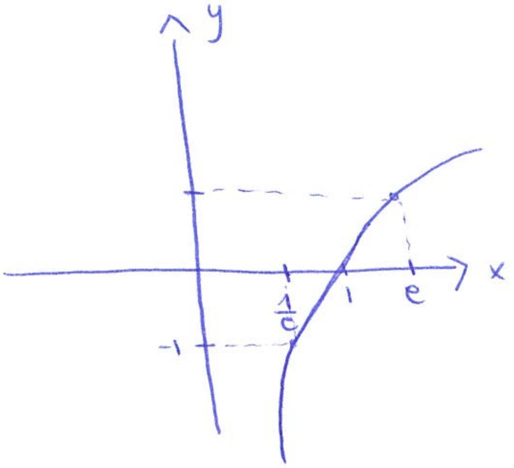
Seminar 1



II1. i) $f(x) = \arcsin(\ln x) + 2\sqrt{1-\ln^2 x}$

$\ln : (0,\infty) \to \mathbb{R} \Rightarrow x > 0$

$\arcsin : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \ln x \in [-1,1]$

$x \in \left[\frac{1}{e}, e\right]$

$1 - \ln^2 x \geq 0 \Rightarrow \ln x \in [-1,1]$

din $\cap \Rightarrow D = \left[\frac{1}{e}, e\right]$

ii) $f(x,y) = \sqrt{(x^2+y^2-1)(4-x^2-y^2)}$

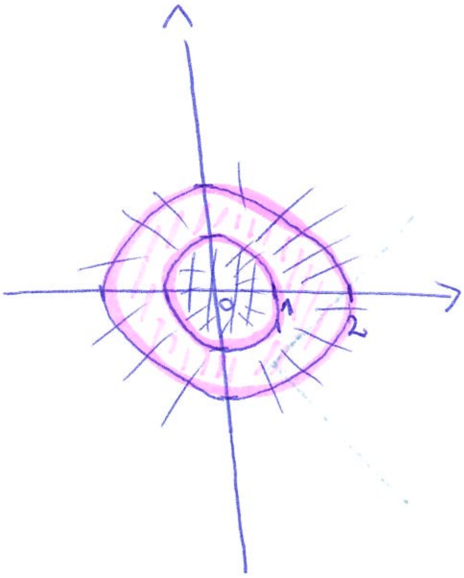
$C(\Omega(a,b),R) : (x-a)^2+(y-b)^2 = R^2$

condiție: $(x^2+y^2-1)(4-x^2-y^2) \geq 0$

caz I : $x^2+y^2-1 \geq 0 \quad \wedge \quad 4-x^2-y^2 \geq 0$

caz II : $x^2+y^2-1 \leq 0 \quad \wedge \quad 4-x^2-y^2 \leq 0$

caz I $\begin{cases} x^2+y^2 \geq 1 \\ x^2+y^2 \leq 4 \end{cases}$



considerăm cercurile:

$x^2+y^2 = 1$ și $x^2+y^2 = 2^2$

$(x-0)^2+(y-0)^2 = 1$

$C_1(\Omega_1(0,0),1)$, $C_2(\Omega_2(0,0),2)$

caz II $\left.\begin{matrix} x^2+y^2 \leq 1 \\ x^2+y^2 \geq 4 \end{matrix}\right\} \Rightarrow \emptyset$

