving differential equations. The next theorem finds the Laplace transform of the delta function.

Theorem 48.1
With $\delta(t)$ defined as above, if $a \leq t_0 < b$

$$\int_a^b f(t)\delta(t-t_0)dt = f(t_0).$$

Proof.
We have

$$\begin{array}{rcl}\int_a^b f(t)\delta(t-t_0) & = & \lim_{\epsilon\to 0^+}\int_a^b f(t)f_\epsilon(t-t_0)dt \\ & = & \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} f(t)dt \\ & = & \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}f(t_0+\beta\epsilon)\epsilon = f(t_0)\end{array}$$

where $0 < \beta < 1$ and the mean-value theorem for integrals has been used ■

Remark 48.1
Since $p_\epsilon(t-t_0) = \frac{1}{\epsilon}$ for $t_0 \leq t \leq t_0+\epsilon$ and 0 otherwise we see that $\int_a^b f(t)\delta(t-a)dt = f(a)$ and $\int_a^b f(t)\delta(t-t_0)dt = 0$ for $t_0 \geq b$.

It follows immediately from the above theorem that

$$\mathcal{L}[\delta(t-t_0)] = \int_0^\infty e^{-st}\delta(t-t_0)dt = e^{-st_0}, \quad t_0 \geq 0.$$

In particular, if $t_0 = 0$ we find

$$\mathcal{L}[\delta(t)] = 1.$$

The following example illustrates the formal use of the delta function.

Example 48.1
A spring-mass system with mass 2, damping 4, and spring constant 10 is subject to a hammer blow at time $t = 0$. The blow imparts a total impulse of 1 to the system, which was initially at rest. Find the response of the system.

Solution.
The situation is modeled by the initial value problem

$$2y'' + 4y' + 10y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Taking Laplace transform of both sides we find

$$2s^2Y(s) + 4sY(s) + 10Y(s) = 1.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{1}{2s^2 + 4s + 10}.$$

The impulsive response is

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{1}{(s+1)^2 + 2^2}\right) = \frac{1}{4}e^{-2t}\sin 2t \ \blacksquare$$

Example 48.2
A 16 lb weight is attached to a spring with a spring constant equal to 2 lb/ft. Neglect damping. The weight is released from rest at 3 ft below the equilibrium position. At $t = 2\pi$ sec, it is struck with a hammer, providing an impulse of 4 lb-sec. Determine the displacement function $y(t)$ of the weight.

Solution.
This situation is modeled by the initial value problem

$$\frac{16}{32}y'' + 2y = 4\delta(t - 2\pi), \quad y(0) = 3, \ y'(0) = 0.$$

Apply Laplace transform to both sides to obtain

$$s^2Y(s) - 3s + 4Y(s) = 8e^{-2\pi s}.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{3s}{s^2 + 4} + \frac{e^{-2\pi s}}{s^2 + 4}.$$

Now take the inverse Laplace transform to get

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 3\cos 2t + 8h(t - 2\pi)f(t - 2\pi)$$

where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{2}\sin 2t.$$

Hence,

$$y(t) = 3\cos 2t + 4h(t - 2\pi)\sin 2(t - 2\pi) = 3\cos 2t + 4h(t - 2\pi)\sin 2t$$

or more explicitly

$$y(t) = \begin{cases} 3\cos 2t, & t < 2\pi \\ 3\cos 2t + 4\sin 2t, & t \geq 2\pi \end{cases} \ \blacksquare$$

Practice Problems

Problem 48.1
Evaluate

(a) $\int_0^3 (1+e^{-t})\delta(t-2)dt.$
(b) $\int_{-2}^1 (1+e^{-t})\delta(t-2)dt.$
(c) $\int_{-1}^2 \begin{bmatrix} \cos 2t \\ te^{-t} \end{bmatrix} \delta(t)dt.$
(d) $\int_{-1}^2 (e^{2t}+t) \begin{bmatrix} \delta(t+2) \\ \delta(t-1) \\ \delta(t-3) \end{bmatrix} dt.$

Problem 48.2
Let $f(t)$ be a function defined and continuous on $0 \le t < \infty$. Determine

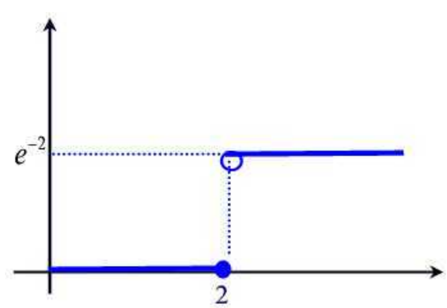
$$(f * \delta)(t) = \int_0^t f(t-s)\delta(s)ds.$$

Problem 48.3
Determine a value of the constant t_0 such that $\int_0^1 \sin^2[\pi(t-t_0)]\delta(t-\frac{1}{2})dt = \frac{3}{4}$.

Problem 48.4
If $\int_1^5 t^n \delta(t-2)dt = 8$, what is the exponent n?

Problem 48.5
Sketch the graph of the function $g(t)$ which is defined by $g(t) = \int_0^t \int_0^s \delta(u-1)duds,\ 0 \le t < \infty$.

Problem 48.6
The graph of the function $g(t) = \int_0^t e^{\alpha t}\delta(t-t_0)dt,\ 0 \le t < \infty$ is shown. Determine the constants α and t_0.



Problem 48.7
(a) Use the method of integarting factor to solve the initial value problem $y' - y = h(t),\ y(0) = 0$.
(b) Use the Laplace transform to solve the initial value problem $\phi' - \phi = \delta(t),\ \phi(0) = 0$.
(c) Evaluate the convolution $\phi * h(t)$ and compare the resulting function with the solution obtained in part(a).

Problem 48.8
Solve the initial value problem

$$y' + y = 2 + \delta(t-1), \quad y(0) = 0,\ 0 \le t \le 6.$$

Graph the solution on the indicated interval.

Problem 48.9
Solve the initial value problem

$$y'' = \delta(t-1) - \delta(t-3), \quad y(0) = 0,\ y'(0) = 0,\ 0 \le t \le 6.$$

Graph the solution on the indicated interval.

Problem 48.10
Solve the initial value problem

$$y'' - 2y' = \delta(t-1), \quad y(0) = 1,\ y'(0) = 0,\ 0 \le t \le 2.$$

Graph the solution on the indicated interval.

Problem 48.11
Solve the initial value problem

$$y'' + 2y' + y = \delta(t-2), \quad y(0) = 0,\ y'(0) = 1,\ 0 \le t \le 6.$$

Graph the solution on the indicated interval.

49 Solving Systems of Differential Equations Using Laplace Transform

In this section we extend the definition of Laplace transform to matrix-valued functions and apply this extension to solving systems of differential equations. Let $y_1(t), y_2(t), \cdots, y_n(t)$ be members of $\mathcal{PE}$. Consider the vector-valued function

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

The Laplace transform of $\mathbf{y}(t)$ is

$$\begin{aligned}
\mathcal{L}[\mathbf{y}(t)] &= \int_0^\infty \mathbf{y}(t)e^{-st}dt \\
&= \begin{bmatrix} \int_0^\infty y_1(t)e^{-st}dt \\ \int_0^\infty y_2(t)e^{-st}dt \\ \vdots \\ \int_0^\infty y_n(t)e^{-st}dt \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{L}[y_1(t)] \\ \mathcal{L}[y_2(t)] \\ \vdots \\ \mathcal{L}[y_n(t)] \end{bmatrix}
\end{aligned}$$

In a similar way, we define the Laplace transform of an $m \times n$ matrix to be the $m \times n$ matrix consisting of the Laplace transforms of the component functions. If the Laplace transform of each component exists then we say $\mathbf{y}(\mathrm{t})$ is **Laplace transformable.**

Example 49.1
Find the Laplace transform of the vector-valued function

$$\mathbf{y}(t) = \begin{bmatrix} t^2 \\ 1 \\ e^t \end{bmatrix}$$

Solution.
The Laplace transform is

$$\mathcal{L}[\mathbf{y}(t)] = \begin{bmatrix} \frac{6}{s^3} \\ \\ \frac{1}{s} \\ \\ \frac{1}{s-1} \end{bmatrix}, \quad s > 1 \blacksquare$$

The linearity property of the Laplace transform can be used to establish the following result.

Theorem 49.1
If $\mathbf{A}$ is a constant $n \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix-valued function then

$$\mathcal{L}[\mathbf{AB}(t)] = \mathbf{A}\mathcal{L}[\mathbf{B}(t)].$$

Proof.
Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B}(t) = (b_{ij}(t))$. Then $\mathbf{AB}(t) = (\sum_{k=1}^n a_{ik}b_{kp})$. Hence,

$$\mathcal{L}[\mathbf{AB}(t)] = [\mathcal{L}(\sum_{k=1}^n a_{ik}b_{kp})] = [\sum_{k=1}^n a_{ik}\mathcal{L}(b_{kp})] = \mathbf{A}\mathcal{L}[\mathbf{B}(t)] \blacksquare$$

Theorem 42.3 can be extended to vector-valued functions.