$D = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2+y^2 \leq 4\}$

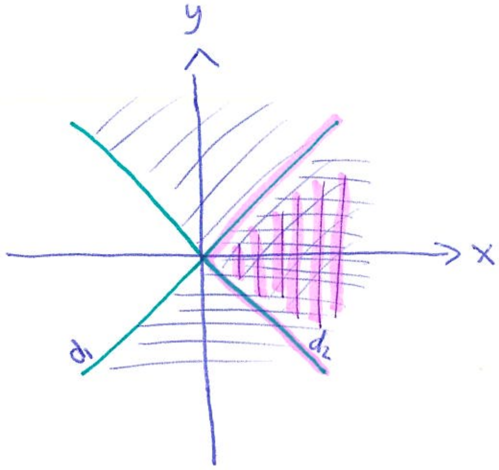
iv) $f(x,y) = \sqrt{x^2 - y^2}$

$x^2 - y^2 \geq 0 \iff x^2 \geq y^2$

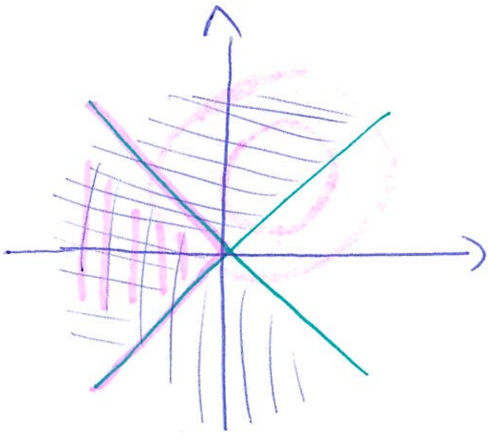
$\iff (x-y)(x+y) \geq 0 \implies \begin{cases} x - y \geq 0 \\ \wedge \\ x + y \geq 0 \end{cases} \vee \begin{cases} x - y \leq 0 \\ \wedge \\ x + y \leq 0 \end{cases}$

caz I : $\begin{cases} x - y \geq 0 \\ \wedge \\ x + y \geq 0 \end{cases} \iff \begin{cases} x \geq y \\ \wedge \\ x \geq -y \end{cases}$ fie dr. $d_1 : x = y$

$d_2 : x = -y$



caz II : $\begin{cases} x - y \leq 0 \\ \wedge \\ x + y \leq 0 \end{cases} \iff \begin{cases} x = y \\ x \leq -y \end{cases}$



$D = \{ (x,y) \in \mathbb{R}^2 \mid x \geq y \wedge x \geq y \wedge x \geq 0 \}$

$\cup \{ (x,y) \in \mathbb{R}^2 \mid x \leq y \wedge x \leq -y \wedge x \leq 0 \}$

II.2.

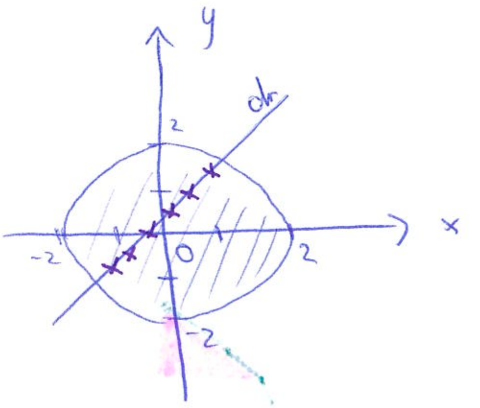
i) $A=\{(x,y) ; x^2+y^2<4 , y \neq x+1\}$

$x^2+y^2<4$

considerăm cercul:

$C(\Omega(0,0), 2)$, $x^2+y^2=2^2$

şi dr. $y=x+1$

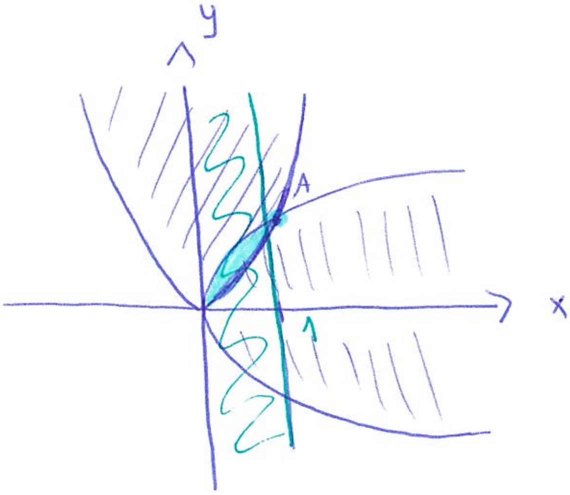


vi) $A=\{(x,y); 0\le x<1 , x^2\le y\le \sqrt{x}\}$;

considerăm dreptele $d_1: x=0$, $d_2: x=1$

şi parabolele $\Gamma_1: x^2=y$; $\Gamma_2: y=\sqrt{x}$

$y^2=x$



$A(x,y)\in\Gamma_1$

$A(a,b)\in\Gamma_1$:

$A\in d_2 \Rightarrow A(1,b)$

$A\in\Gamma_1 \Rightarrow 1=b=y \Rightarrow A(1,1)$

$A\in\Gamma_2 \Leftrightarrow 1^2=1$ (A) $\Rightarrow A\in\Gamma_2$

iii) $A=\{(x,y); x^2+y^2\le 4 , x^2+\frac{1}{4}y^2\ge 1 , x\ge 0\}$;

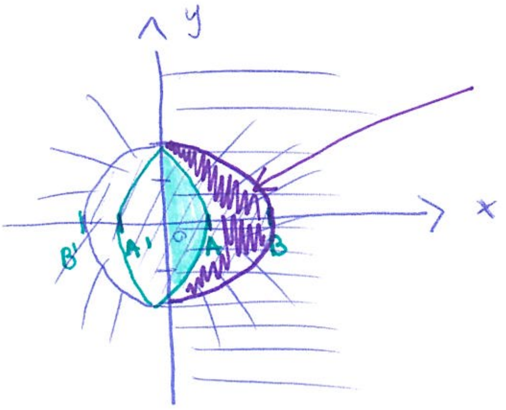
Considerăm $\mathcal{C}(\Omega(0,0), R)$

$x^2+y^2=2^2$

considerăm elipsa $\frac{x^2}{1}+\frac{y^2}{4}=1$

considerăm $d: x=0$

$y=0 \Rightarrow x^2=1 \Rightarrow x=\pm1$

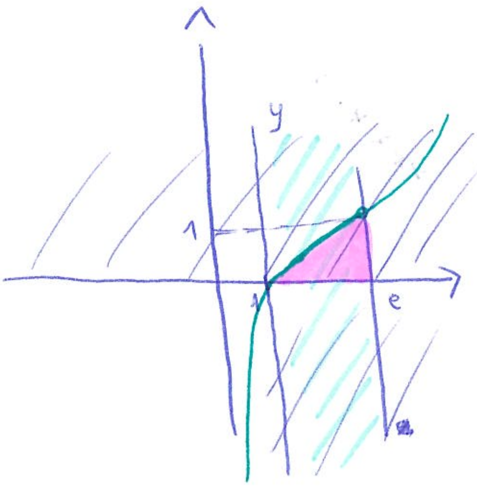


$x^2+\frac{y^2}{4}=1$

$x=0 \Rightarrow y^2=4 \Rightarrow y=\pm2$

$A(0,2), B(0,-2)$

v) $A = \{ (x,y) ; 1 \le x \le e , 0 \le y \le \ln x \}$



considerăm $d_1 : x = 1$

$d_2 : x = e$

$d_3 : y = 0$

Seminar 2

II.2.

IV) $A=\{(x,y);\ 1\le xy\le 3\ ,\ 1\le \frac{y}{x}\le 4, x>0\}$

$1\le xy\le 3$

$1=xy \Rightarrow y=\frac{1}{x}$

$3=xy \Rightarrow y=\frac{3}{x}$

$1\le \frac{y}{x}\le 4$

$y=x$

$y=4x$

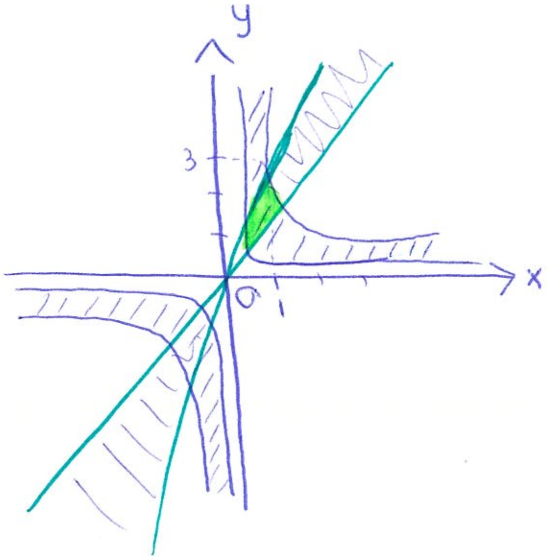
fie $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}\ ,\ f(x)=\frac{1}{x} \Rightarrow f'(x)=-\frac{1}{x^2}$

$f'(x)<0,\ \forall x\in\mathbb{R}\setminus\{0\}$

$\lim_{x\searrow 0} f(x)=\frac{1}{+0}=+\infty$

$\lim_{x\nearrow} f(x)=\frac{1}{-0}=-\infty$

$\lim_{x\to\pm\infty} f(x)=0$, $y=0$ asimptotă oriz.