Theorem 49.2
(a) Suppose that $\mathbf{y}(t)$ is continuous for $t \geq 0$ and let the components of the derivative vector $\mathbf{y}'$ be members of $\mathcal{PE}$. Then

$$\mathcal{L}[\mathbf{y}'(t)] = s\mathcal{L}[\mathbf{y}(t)] - \mathbf{y}(0).$$

(b) Let $\mathbf{y}'(t)$ be continuous for $t \geq 0$, and let the entries of $\mathbf{y}''(t)$ be members of $\mathcal{PE}$. Then

$$\mathcal{L}[\mathbf{y}''(t)] = s^2\mathcal{L}[\mathbf{y}(t)] - s\mathbf{y}(0) - \mathbf{y}'(0).$$

(c) Let the entries of $\mathbf{y}(t)$ be members of $\mathcal{PE}$. Then

$$\mathcal{L}\left\{\int_0^t \mathbf{y}(s)ds\right\} = \frac{\mathcal{L}[\mathbf{y}(t)]}{s}.$$

Proof.
(a) We have

$$
\begin{aligned}
\mathcal{L}[\mathbf{y}'(t)] &= \begin{bmatrix} \mathcal{L}[y_1'(t)] \\ \mathcal{L}[y_2'(t)] \\ \vdots \\ \mathcal{L}[y_n'(t)] \end{bmatrix} \\
&= \begin{bmatrix} s\mathcal{L}[y_1(t)] - y_1(0) \\ s\mathcal{L}[y_2(t)] - y_2(0) \\ \vdots \\ s\mathcal{L}[y_n(t)] - y_n(0) \end{bmatrix} \\
&= s\mathcal{L}[\mathbf{y}(t)] - \mathbf{y}(0)
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
\mathcal{L}[\mathbf{y}''(t)] &= s\mathcal{L}[\mathbf{y}'(t)] - \mathbf{y}'(0) \\
&= s(s\mathcal{L}[\mathbf{y}(t)] - \mathbf{y}(0)) - \mathbf{y}'(0) \\
&= s^2\mathcal{L}[\mathbf{y}(t)] - s\mathbf{y}(0) - \mathbf{y}'(0)
\end{aligned}
$$

(c) We have

$$\mathcal{L}[\mathbf{y}(t)] = s\mathcal{L}\left\{\int_0^t \mathbf{y}(s)ds\right\}$$

so that

$$\mathcal{L}\left\{\int_0^t \mathbf{y}(s)ds\right\} = \frac{\mathcal{L}[\mathbf{y}(t)]}{s} \blacksquare$$

The above two theorems can be used for solving the following initial value problem

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}(0) = \mathbf{y}_0, \quad t > 0 \tag{8}$$

where $\mathbf{A}$ is a constant matrix and the components of $\mathbf{g}(t)$ are members of $\mathcal{PE}$.
Using the above theorems we can write

$$s\mathbf{Y}(s) - \mathbf{y}_0 = \mathbf{A}\mathbf{Y}(s) + \mathbf{G}(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{Y}(s) = \mathbf{y}_0 + \mathbf{G}(s)$$

where $\mathcal{L}[\mathbf{g}(t)] = \mathbf{G}(s)$. If s is not an eigenvalue of $\mathbf{A}$ then the matrix $s\mathbf{I} - \mathbf{A}$ is invertible and in this case we have

$$\mathbf{Y}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{y}_0 + \mathbf{G}(s)]. \tag{9}$$

To compute $\mathbf{y}(t) = \mathcal{L}^{-1}[\mathbf{Y}(s)]$ we compute the inverse Laplace transform of each component of $\mathbf{Y}(s)$. We illustrate the above discussion in the next example.

Example 49.2
Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{2t} \\ -2t \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Solution.
We have

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s-3)} \begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix}$$

and

$$\mathbf{G}(s) = \begin{bmatrix} \frac{1}{s-2} \\ -\frac{2}{s^2} \end{bmatrix}.$$

Thus,

$$\begin{aligned}
\mathbf{Y}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{y}_0 + \mathbf{G}(s)] \\
&= \frac{1}{(s+1)(s-3)} \begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} 1 + \frac{1}{s-2} \\ -2 - \frac{2}{s^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{s^4-6s^3+9s^2-4s+8}{s^2(s+1)(s-2)(s-3)} \\ \frac{-2s^4+8s^3-8s^2+6s-4}{s^2(s+1)(s-2)(s-3)} \end{bmatrix}
\end{aligned}$$

Using the method of partial fractions we can write

$$\begin{aligned}
Y_1(s) &= \frac{4}{3}\frac{1}{s^2} - \frac{8}{9}\frac{1}{s} + \frac{7}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s-2} - \frac{1}{9}\frac{1}{s-3} \\
Y_2(s) &= -\frac{2}{3}\frac{1}{s^2} + \frac{10}{9}\frac{1}{s} - \frac{7}{3}\frac{1}{s+1} - \frac{2}{3}\frac{1}{s-2} - \frac{1}{9}\frac{1}{s-3}
\end{aligned}$$

Therefore

$$\begin{aligned}
y_1(t) &= \mathcal{L}^{-1}[Y_1(s)] = \frac{4}{3}t - \frac{8}{9} + \frac{7}{3}e^{-t} - \frac{1}{3}e^{2t} - \frac{1}{9}e^{3t} \\
y_2(t) &= \mathcal{L}^{-1}[Y_2(s)] = -\frac{2}{3}t + \frac{10}{9} - \frac{7}{3}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{9}e^{3t}, \quad t \ge 0
\end{aligned}$$

Hence, for $t \geq 0$

$$\mathbf{y}(t) = t\begin{bmatrix} \frac{4}{3} \\ -\frac{2}{3} \end{bmatrix} + \begin{bmatrix} -\frac{8}{9} \\ \frac{10}{9} \end{bmatrix} + e^{-t}\begin{bmatrix} \frac{7}{3} \\ \frac{7}{3} \end{bmatrix} + e^{2t}\begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} + e^{3t}\begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{9} \end{bmatrix} \blacksquare$$

System Transfer Matrix and the Laplace Transform of $e^{t\mathbf{A}}$

The vector equation (8) is a linear time invariant system whose Laplace input is given by $\mathbf{y}_0 + G(s)$ and the Laplace output $\mathbf{Y}(s)$. According to (9) the system tranform matrix is given by $(s\mathbf{I} - \mathbf{A})^{-1}$. We will show that this matrix is the Laplace transform of the exponential matrix function $e^{t\mathbf{A}}$. Indeed, $e^{t\mathbf{A}}$ is the solution to the initial value problem

$$\mathbf{\Phi}'(t) = \mathbf{A}\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I},$$

where $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{A}$ is a constant $n \times n$ matrix. Taking Laplace of both sides yields

$$s\mathcal{L}[\Phi(t)] - \mathbf{I} = \mathbf{A}\mathcal{L}[\Phi(t)].$$

Solving for $\mathcal{L}[\Phi(t)]$ we find

$$\mathcal{L}[\Phi(t)] = (s\mathbf{I} - \mathbf{A})^{-1} = \mathcal{L}[e^{t\mathbf{A}}].$$

Practice Problems

Problem 49.1
Find $\mathcal{L}[\mathbf{y}(t)]$ where

$$\mathbf{y}(t) = \frac{d}{dt}\left[\begin{array}{c} e^{-t}\cos 2t \\ 0 \\ t+e^t \end{array}\right]$$

Problem 49.2
Find $\mathcal{L}[\mathbf{y}(t)]$ where

$$\mathbf{y}(t) = \int_0^t \left[\begin{array}{c} 1 \\ u \\ e^{-u} \end{array}\right] du$$

Problem 49.3
Find $\mathcal{L}^{-1}[\mathbf{Y}(s)]$ where

$$\mathbf{Y}(s) = \left[\begin{array}{c} \frac{1}{s} \\ \frac{2}{s^2+2s+2} \\ \frac{1}{s^2+s} \end{array}\right]$$

Problem 49.4
Find $\mathcal{L}^{-1}[\mathbf{Y}(s)]$ where

$$\mathbf{Y}(s) = \left[\begin{array}{ccc} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{array}\right]\left[\begin{array}{c} \mathcal{L}[t^3] \\ \mathcal{L}[e^{2t}] \\ \mathcal{L}[\sin t] \end{array}\right]$$

Problem 49.5
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \left[\begin{array}{cc} 5 & -4 \\ 5 & -4 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} 0 \\ 1 \end{array}\right], \quad \mathbf{y}(0) = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Problem 49.6
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \left[\begin{array}{cc} 5 & -4 \\ 3 & -2 \end{array}\right]\mathbf{y}, \quad \mathbf{y}(0) = \left[\begin{array}{c} 3 \\ 2 \end{array}\right]$$

Problem 49.7
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 0 \\ 3e^t \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Problem 49.8
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}'' = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{y}'(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Problem 49.9
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}'' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{y}'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 49.10
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} e^t \\ 1 \\ -2t \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 49.11
The Laplace transform was applied to the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$, where $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, $\mathbf{A}$ is a 2×2 constant matrix, and $\mathbf{y}_0 = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix}$.
The following transform domain solution was obtained

$$\mathcal{L}[\mathbf{y}(t)] = \mathbf{Y}(s) = \frac{1}{s^2 - 9s + 18}\begin{bmatrix} s-2 & -1 \\ 4 & s-7 \end{bmatrix}\begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix}.$$

(a) what are the eigenvalues of $\mathbf{A}$?
(b) Find $\mathbf{A}$.

50 Solutions to Problems

Section 43

Problem 43.1
Determine whether the integral $\int_0^\infty \frac{1}{1+t^2}dt$ converges. If the integral converges, give its value.

Solution.
We have

$$\begin{aligned}\int_0^\infty \frac{1}{1+t^2}dt = &\lim_{A\to\infty}\int_0^A \frac{1}{1+t^2}dt = \lim_{A\to\infty}[\arctan t]_0^A\\ =&\lim_{A\to\infty}\arctan A = \frac{\pi}{2}\end{aligned}$$

So the integral is convergent ■

Problem 43.2
Determine whether the integral $\int_0^\infty \frac{t}{1+t^2}dt$ converges. If the integral converges, give its value.

Solution.
We have

$$\begin{aligned}\int_0^\infty \frac{t}{1+t^2}dt =&\frac{1}{2}\lim_{A\to\infty}\int_0^A \frac{2t}{1+t^2}dt = \frac{1}{2}\lim_{A\to\infty}\left[\ln\left(1+t^2\right)\right]_0^A\\ =&\frac{1}{2}\lim_{A\to\infty}\ln\left(1+A^2\right) = \infty\end{aligned}$$

Hence, the integral is divergent ■

Problem 43.3
Determine whether the integral $\int_0^\infty e^{-t}\cos\left(e^{-t}\right)dt$ converges. If the integral converges, give its value.

Solution.
Using the substitution $u = e^{-t}$ we find

$$\begin{aligned}\int_0^\infty e^{-t}\cos(e^{-t})dt =& \lim_{A\to\infty}\int_1^{e^{-A}} -\cos u du\\ =& \lim_{A\to\infty}[-\sin u]_1^{e^{-A}} = \lim_{A\to\infty}[\sin 1 - \sin(e^{-A})]\\ =& \sin 1\end{aligned}$$

Hence, the integral is convergent ■

Problem 43.4
Using the definition, find $\mathcal{L}[e^{3t}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Solution.
We have

$$\begin{aligned}\mathcal{L}[e^{3t}] =& \lim_{A\to\infty}\int_0^A e^{3t}e^{-st}dt = \lim_{A\to\infty}\int_0^A e^{t(3-s)}dt\\ =& \lim_{A\to\infty}\left[\frac{e^{t(3-s)}}{3-s}\right]_0^A\\ =& \lim_{A\to\infty}\left[\frac{e^{A(3-s)}}{3-s} - \frac{1}{3-s}\right]\\ =& \frac{1}{s-3}, \ \ s > 3\end{aligned}$$ ■

Problem 43.5
Using the definition, find $\mathcal{L}[t-5]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Solution.
Using integration by parts we find

$$\begin{aligned}\mathcal{L}[t-5] =& \lim_{A\to\infty}\int_0^A (t-5)e^{-st}dt = \lim_{A\to\infty}\left\{\left[\frac{-(t-5)e^{-st}}{s}\right]_0^A + \frac{1}{s}\int_0^A e^{-st}dt\right\}\\ =& \lim_{A\to\infty}\left\{\frac{-(A-5)e^{-sA}+5}{s} - \left[\frac{e^{-st}}{s^2}\right]_0^A\right\}\\ =& \frac{1}{s^2} - \frac{5}{s}, \ \ s > 0\end{aligned}$$ ■

Problem 43.6
Using the definition, find $\mathcal{L}[e^{(t-1)^2}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Solution.
We have

$$\int_0^\infty e^{(t-1)^2}e^{-st}dt = \int_0^\infty e^{(t-1)^2-st}dt.$$

Since $\lim_{t\to\infty}(t-1)^2 - st = \lim_{t\to\infty} t^2\left(1 - \frac{(2+s)}{t} + \frac{1}{t^2}\right) = \infty$, for any fixed s we can choose a positive C such that $(t-1)^2 - st \ge 0$ for $t \ge C$. In this case, $e^{(t-1)^2-st} \ge 1$ and this implies that $\int_0^\infty e^{(t-1)^2-st}dt \ge \int_C^\infty dt$. The integral on the right is divergent so that the integral on the left is also divergent by the comparison theorem of improper integrals. Hence, $f(t) = e^{(t-1)^2}$ does not have a Laplace transform ■

Problem 43.7
Using the definition, find $\mathcal{L}[(t-2)^2]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

Solution.
We have

$$\mathcal{L}[(t-2)^2] = \lim_{T\to\infty}\int_0^T (t-2)^2e^{-st}dt.$$

Using integration by parts with $u' = e^{-st}$ and $v = (t-2)^2$ we find

$$\begin{aligned}\int_0^T (t-2)^2e^{-st}dt =& -\left[\frac{(t-2)^2e^{-st}}{s}\right]_0^T + \frac{2}{s}\int_0^T (t-2)e^{-st}dt\\ =&\frac{4}{s} - \frac{(T-2)^2e^{-sT}}{s} + \frac{2}{s}\int_0^T (t-2)e^{-st}dt.\end{aligned}$$

Thus,

$$\lim_{T\to\infty}\int_0^T (t-2)^2e^{-st}dt = \frac{4}{s} + \frac{2}{s}\lim_{T\to\infty}\int_0^T (t-2)e^{-st}dt.$$

Using by parts with $u' = e^{-st}$ and $v = t-2$ we find

$$\int_0^T (t-2)e^{-st}dt = \left[-\frac{(t-2)e^{-st}}{s} - \frac{1}{s^2}e^{-st}\right]_0^T.$$

Letting $T \to \infty$ in the above expression we find

$$\lim_{T\to\infty}\int_0^T (t-2)e^{-st}dt = -\frac{2}{s} + \frac{1}{s^2}, \quad s > 0.$$

Hence,

$$F(s) = \frac{4}{s} + \frac{2}{s}\left(-\frac{2}{s} + \frac{1}{s^2}\right) = \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}, \quad s > 0 \blacksquare$$

Problem 43.8
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & t \ge 1 \end{cases}$$

Solution.
We have

$$\mathcal{L}[f(t)] = \lim_{T\to\infty}\int_1^T (t-1)e^{-st}dt.$$

Using integration by parts with $u' = e^{-st}$ and $v = t-1$ we find

$$\lim_{T\to\infty}\int_1^T (t-1)e^{-st}dt = \lim_{T\to\infty}\left[-\frac{(t-1)e^{-st}}{s} - \frac{1}{s^2}e^{-st}\right]_1^T = \frac{e^{-s}}{s^2}, \quad s > 0 \blacksquare$$

Problem 43.9
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$$

Solution.
We have

$$\begin{aligned}\mathcal{L}[f(t)] = &\int_1^2 (t-1)e^{-st}dt = \left[-\frac{(t-1)e^{-st}}{s} - \frac{1}{s^2}e^{-st}\right]_1^2 \\ = &-\frac{e^{-2s}}{s} + \frac{1}{s^2}(e^{-s} - e^{-2s}), \quad s \ne 0 \blacksquare\end{aligned}$$

Problem 43.10
Let n be a positive integer. Using integration by parts establish the reduction formula

$$\int t^n e^{-st}dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s}\int t^{n-1}e^{-st}dt, \quad s > 0.$$

Solution.
Let $u' = e^{-st}$ and $v = t^n$. Then $u = -\frac{e^{-st}}{s}$ and $v' = nt^{n-1}$. Hence,

$$\int t^n e^{-st}dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s}\int t^{n-1}e^{-st}dt, \quad s > 0 \blacksquare$$

Problem 43.11
For $s > 0$ and n a positive integer evaluate the limits

(a) $\lim_{t\to 0} t^n e^{-st}$ (b) $\lim_{t\to\infty} t^n e^{-st}$

Solution.
(a) $\lim_{t\to 0} t^n e^{-st} = \lim_{t\to 0} \frac{t^n}{e^{st}} = \frac{0}{1} = 0.$
(b) Using L'Hôpital's rule repeatedly we find

$$\lim_{t\to\infty} t^n e^{-st} = \cdots = \lim_{t\to\infty} \frac{n!}{s^n e^{st}} = 0 \blacksquare$$

Problem 43.12
(a) Use the previous two problems to derive the reduction formula for the Laplace transform of $f(t) = t^n$,

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}], \quad s > 0.$$

(b) Calculate $\mathcal{L}[t^k]$, for $k = 1, 2, 3, 4, 5$.
(c) Formulate a conjecture as to the Laplace transform of $f(t), t^n$ with n a positive integer.