IX) $A=\{(x,y)\in\mathbb{R}^2 \mid 1\le x\le 2, 2-x\le y\le \sqrt{2x-x^2}\}$

fie $y=2-x$

$y=\sqrt{2x-x^2}$

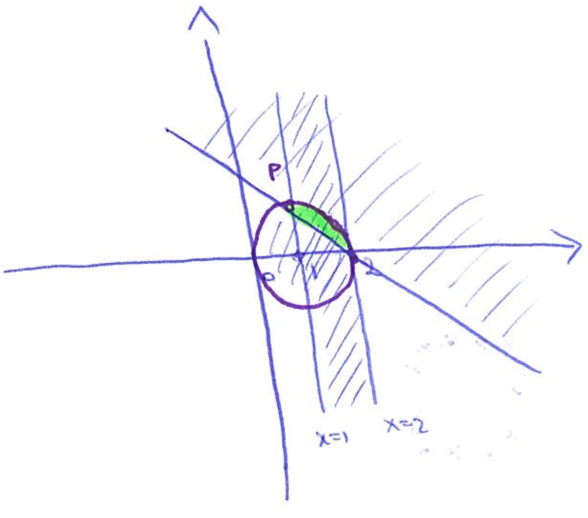
$2x-x^2\ge 0 \Rightarrow$

$\Rightarrow x(2-x)\ge 0$

caz I $\begin{cases} x\ge 0 \\ 2-x\ge 0 \end{cases}$

$\} \Rightarrow x\in[1,2]$

caz II $\begin{cases} x\le 0 \\ 2-x\le 0 \end{cases}$



$y=\sqrt{2x-x^2}$

$\Leftrightarrow y^2=2x-x^2 \Leftrightarrow y^2+x^2-2x=0 \Leftrightarrow y^2+x^2-2x+1-1=0$

$\Leftrightarrow y^2+(x-1)^2=1^2$

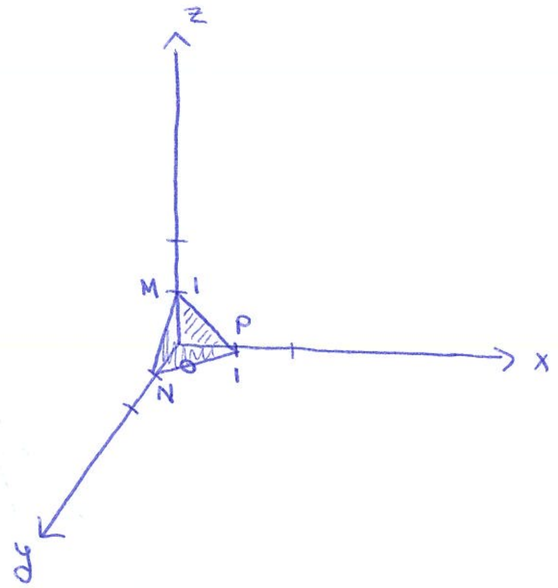
$\Omega(1,0), R=1$, $C(\Omega,1)$

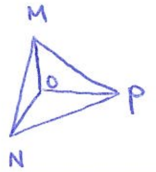
$\{P\}=(x=1)\cap(y=2-x)$, dar $P(a,b) \Rightarrow a,b$ verif. cele 2 ec. $\Rightarrow \begin{cases} a=1 \\ b=2-a=1 \end{cases} \Rightarrow P(1,1).$

vrem să verificăm că $P \in C(\Omega, 1)$:

$1^2 + (1-1)^2 = 1^2$ (A) $\Rightarrow P \in C(\Omega, 1)$.

xv) $A = \{ (x, y, z) ;\ x+y+z < 1 ,\ x > 0 ,\ y > 0 ,\ z \geq 0 \} ;$



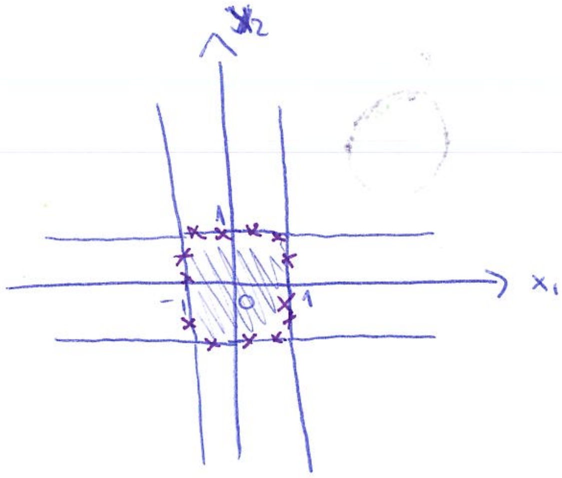


$x + y + z = 1$.