Solution.
(a) Using the two previous problems we find

$$\begin{aligned}\mathcal{L}[t^n] =& \lim_{T\to\infty}\int_0^T t^n e^{-st}dt = \lim_{T\to\infty}\left\{-\left[\frac{t^n e^{-st}}{s}\right]_0^T + \frac{n}{s}\int_0^T t^{n-1}e^{-st}dt\right\}\\ =&\frac{n}{s}\lim_{T\to\infty}\int_0^T t^{n-1}e^{-st}dt = \frac{n}{s}\mathcal{L}[t^{n-1}], \quad s > 0\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathcal{L}[t] =& \frac{1}{s^2}\\
\mathcal{L}[t^2] =& \frac{2}{s}\mathcal{L}[t] = \frac{2}{s^3}\\
\mathcal{L}[t^3] =& \frac{3}{s}\mathcal{L}[t^2] = \frac{6}{s^4}\\
\mathcal{L}[t^4] =& \frac{4}{s}\mathcal{L}[t^3] = \frac{24}{s^5}\\
\mathcal{L}[t^5] =& \frac{5}{s}\mathcal{L}[t^4] = \frac{120}{s^5}
\end{aligned}$$

(c) By induction, one can easily show that for $n = 0, 1, 2, \cdots$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0 \blacksquare$$

From a table of integrals,

$$\begin{aligned}
\int e^{\alpha u} \sin \beta u du =& e^{\alpha u} \frac{\alpha \sin \beta u - \beta \cos \beta u}{\alpha^2 + \beta^2}\\
\int e^{\alpha u} \cos \beta u du =& e^{\alpha u} \frac{\alpha \cos \beta u + \beta \sin \beta u}{\alpha^2 + \beta^2}
\end{aligned}$$

Problem 43.13
Use the above integrals to find the Laplace transform of $f(t) = \cos \omega t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Solution.
We have

$$\mathcal{L}[\cos \omega t] = \lim_{T \to \infty} -\left\{ e^{-st} \left[\frac{-s \cos \omega t + \omega \sin \omega t}{s^2 + \omega^2} \right]_0^T \right\} = \frac{s}{s^2 + \omega^2}, \quad s > 0 \blacksquare$$

Problem 43.14
Use the above integrals to find the Laplace transform of $f(t) = \sin \omega t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Solution.
We have

$$\mathcal{L}[\sin \omega t] = \lim_{T \to \infty} -\left\{ e^{-st} \left[\frac{-s \sin \omega t + \omega \cos \omega t}{s^2 + \omega^2} \right]_0^T \right\} = \frac{\omega}{s^2 + \omega^2}, \quad s > 0 \blacksquare$$

Problem 43.15
Use the above integrals to find the Laplace transform of $f(t) = \cos\omega(t-2)$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Solution.
Using a trigonometric identity we can write $f(t) = \cos\omega(t-2) = \cos\omega t\cos 2\omega + \sin\omega t\sin 2\omega$. Thus, using the previous two problems we find

$$\mathcal{L}[\cos\omega(t-2)] = \frac{s\cos 2\omega + \omega\sin 2\omega}{s^2+\omega^2}, \quad s > 0 \blacksquare$$

Problem 43.16
Use the above integrals to find the Laplace transform of $f(t) = e^{3t}\sin t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

Solution.
We have

$$\begin{aligned}\mathcal{L}[e^{3t}\sin t] =& \lim_{T\to\infty}\int_0^T e^{-(s-3)t}\sin t dt\\ =& \lim_{T\to\infty} -\left\{e^{-(s-3)t}\left[\frac{(s-3)\sin t+\cos t}{(s-3)^2+1}\right]_0^T\right\}\\ =& \frac{1}{(s-3)^2+1}, \quad s > 3 \blacksquare\end{aligned}$$

Problem 43.17
Use the linearity property of Laplace transform to find $\mathcal{L}[5e^{-7t} + t + 2e^{2t}]$. Find the domain of $F(s)$.

Solution.
We have $\mathcal{L}[e^{-7t}] = \frac{1}{s+7},\ s > -7$, $\mathcal{L}[t] = \frac{1}{s^2},\ s > 0$, and $\mathcal{L}[e^{2t}] = \frac{1}{s-2},\ s > 2$. Hence,

$$\mathcal{L}[5e^{-7t} + t + 2e^{2t}] = 5\mathcal{L}[e^{-7t}] + \mathcal{L}[t] + 2\mathcal{L}[e^{2t}] = \frac{5}{s+7} + \frac{1}{s^2} + \frac{2}{s-2}, \ s > 2 \blacksquare$$

Problem 43.18
Consider the function $f(t) = \tan t$.
(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

Solution.
(a) Since $f(t) = \tan t = \frac{\sin t}{\cos t}$ and this function is discontinuous at $t = (2n+1)\frac{\pi}{2}$. Since this function has vertical asymptotes there it is not piecewise continuous.
(b) The graph of the function does not show that it can be bounded by exponential functions. Hence, no such numbers a and M ■

Problem 43.19
Consider the function $f(t) = t^2e^{-t}$.
(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

Solution.
(a) Since t^2 and e^{-t} are continuous everywhere, $f(t) = t^2e^{-t}$ is continuous on $0 \le t < \infty$.
(b) By L'Hôpital's rule one has

$$\lim_{t\to\infty} \frac{t^2}{e^t} = 0$$

Since $f(0) = 0$, $f(t)$ is bounded. Since $f'(t) = (2t - t^2)e^{-t}$, $f(t)$ has a maximum when $t = 2$. The value of this maximum is $f(2) = 4e^{-2}$. Hence, $M = 4e^{-2}$ and $a = 0$ ■

Problem 43.20
Consider the function $f(t) = \frac{e^{t^2}}{e^{2t}+1}$.
(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

Solution.
(a) Since et^2 and $e^{2t}+1$ are continuous everywhere, $f(t) = \frac{e^{t^2}}{e^{2t}+1}$ is continuous on $0 \le t < \infty$.
(b) Since $e^{2t} + 1 \le e^{2t} + e^{2t} = 2e^{2t}$, $f(t) \ge \frac{1}{2}e^{t^2}e^{-2t} = \frac{1}{2}e^{t^2-2t}$. But for $t \ge 4$ we have $t^2 - 2t > \frac{t^2}{2}$. Hence, $f(t) > \frac{1}{2}e^{\frac{t^2}{2}}$. So $f(t)$ is not of exponential order at infinity ■

Problem 43.21
Consider the floor function $f(t) = \lfloor t \rfloor$, where for any integer n we have $\lfloor t \rfloor = n$ for all $n \leq t < n+1$.
(a) Is $f(t)$ continuous on $0 \leq t < \infty$, discontinuous but piecewise continuous on $0 \leq t < \infty$, or neither?
(b) Are there fixed numbers a and M such that $|f(t)| \leq Me^{at}$ for $0 \leq t < \infty$?

Solution.
(a) The floor function is a piecewise continuous function on $0 \leq t < \infty$.
(b) Since $\lfloor t \rfloor \leq t < e^t$ for $0 \geq t < \infty$ we find $M = 1$ and $a = 1$ ■

Problem 43.22
Find $\mathcal{L}^{-1}\left(\frac{3}{s-2}\right)$.

Solution.
Since $\mathcal{L}\left(\frac{1}{s-a}\right) = \frac{1}{s-a}, \ s > a$ we find

$$\mathcal{L}^{-1}\left(\frac{3}{s-2}\right) = 3\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = 3e^{2t}, \ t \geq 0 \ ■$$

Problem 43.23
Find $\mathcal{L}^{-1}\left(-\frac{2}{s^2} + \frac{1}{s+1}\right)$.

Solution.
Since $\mathcal{L}[t] = \frac{1}{s^2}, \ s > 0$ and $\mathcal{L}\left(\frac{1}{s-a}\right) = \frac{1}{s-a}, \ s > a$ we find

$$\begin{aligned}\mathcal{L}^{-1}\left(-\frac{2}{s^2} + \frac{1}{s+1}\right) &= -2\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)\\ &= -2t + e^{-t}, \ t \geq 0 \ ■\end{aligned}$$

Problem 43.24
Find $\mathcal{L}^{-1}\left(\frac{2}{s+2} + \frac{2}{s-2}\right)$.

Solution.
We have

$$\mathcal{L}^{-1}\left(\frac{2}{s+2} + \frac{2}{s-2}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = 2(e^{-2t} + e^{2t}), \ t \geq 0 \ ■$$

Section 44

Problem 44.1
Use Table $\mathcal{L}$ to find $\mathcal{L}[2e^t + 5]$.

Solution.

$$\mathcal{L}[2e^t + 5] = 2\mathcal{L}[e^t] + 5\mathcal{L}[1] = \frac{2}{s-1} + \frac{5}{s}, \quad s > 1 \blacksquare$$

Problem 44.2
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{3t-3}h(t-1)]$.

Solution.

$$\mathcal{L}[e^{3t-3}h(t-1)] = \mathcal{L}[e^{3(t-1)}h(t-1)] = e^{-s}\mathcal{L}[e^{3t}] = \frac{e^{-s}}{s-3}, \quad s > 3 \blacksquare$$

Problem 44.3
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin^2 \omega t]$.

Solution.

$$\mathcal{L}[\sin^2 \omega t] = \mathcal{L}[\frac{1-\cos 2\omega t}{2}] = \frac{1}{2}(\mathcal{L}[1] - \mathcal{L}[\cos 2\omega t]) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4\omega^2}\right), \quad s > 0 \blacksquare$$

Problem 44.4
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin 3t \cos 3t]$.

Solution.

$$\mathcal{L}[\sin 3t \cos 3t] = \mathcal{L}\left[\frac{\sin 6t}{2}\right] = \frac{1}{2}\mathcal{L}[\sin 6t] = \frac{3}{s^2+36}, \quad s > 0 \blacksquare$$

Problem 44.5
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{2t}\cos 3t]$.

Solution.

$$\mathcal{L}[e^{2t}\cos 3t] = \frac{s-2}{(s-2)^2+9}, \quad s > 2 \blacksquare$$

Problem 44.6
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{4t}(t^2+3t+5)]$.

Solution.

$$\mathcal{L}[e^{4t}(t^2+3t+5)] = \mathcal{L}[e^{4t}t^2]+3\mathcal{L}[e^{4t}t]+5\mathcal{L}[e^{4t}] = \frac{2}{(s-4)^3}+\frac{3}{(s-4)^2}+\frac{5}{s-4}, \ \ s>4 \blacksquare$$

Problem 44.7
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{10}{s^2+25}+\frac{4}{s-3}]$.

Solution.

$$\mathcal{L}^{-1}[\frac{10}{s^2+25}+\frac{4}{s-3}] = 2\mathcal{L}^{-1}[\frac{5}{s^2+25}]+4\mathcal{L}^{-1}[\frac{1}{s-3}] = 2\sin 5t+4e^{3t}, \ \ t\geq 0 \blacksquare$$

Problem 44.8
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{5}{(s-3)^4}]$.

Solution.

$$\mathcal{L}^{-1}[\frac{5}{(s-3)^4}] = \frac{5}{6}\mathcal{L}^{-1}[\frac{3!}{(s-3)^4}] = \frac{5}{6}e^{3t}t^3, \ \ t\geq 0 \blacksquare$$

Problem 44.9
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}]$.

Solution.

$$\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}] = e^{9(t-2)}h(t-2) = \begin{cases} 0, & 0\leq t<2 \\ e^{9(t-2)}, & t\geq 2 \end{cases} \blacksquare$$

Problem 44.10
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-3s}(2s+7)}{s^2+16}]$.

Solution.
We have

$$\begin{aligned}\mathcal{L}^{-1}[\frac{e^{-3s}(2s+7)}{s^2+16}] =& 2\mathcal{L}^{-1}[\frac{e^{-3s}s}{s^2+16}] + \frac{7}{4}\mathcal{L}^{-1}[\frac{e^{-3s}4}{s^2+16}] \\ =& 2\cos 4(t-3)h(t-3) + \frac{7}{4}\sin 4(t-3)h(t-3), \ t\geq 0 \blacksquare\end{aligned}$$

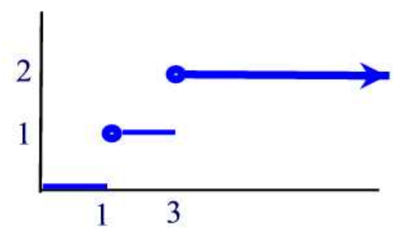
Problem 44.11
Graph the function $f(t) = h(t-1) + h(t-3)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 3 \\ 2, & t \geq 3 \end{cases}$$

The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find

$$\mathcal{L}[f(t)] = \mathcal{L}[h(t-1)] + \mathcal{L}[h(t-3)] = \frac{e^{-s}}{s} + \frac{e^{-3s}}{s}, \ s > 0 \ \blacksquare$$



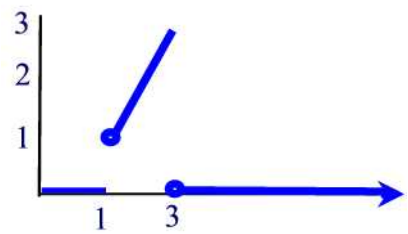
Problem 44.12
Graph the function $f(t) = t[h(t-1) - h(t-3)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t, & 1 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find

$$\begin{aligned} \mathcal{L}[f(t)] =& \mathcal{L}[(t-1)h(t-1) + h(t-1) - (t-3)h(t-3) - 3h(t-3)] \\ =& \mathcal{L}[(t-1)h(t-1)] + \mathcal{L}[h(t-1)] - \mathcal{L}[(t-3)h(t-3)] - 3\mathcal{L}[h(t-3)] \\ =& \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-3s}}{s^2} - \frac{3e^{-3s}}{s}, \quad s > 1 \ \blacksquare \end{aligned}$$



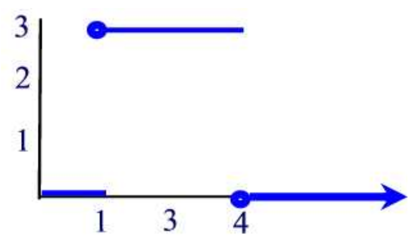
Problem 44.13
Graph the function $f(t) = 3[h(t-1) - h(t-4)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 3, & 1 \leq t < 4 \\ 0, & t \geq 4 \end{cases}$$

The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find

$$\mathcal{L}[f(t)] = 3\mathcal{L}[h(t-1)] - 3\mathcal{L}[h(t-4)] = \frac{3e^{-s}}{s} - \frac{3e^{-4s}}{s}, \ s > 0 \blacksquare$$



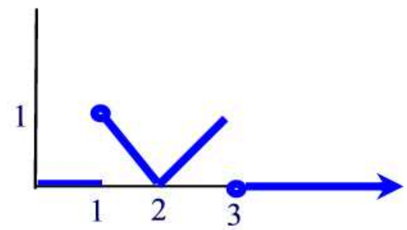
Problem 44.14
Graph the function $f(t) = |2-t|[h(t-1) - h(t-3)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ |2-t|, & 1 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find

$$\begin{aligned}\mathcal{L}[f(t)] =&(2-t)h(t-1)+2(t-2)h(t-2)-(t-2)h(t-3)\\ =&\mathcal{L}[-(t-1)h(t-1)+h(t-1)+2(t-2)h(t-2)-(t-3)h(t-3)-h(t-3)]\\ =&-\mathcal{L}[(t-1)h(t-1)]+\mathcal{L}[h(t-1)]+2\mathcal{L}[(t-2)h(t-2)]\\ &-\mathcal{L}[(t-3)h(t-3)]-\mathcal{L}[h(t-3)]\\ =&-\frac{e^{-s}}{s^2}+\frac{e^{-s}}{s}+\frac{2e^{-2s}}{s^2}-\frac{e^{-3s}}{s^2}-\frac{e^{-3s}}{s},\ s>0\ \blacksquare\end{aligned}$$



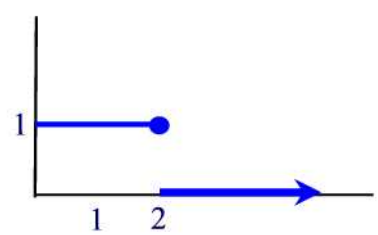
Problem 44.15
Graph the function $f(t)=h(2-t)$ for $t\geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

$$f(t)=\begin{cases}1, & 0\leq t\leq 2\\ 0, & t>2\end{cases}$$

The graph of $f(t)$ is shown below. From this graph we see that $f(t)=h(t)-h(t-2)h(t-2)$. Using Table $\mathcal{L}$ we find

$$\mathcal{L}[f(t)]=\mathcal{L}[h(t)]-\mathcal{L}[h(t-1)h(t-1)]=\frac{1-e^{-2s}}{s},\ s>0\ \blacksquare$$



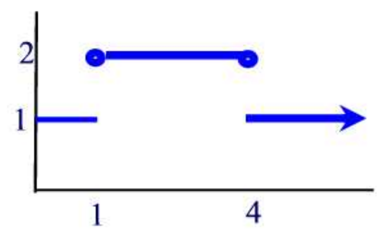
Problem 44.16
Graph the function $f(t) = h(t-1) + h(4-t)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

Solution.
Note that

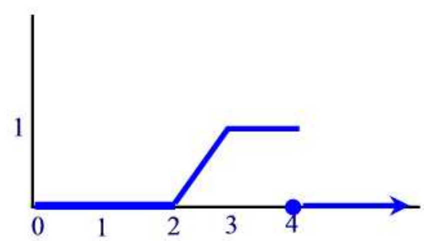
$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 2, & 1 \leq t \leq 4 \\ 1, & t > 4 \end{cases}$$

The graph of $f(t)$ is shown below. Using Table $\mathcal{L}$ we find

$$\mathcal{L}[f(t)] = \mathcal{L}[h(t-1)] + \mathcal{L}[h(4-t)] = \frac{e^{-s}}{s} + \int_0^4 e^{-st}dt = \frac{1 + e^{-s} - e^{-4s}}{s}, \ s > 0 \blacksquare$$



Problem 44.17
The graph of $f(t)$ is given below. Represent $f(t)$ as a combination of Heaviside step functions, and use Table $\mathcal{L}$ to calculate the Laplace transform of $f(t)$.



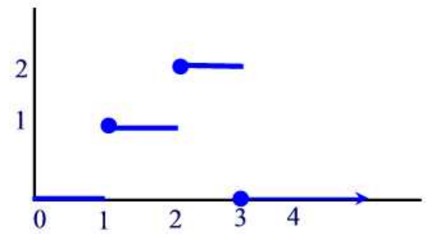
Solution.
From the graph we see that

$$f(t) = (t-2)h(t-2) - (t-3)h(t-3) - h(t-4)$$

Thus,

$$\mathcal{L}[f(t)] = \mathcal{L}[(t-2)h(t-2)] - \mathcal{L}[(t-3)h(t-3)] - \mathcal{L}[h(t-4)] = \frac{e^{-2s} - e^{-3s}}{s^2} - \frac{e^{-4s}}{s}, \ s > 0 \blacksquare$$

Problem 44.18
The graph of $f(t)$ is given below. Represent $f(t)$ as a combination of Heaviside step functions, and use Table $\mathcal{L}$ to calculate the Laplace transform of $f(t)$.