$x = 0 \quad y = 0 \Rightarrow z = 1$
$x = 0 \quad z = 0 \Rightarrow y' = 1$
$y = 0 \quad z = 0 \Rightarrow x = 1$

- tetraedrul MNOP
  fără fața MON
  iar $y = 0$ fără fața MOP

xii) $A = \{ (x_1, x_2) ;\ \max\{ |x_1| , |x_2| \} < 1 \} = \int_{d_2} (0,0,1)$



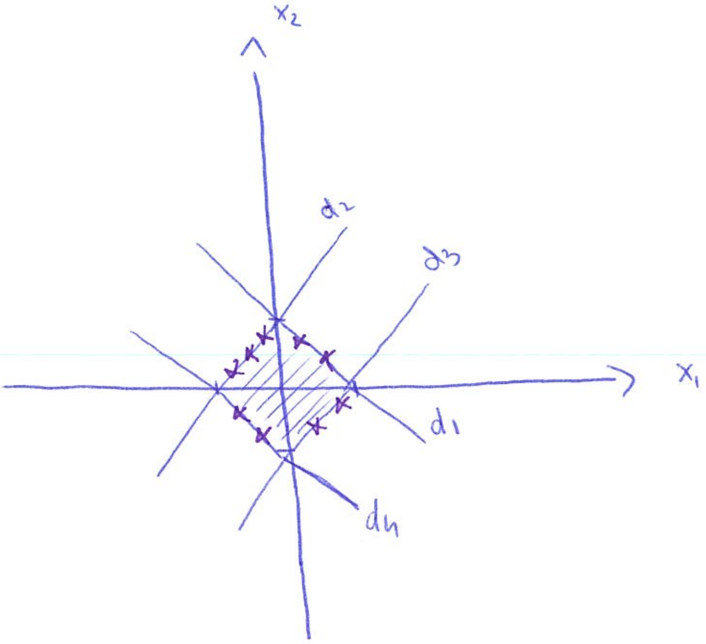
$\max\{ |x_1| , |x_2| \} = 1$

$$\Rightarrow \begin{cases} |x_1| = 1 \\ \text{sau} \\ |x_2| = 1 \end{cases} \Rightarrow \begin{cases} -1 \leq x_1 \leq 1 \\ \text{sau} \\ -1 \leq x_2 \leq 1 \end{cases} \begin{cases} \begin{cases} x_1 = 1 \\ \text{sau} \\ x_1 = -1 \end{cases} \\ \text{sau} \\ \begin{cases} x_2 = 1 \\ \text{sau} \\ x_2 = -1 \end{cases} \end{cases}$$

↓ facem reuniune

xiii) $A = \{(x_1, x_2) ; |x_1| + |x_2| < 1\}$

$|x_1| + |x_2| = 1 \Rightarrow |x_2| = 1 - x_1 \Rightarrow \begin{cases} x_2 = 1 - |x_1| \\ \text{sau} \\ x_2 = |x_1| - 1 \end{cases} \Leftrightarrow \begin{cases} \overbrace{x_2 = 1 - x_1}^{d_1} \text{ sau } \overbrace{x_2 = 1 + x_1}^{d_2} \\ \text{sau} \\ \underbrace{x_2 = x_1 - 1}_{d_3} \text{ sau } \underbrace{x_2 = -x_1 - 1}_{d_4} \end{cases}$