Solution.
From the graph we see that

$$f(t) = h(t-1) + h(t-2) - 2h(t-3).$$

Thus,

$$\mathcal{L}[f(t)] = \mathcal{L}[h(t-1)] - 2\mathcal{L}[h(t-3)] + \mathcal{L}[h(t-2)] = \frac{e^{-s} - 2e^{-3s} + e^{-2s}}{s}, \ s > 0 \ \blacksquare$$

Problem 44.19
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{12}{(s-3)(s+1)}\right]$.

Solution.
Write

$$\frac{12}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}.$$

Multiply both sides of this equation by $s-3$ and cancel common factors to obtain

$$\frac{12}{s+1} = A + \frac{B(s-3)}{s+1}.$$

Now, find A by setting $s=3$ to obtain $A=3$. Similarly, by multiplying both sides by $s+1$ and then setting $s=-1$ in the resulting equation leads to $B=-3$. Hence,

$$\frac{12}{(s-3)(s+1)} = 3\left(\frac{1}{s-3} - \frac{1}{s+1}\right).$$

Finally,

$$\mathcal{L}^{-1}\left[\frac{12}{(s-3)(s+1)}\right] =3\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$
$$=3e^{3t} - 3e^{-t},\ t \geq 0 \ \blacksquare$$

Problem 44.20
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{24e^{-5s}}{s^2-9}\right]$.

Solution.
Write

$$\frac{24}{(s-3)(s+3)} = \frac{A}{s-3} + \frac{B}{s+3}.$$

Multiply both sides of this equation by $s-3$ and cancel common factors to obtain

$$\frac{24}{s+3} = A + \frac{B(s-3)}{s+3}.$$

Now, find A by setting $s = 3$ to obtain $A = 4$. Similarly, by multiplying both sides by $s+3$ and then setting $s = -3$ in the resulting equation leads to $B = -4$. Hence,

$$\frac{24}{(s-3)(s+3)} = 4\left(\frac{1}{s-3} - \frac{1}{s+3}\right).$$

Finally,

$$\mathcal{L}^{-1}\left[\frac{24e^{-5s}}{(s-3)(s+3)}\right] =4\mathcal{L}^{-1}\left[\frac{e^{-5s}}{s-3}\right] - 4\mathcal{L}^{-1}\left[\frac{e^{-5s}}{s+3}\right]$$
$$=4[e^{3(t-5)} - e^{-3(t-5)}]h(t-5),\ t \geq 0 \ \blacksquare$$

Problem 44.21
Use Laplace transform technique to solve the initial value problem

$$y' + 4y = g(t), \quad y(0) = 2$$

where

$$g(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 12, & 1 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

Solution.
Note first that $g(t) = 12[h(t-1) - h(t-3)]$ so that

$$\mathcal{L}[g(t)] = 12\mathcal{L}[h(t-1)] - 12\mathcal{L}[h(t-3)] = \frac{12(e^{-s} - e^{-3s})}{s}, \; s > 0.$$

Now taking the Laplace transform of the DE and using linearity we find

$$\mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}[g(t)].$$

But $\mathcal{L}[y'] = s\mathcal{L}[y] - y(0) = s\mathcal{L}[y] - 2$. Letting $\mathcal{L}[y] = Y(s)$ we obtain

$$sY(s) - 2 + 4Y(s) = 12\frac{e^{-s} - e^{-3s}}{s}.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{2}{s+4} + 12\frac{e^{-s} - e^{-3s}}{s(s+4)}.$$

But

$$\mathcal{L}^{-1}\left[\frac{2}{s+4}\right] = 2e^{-4t}$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left[12\frac{e^{-s} - e^{-3s}}{s(s+4)}\right] =& 3\mathcal{L}^{-1}\left[(e^{-s} - e^{-3s})\left(\frac{1}{s} - \frac{1}{s+4}\right)\right] \\ =& 3\mathcal{L}^{-1}\left[\frac{e^{-s}}{s}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-s}}{s+4}\right] + 3\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s+4}\right] \\ =& 3h(t-1) - 3h(t-3) - 3e^{-4(t-1)}h(t-1) + 3e^{-4(t-3)}h(t-3)\end{aligned}$$

Hence,

$$y(t) = 2e^{-4t} + 3[h(t-1) - h(t-3)] - 3[e^{-4(t-1)}h(t-1) - e^{-4(t-3)}h(t-3)], \; t \geq 0 \blacksquare$$

Problem 44.22
Use Laplace transform technique to solve the initial value problem

$$y'' - 4y = e^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution.
Taking the Laplace transform of the DE and using linearity we find

$$\mathcal{L}[y''] - 4\mathcal{L}[y] = \mathcal{L}[e^{3t}].$$

But $\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0) = s^2\mathcal{L}[y]$. Letting $\mathcal{L}[y] = Y(s)$ we obtain

$$s^2Y(s) - 4Y(s) = \frac{1}{s-3}.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{1}{(s-3)(s-2)(s+2)}.$$

Using partial fraction decomposition

$$\frac{1}{(s-3)(s-2)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} + \frac{C}{s-2}$$

we find $A = \frac{1}{5}$, $B = \frac{1}{20}$, and $C = -\frac{1}{4}$. Thus,

$$\begin{aligned}
y(t) =&\mathcal{L}^{-1}\left[\frac{1}{(s-3)(s-2)(s+2)}\right] = \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] + \frac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]\\
=&\frac{1}{5}e^{3t} + \frac{1}{20}e^{-2t} - \frac{1}{4}e^{2t}, \ t \geq 0 \ \blacksquare
\end{aligned}$$

Problem 44.23
Obtain the Laplace transform of the function $\int_2^t f(\lambda)d\lambda$ in terms of $\mathcal{L}[f(t)] = F(s)$ given that $\int_0^2 f(\lambda)d\lambda = 3$.

Solution.
We have

$$\begin{aligned}
\mathcal{L}\left[\int_2^t f(\lambda)d\lambda\right] =&\mathcal{L}\left[\int_0^t f(\lambda)d\lambda - \int_0^2 f(\lambda)d\lambda\right]\\
=&\frac{F(s)}{s} - \mathcal{L}[3]\\
=&\frac{F(s)}{s} - \frac{3}{s}, \quad s > 0 \ \blacksquare
\end{aligned}$$

Section 45

In Problems 45.1 - 45.4, give the form of the partial fraction expansion for $F(s)$. You need not evaluate the constants in the expansion. However, if the denominator has an irreducible quadratic expression then use the completing the square process to write it as the sum/difference of two squares.

Problem 45.1

$$F(s) = \frac{s^3 + 3s + 1}{(s-1)^3(s-2)^2}.$$

Solution.

$$F(s) = \frac{A_1}{(s-1)^3} + \frac{A_2}{(s-1)^2} + \frac{A_3}{s-1} + \frac{B_1}{(s-2)^2} + \frac{B_2}{s-2} \blacksquare$$

Problem 45.2

$$F(s) = \frac{s^2 + 5s - 3}{(s^2+16)(s-2)}.$$

Solution.

$$F(s) = \frac{A_1 s + A_2}{s^2+16} + \frac{B_1}{s-2} \blacksquare$$

Problem 45.3

$$F(s) = \frac{s^3 - 1}{(s^2+1)^2(s+4)^2}.$$

Solution.

$$F(s) = \frac{A_1 s + A_2}{(s^2+1)^2} + \frac{A_3 s + A_4}{s^2+1} + \frac{B_1}{(s+4)^2} + \frac{B_2}{s+4} \blacksquare$$

Problem 45.4

$$F(s) = \frac{s^4 + 5s^2 + 2s - 9}{(s^2+8s+17)(s-2)^2}.$$

Solution.

$$F(s) = \frac{A_1}{(s-2)^2} + \frac{A_2}{s-2} + \frac{B_1 s + B_2}{(s+4)^2+1} \blacksquare$$

Problem 45.5
Find $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3}\right]$.

Solution.
Using Table $\mathcal{L}$ we find $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3}\right] = \frac{1}{2}e^{-t}t^2, \ t \ge 0 \blacksquare$

Problem 45.6
Find $\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-3s+2}\right]$.

Solution.
We factor the denominator and split the rational function into partial fractions:

$$\frac{2s-3}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

Multiplying both sides by $(s-1)(s-2)$ and simplifying to obtain

$$\begin{aligned} 2s-3 =& A(s-2) + B(s-1) \\ =& (A+B)s - 2A - B. \end{aligned}$$

Equating coefficients of like powers of s we obtain the system

$$\left\{ \begin{array}{rcr} A+B & = & 2 \\ -2A-B & = & -3. \end{array} \right.$$

Solving this system by elimination we find $A = 1$ and $B = 1$. Now finding the inverse Laplace transform to obtain

$$\mathcal{L}^{-1}\left[\frac{2s-3}{(s-1)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = e^t + e^{2t}, \ t \ge 0. \blacksquare$$

Problem 45.7
Find $\mathcal{L}^{-1}\left[\frac{4s^2+s+1}{s^3+s}\right]$.

Solution.
We factor the denominator and split the rational function into partial fractions:

$$\frac{4s^2+s+1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Multiplying both sides by $s(s^2+1)$ and simplifying to obtain

$$\begin{aligned}4s^2+s+1 =& A(s^2+1) + (Bs+C)s\\ =&(A+B)s^2 + Cs + A.\end{aligned}$$

Equating coefficients of like powers of s we obtain $A+B=4,\ C=1,\ A=1.$ Thus, $B=3$. Now finding the inverse Laplace transform to obtain

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{4s^2+s+1}{s(s^2+1)}\right] =& \mathcal{L}^{-1}\left[\frac{1}{s}\right] + 3\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]\\ =&1+3\cos t + \sin t,\ t \geq 0 \ \blacksquare\end{aligned}$$

Problem 45.8
Find $\mathcal{L}^{-1}\left[\frac{s^2+6s+8}{s^4+8s^2+16}\right]$.

Solution.
We factor the denominator and split the rational function into partial fractions:

$$\frac{s^2+6s+8}{(s^2+4)^2} = \frac{B_1 s + C_1}{s^2+4} + \frac{B_2 s + C_2}{(s^2+4)^2}.$$

Multiplying both sides by $(s^2+4)^2$ and simplifying to obtain

$$\begin{aligned}s^2+6s+8 =& (B_1 s + C_1)(s^2+4) + B_2 s + C_2\\ =&B_1 s^3 + C_1 s^2 + (4B_1 + B_2)s + 4C_1 + C_2.\end{aligned}$$

Equating coefficients of like powers of s we obtain $B_1 = 0, C_1 = 1, B_2 = 6,$ and $C_2 = 4$. Now finding the inverse Laplace transform to obtain

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s^2+6s+8}{(s^2+4)^2}\right] =& \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] + 6\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{(s^2+4)^2}\right]\\ =&\frac{1}{2}\sin 2t + 6\left(\frac{t}{4}\sin 2t\right) + 4\left(\frac{1}{16}[\sin 2t - 2t\cos 2t]\right)\\ =&\frac{3}{2}t\sin 2t + \frac{3}{4}\sin 2t - \frac{1}{2}t\cos 2t,\ t \geq 0 \ \blacksquare\end{aligned}$$

Problem 45.9
Use Laplace transform to solve the initial value problem

$$y' + 2y = 26\sin 3t,\ y(0) = 3.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = 26\mathcal{L}[\sin 3t].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$sY(s) - y(0) + 2Y(s) = 26\left(\frac{3}{s^2+9}\right).$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{3}{s+2} + \frac{78}{(s+2)(s^2+9)}.$$

Using the partial fraction decomposition we can write

$$\frac{1}{(s+2)(s^2+9)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+9}.$$

Multipliying both sides by $(s+2)(s^2+9)$ to obtain

$$\begin{aligned}1 =& A(s^2+9) + (Bs+C)(s+2)\\ =& (A+B)s^2 + (2B+C)s + 9A + 2C.\end{aligned}$$

Equating coefficients of like powers of s we find $A+B=0,\ 2B+C=0$, and $9A+2C=1$. Solving this system we find $A=\frac{1}{13},\ B=-\frac{1}{13}$, and $C=\frac{2}{13}$. Thus,

$$Y(s) = \frac{9}{s+2} - 6\left(\frac{s}{s^2+9}\right) + 4\left(\frac{3}{s^2+9}\right).$$

Finally,

$$\begin{aligned}y(t) = \mathcal{L}^{-1}[Y(s)] = 9\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 6\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] + 4\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right]\\ = 9e^{-2t} - 6\cos 3t + 4\sin 3t,\ t \geq 0 \blacksquare\end{aligned}$$

Problem 45.10
Use Laplace transform to solve the initial value problem

$$y' + 2y = 4t, \; y(0) = 3.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = 4\mathcal{L}[t].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s^2}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{3}{s+2} + \frac{4}{(s+2)s^2}.$$

Using the partial fraction decomposition we can write

$$\frac{1}{(s+2)s^2} = \frac{A}{s+2} + \frac{Bs+C}{s^2}.$$

Multipliying both sides by $(s+2)s^2$ to obtain

$$\begin{aligned} 1 &= As^2 + (Bs+C)(s+2) \\ &= (A+B)s^2 + (2B+C)s + 2C \end{aligned}$$

Equating coefficients of like powers of s we find $A+B=0, \; 2B+C=0$, and $2C=1$. Solving this system we find $A=\frac{1}{4}, \; B=-\frac{1}{4}$, and $C=\frac{1}{2}$. Thus,

$$Y(s) = \frac{4}{s+2} - \frac{1}{s} + 2\frac{1}{s^2}.$$

Finally,

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)] &= 4\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \\ &= 4e^{-2t} - 1 + 2t, \; t \geq 0 \; \blacksquare \end{aligned}$$

Problem 45.11
Use Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = 6e^{-t},\ y(0) = 1,\ y'(0) = 2.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = 6\mathcal{L}[e^{-t}].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{6}{s+1}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{s+5}{(s+1)(s+2)} + \frac{6}{(s+2)(s+1)^2} = \frac{s^2+6s+11}{(s+1)^2(s+2)}.$$

Using the partial fraction decomposition we can write

$$\frac{s^2+6s+11}{(s+2)(s+1)^2} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}.$$

Multipliying both sides by $(s+2)(s+1)^2$ to obtain

$$\begin{aligned} s^2+6s+11 &= A(s+1)^2 + B(s+1)(s+2) + C(s+2) \\ &= (A+B)s^2 + (2A+3B+C)s + A + 2B + 2C \end{aligned}$$

Equating coefficients of like powers of s we find $A+B=1,\ 2A+3B+C=6,$ and $A+2B+2C=11$. Solving this system we find $A=3,\ B=-2,$ and $C=6$. Thus,

$$Y(s) = \frac{3}{s+2} - \frac{2}{s+1} + \frac{6}{(s+1)^2}.$$

Finally,

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)] &= 3\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + 6\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] \\ &= 3e^{-2t} - 2e^{-t} + 6te^{-t}, t \ge 0 \ \blacksquare \end{aligned}$$

Problem 45.12
Use Laplace transform to solve the initial value problem

$$y'' + 4y = \cos 2t, \;\; y(0) = 1, \;\; y'(0) = 1.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[\cos 2t].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{s}{s^2+4}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{s+1}{s^2+4} + \frac{s}{(s^2+4)^2}.$$

Using Table $\mathcal{L}$ again we find

$$\begin{aligned} y(t) =& \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] + \mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right] \\ =& \cos 2t + \frac{1}{2}\sin 2t + \frac{t}{4}\sin 2t, \;\; t \geq 0 \blacksquare \end{aligned}$$

Problem 45.13
Use Laplace transform to solve the initial value problem

$$y'' - 2y' + y = e^{2t}, \;\; y(0) = 0, \;\; y'(0) = 0.$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y''] - 2\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[e^{2t}].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + Y(s) = \frac{1}{s-2}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{1}{(s-1)^2(s-2)}.$$

Using the partial fraction decomposition, we can write

$$Y(s) = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2}.$$

Multipliying both sides by $(s-2)(s-1)^2$ to obtain

$$\begin{aligned} 1 &= A(s-1)(s-2) + B(s-2) + C(s-1)^2 \\ &= (A+C)s^2 + (-3A+B-2C)s + 2A - 2B + C \end{aligned}$$

Equating coefficients of like powers of s we find $A+C=0,\ -3A+B-2C=0$, and $2A-2B+C=1$. Solving this system we find $A=-1,\ B=-1$, and $C=1$. Thus,

$$Y(s) = -\frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{s-2}.$$

Finally,

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)] &= -\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] \\ &= -e^t - te^t + e^{2t},\ t \geq 0 \ \blacksquare \end{aligned}$$

Problem 45.14
Use Laplace transform to solve the initial value problem

$$y'' + 9y = g(t),\ y(0) = 1,\ y'(0) = 3$$

where

$$g(t) = \begin{cases} 6, & 0 \leq t < \pi \\ 0, & \pi \leq t < \infty \end{cases}$$

Solution.
Taking the Laplace of both sides to obtain

$$\mathcal{L}[y''] + 9\mathcal{L}[y] = \mathcal{L}[g(t)] = 6\mathcal{L}[h(t) - h(t-\pi)].$$

Using Table $\mathcal{L}$ the last equation reduces to

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{6}{s} - \frac{6e^{-\pi s}}{s}.$$

Solving this equation for $Y(s)$ we find

$$Y(s) = \frac{s+3}{s^2+9} + \frac{6}{s(s^2+9)}(1 - e^{-\pi s}).$$

Using the partial fraction decomposition, we can write

$$\frac{6}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}.$$

Multipliying both sides by $s(s^2+9)$ to obtain

$$\begin{aligned} 6 &= A(s^2+9) + (Bs+C)s \\ &= (A+B)s^2 + Cs + 9A \end{aligned}$$

Equating coefficients of like powers of s we find $A + B = 0,\ C = 0$, and $9A = 6$. Solving this system we find $A = \frac{2}{3},\ B = -\frac{2}{3}$, and $C = 0$. Thus,

$$Y(s) = \frac{s}{s^2+9} + \frac{3}{s^2+9} + (1 - e^{-\pi s})\left(\frac{2}{3}\frac{1}{s} - \frac{2}{3}\frac{s}{s^2+9}\right).$$

Finally,

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)] &= \cos 3t + \sin 3t + \frac{2}{3}(1 - \cos 3t) - \frac{2}{3}(1 - \cos 3(t-\pi))h(t-\pi) \\ &= \cos 3t + \sin 3t + \frac{2}{3}(1 - \cos 3t) - \frac{2}{3}(1 + \cos 3t)h(t-\pi),\ t \geq 0 \ \blacksquare \end{aligned}$$