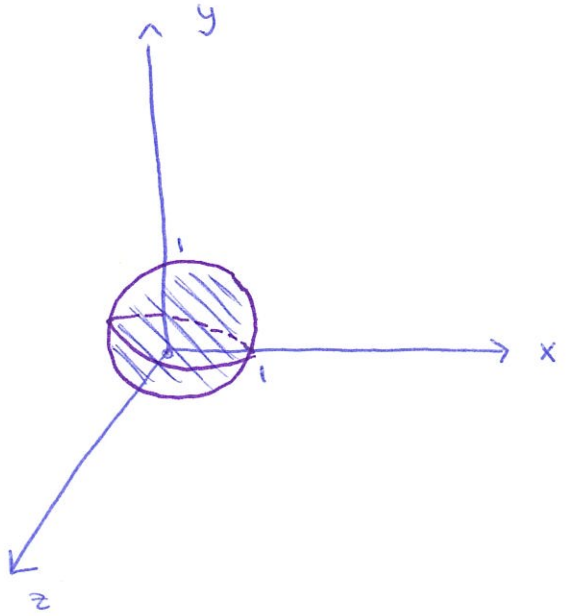
xi) $A = \{(x, y, z) ; x^2 + y^2 + z^2 \le 1\}$

$S(\Omega, R)$

$\Omega(x_0, y_0, z_0)$

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$

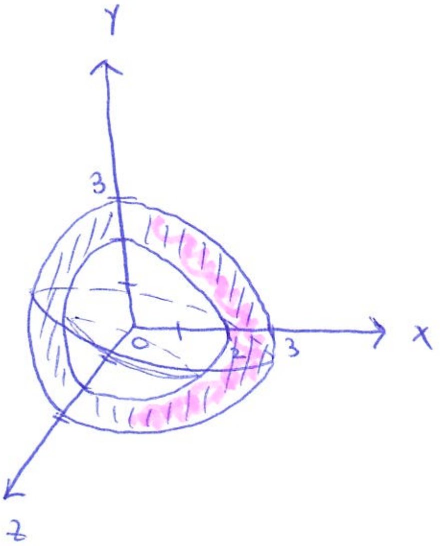
$A_1 = \{(x, y, z) \mid 4 \le x^2 + y^2 + z^2 \le 9\}$ $x \geq 0$



ex. $4 \le x^2 + y^2 + z^2 \le 9$

$x^2 + y^2 + z^2 = 2^2$

$x^2 + y^2 + z^2 = 3^2$



I.5. (X, d) sp. metric

Arătați că: $d_1 : X \times X \to \mathbb{R}_+$

$$d_1(x,y) = \ln(1 + d(x,y)) \quad , \forall x, y \in X$$

este o metrică pe X.

1) $d(x,y) \geq 0 \quad , \forall x, y \in X$

~~$\ln(1+\alpha) > 0 \Leftrightarrow 1+\alpha$~~

$d_1(x,y) = \ln(1 + d(x,y)) \geq \ln 1 = 0$

deoarece $\left.\begin{matrix} 1 + d(x,y) \\ (\ln x) \uparrow \end{matrix}\right\} \Rightarrow d_1(x,y) \geq 0$

$d_1(x,y) = 0 \Leftrightarrow \ln(1 + d(x,y)) = 0 \Rightarrow$

$1 + d(x,y) = e^0 \Leftrightarrow d(x,y) = 0 \Rightarrow x = y.$

2) simetria

$d_1(x,y) = \ln(1 + d(x,y)) = \ln(1 + d(y,x)) = d_1(y,x)$

3) ineg. Δ:

$d_1(x,y) = \ln(1 + d(x,y)) \leq \ln$

$d_1(x,y) \leq d_1(x,z) + d_1(z,y) \quad , \forall x, y, z \in X$

$\ln(1 + d(x,y)) \leq \ln(1 + d(x,z)) + \ln(1 + d(z,y))$

știm că: $d(x,y) \leq d(x,z) + d(z,y) \quad |+1$

$\Leftrightarrow 1 + d(x,y) \leq 1 + d(x,z) + d(z,y) \cancel{\leq 1 + d(x,z) + 1 + d(z,y)}$

$\ln(1 + d(x,y)) \leq \ln(1 + d(x,z) + d(z,y)) \quad (***)$

$$1 + d(x,z) + d(z,y) \overset{(*)}{\leq} (1 + d(x,y))(1 + d(z,y)) = 1 + d(z,y) + d(x,z) + d(x,z) \cdot d(z,y) \quad (A)$$