Problem 45.15
Determine the constants α, β, y_0, and y'_0 so that $Y(s) = \frac{2s-1}{s^2+s+2}$ is the Laplace transform of the solution to the initial value problem

$$y'' + \alpha y' + \beta y = 0,\ y(0) = y_0,\ y'(0) = y'_0.$$

Solution.
Taking the Laplace transform of both sides we find

$$s^2Y(s) - sy_0 - y'_0 + \alpha s Y(s) - \alpha y_0 + \beta Y(s) = 0.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{sy_0 + (y_0' + \alpha y_0)}{s^2 + \alpha s + \beta} = \frac{2s - 1}{s^2 + s + 2}.$$

By identification we find $\alpha = 1,\ \beta = 2,\ y_0 = 2$, and $y_0' = -3$ ■

Problem 45.16
Determine the constants α, β, y_0, and y_0' so that $Y(s) = \frac{s}{(s+1)^2}$ is the Laplace transform of the solution to the initial value problem

$$y'' + \alpha y' + \beta y = 0,\ y(0) = y_0,\ y'(0) = y_0'.$$

Solution.
Taking the Laplace transform of both sides we find

$$s^2 Y(s) - sy_0 - y_0' + \alpha s Y(s) - \alpha y_0 + \beta Y(s) = 0.$$

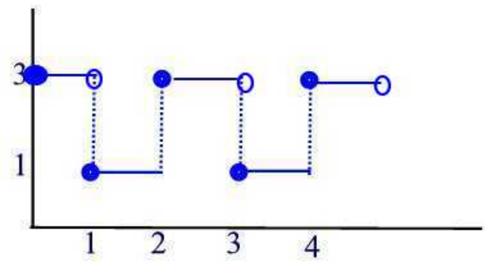
Solving for $Y(s)$ we find

$$Y(s) = \frac{sy_0 + (y_0' + \alpha y_0)}{s^2 + \alpha s + \beta} = \frac{s}{s^2 + 2s + 1}.$$

By identification we find $\alpha = 2,\ \beta = 1,\ y_0 = 1$, and $y_0' = -2$ ■

Section 46

Problem 46.1
Find the Laplace transform of the periodic function whose graph is shown.



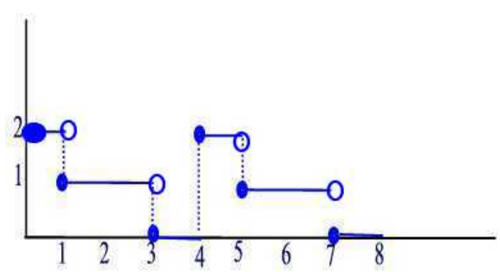
Solution.
The function is of period $T = 2$. Thus,

$$3\int_0^1 e^{-st}dt + \int_1^2 e^{-st}dt = \left[-\frac{3}{s}e^{-st}\right]_0^1 - \left[\frac{e^{-st}}{s}\right]_1^2 = \frac{1}{s}(3 - 2e^{-s} - e^{-2s}).$$

Hence,

$$\mathcal{L}[f(t)] = \frac{3 - 2e^{-s} - e^{-2s}}{s(1 - e^{-2s})} \blacksquare$$

Problem 46.2
Find the Laplace transform of the periodic function whose graph is shown.



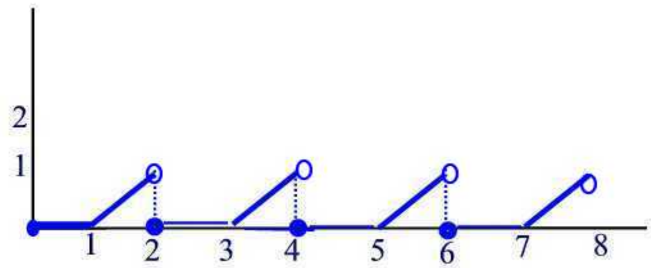
Solution.
The function is of period $T = 4$. Thus,

$$2\int_0^1 e^{-st}dt + \int_1^3 e^{-st}dt = \left[-\frac{2}{s}e^{-st}\right]_0^1 - \left[\frac{e^{-st}}{s}\right]_1^3 = \frac{1}{s}(2 - e^{-s} - e^{-3s}).$$

Hence,

$$\mathcal{L}[f(t)] = \frac{2 - e^{-s} - e^{-3s}}{s(1 - e^{-4s})} \blacksquare$$

Problem 46.3
Find the Laplace transform of the periodic function whose graph is shown.



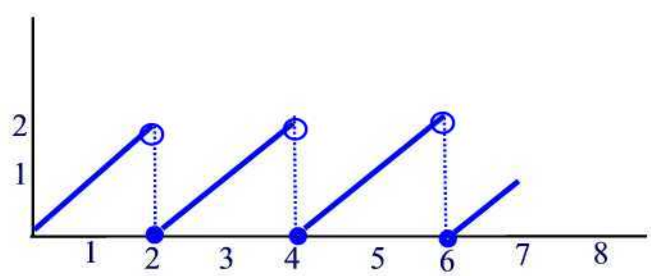
Solution.
The function is of period $T = 2$. Thus,

$$\begin{aligned}\int_1^2 (t-1)e^{-st}dt &= \int_1^2 te^{-st}dt - \int_1^2 e^{-st}dt \\ &= \left[-\frac{t}{s}e^{-st} - \frac{e^{-st}}{s^2} + \frac{e^{-st}}{s}\right]_1^2 \\ &= -\frac{e^{-s}}{s^2}[(1+s)e^{-s} - 1]\end{aligned}$$

Hence,

$$\mathcal{L}[f(t)] = \frac{e^{-s}}{s^2(1-e^{-2s})}[1 - (s+1)e^{-s}] \blacksquare$$

Problem 46.4
Find the Laplace transform of the periodic function whose graph is shown.



Solution.
The function is of period $T = 2$. Thus,

$$\int_0^2 te^{-st}dt = \left[-\frac{1}{s^2}(st+1)e^{-st}\right]_0^2 = -\frac{1}{s^2}[(2s+1)e^{-2s} - 1].$$

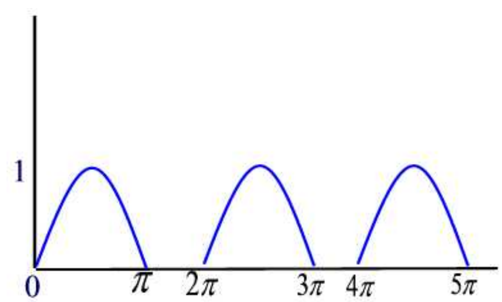
Hence,

$$\mathcal{L}[f(t)] = \frac{1}{s^2(1-e^{-2s})}[1-(2s+1)e^{-2s}] \blacksquare$$

Problem 46.5
State the period of the function $f(t)$ and find its Laplace transform where

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \end{cases} \qquad f(t+2\pi) = f(t), \ t \ge 0.$$

Solution.
The graph of $f(t)$ is shown below.



The function $f(t)$ is of period $T = 2\pi$. The Laplace transform of $f(t)$ is

$$\mathcal{L}[f(t)] = \frac{\int_0^\pi \sin t e^{-st}dt}{1-e^{-2\pi s}}$$

Using integration by parts twice we find

$$\int \sin t e^{-st}dt = -\frac{e^{-st}}{1+s^2}(\cos t + s\sin t)$$

Thus,

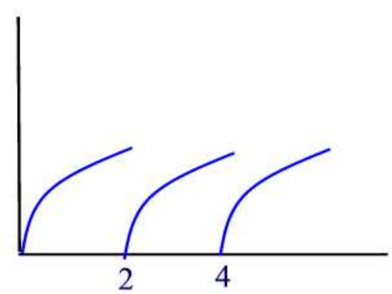
$$\begin{aligned}\int_0^{\pi} \sin t e^{-st} dt =& \left[-\frac{e^{-st}}{1+s^2}(\cos t + s\sin t)\right]_0^{\pi}\\ =&\frac{e^{-\pi s}}{1+s^2} + \frac{1}{1+s^2}\\ =&\frac{1+e^{-\pi s}}{1+s^2}\end{aligned}$$

Hence,

$$\mathcal{L}[f(t)] = \frac{1+e^{-\pi s}}{(1+s^2)(1-e^{-2\pi s})} \blacksquare$$

Problem 46.6
State the period of the function $f(t) = 1 - e^{-t},\ 0 \le t < 2, \quad f(t+2) = f(t),$ and find its Laplace transform.

Solution.
The graph of $f(t)$ is shown below



The function is periodic of period $T = 2$. Its Laplace transform is

$$\mathcal{L}[f(t)] = \frac{\int_0^2 (1-e^{-t})e^{-st}dt}{1-e^{-2s}}.$$

But

$$\int_0^2 (1-e^{-t})e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^2 + \left[\frac{e^{-(s+1)t}}{s+1}\right]_0^2 = \frac{1}{s}(1-e^{-2s}) - \frac{1}{s+1}(1-e^{-2(s+1)}).$$

Hence,

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{1-e^{-2(s+1)}}{(s+1)(1-e^{-2s})} \blacksquare$$