Aplicăm logaritmul în (*)

$$\ln(1 + d(x,z) + d(z,y)) \leq \ln(1 + d(x,z))(1 + d(z,y))$$

$$= \ln(1 + (x,z)) + \ln(1 + (z,y))$$

din $(**)$ și $(***) \to d_1(x,y) \leq d_1(x,z) + d_1(z,y)$

Seminar 3

proprietățile normei

1) poz def $\|x\|\geq 0$, $\forall x\in\mathbb{R}$

$\|x\|=0 \Leftrightarrow x=0$

2) omogenitate

$\|\lambda x\| = |\lambda|\,\|x\|$ $\forall \lambda\in\mathbb{R}, \forall x\in\mathbb{R}$.

3) ineg Δ:

$\|x+y\|\leq\|x\|+\|y\|$, $\forall x,y\in\mathbb{R}$.

① Determinați toate normele de pe $\mathbb{R}$:

$\|\cdot\|$ este o normă pe $\mathbb{R}$ de forma $\|x\| = c\|x\|$, $c>0$

$\|\ \|:\mathbb{R}\to\mathbb{R}_+$

în relația 2) alegem $\lambda=x$ și $x=1$ $\Rightarrow$

$$\Rightarrow \|x\cdot 1\| = |x|\cdot\|1\| \Rightarrow \|x\| = \underset{=c>0}{\boxed{\|1\|}}\cdot|x|,\ \forall x\in\mathbb{R}$$

$$= c|x|$$

② $d:\mathbb{R}\times\mathbb{R}\to\mathbb{R}_+$

$$d(x,y)=\frac{|x-y|}{1+|x-y|}$$

este o dist. care nu provine dintr-o $\underset{\#}{\underline{\text{normă}}}$

1. Arătăm că este o metrică

a) pozitiv definită: $d(x,y)\geq 0$, $\forall x,y\in\mathbb{R}$

$\frac{|x-y|}{1+|x-y|}\geq 0$, $|x-y|\geq 0$ (A).

$d(x,y)=0 \Rightarrow \frac{|x-y|}{1+|x-y|}=0 \Rightarrow |x-y|=0$

$\Rightarrow x=y$ (A)

b) simetria $d(x,y)=d(y,x)$, $\forall x,y\in\mathbb{R}$

$$\left.\begin{array}{l} d(x,y)=\dfrac{|x-y|}{1+|x-y|} \\ |x-y|=|y-x| \end{array}\right\} \Rightarrow d(x,y)=\frac{|y-x|}{1+|y-x|}=d(y,x)$$

c) inegalitatea triunghiulară:

① $d(x,y)\leq d(x,z)+d(z,y)$, $\forall x,y,z\in\mathbb{R}$.

$$\frac{|x-y|}{1+|x-y|}\leq\frac{|x-z|}{1+|x-z|}+\frac{|z-y|}{1+|z-y|} \Leftrightarrow (1)$$

$|x-y|\leq|x-z+z-y| \Rightarrow$

$|x-y|\leq|x-z|+|z-y|$ (2)

fie $f : \mathbb{R}_+ \to \mathbb{R}_+$

$$f(x) = \frac{x}{1+x}$$

$$f'(x) = \frac{x'(1+x) - x(1+x)'}{(1+x)^2} = \frac{1+x-x}{(1+x)^2} = \frac{1}{(1+x)^2} > 0 \Rightarrow f \text{ crescătoare}$$

$$\forall t_1 \le t_2 \Rightarrow f(t_1) \le f(t_2) \quad (3)$$

în relația (3) luăm $t_1 = |x-y|$ și $t_2 = |x-z| + |z-y|$

$$\frac{|x-y|}{1+|x-y|} \le \frac{|x-z|+|z-y|}{1+|x-z|+|z-y|}$$

$$\frac{|x-y|}{1+|x-y|} \le \frac{|x-z|}{1+|x-z|+|z-y|} + \frac{|z-y|}{1+|x-z|+|z-y|} \le \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|}$$

(A)

O metrică care provine dintr-o normă are proprietățile:

1) omogenitatea $d(\lambda x, \lambda y) = |\lambda| d(x,y)$, $\forall x, y \in X$, $\forall \lambda \in \mathbb{R}$.

2) invarianța la translație $d(x+z, y+z) = d(x,y)$, $\forall x, y, z \in X$

$$d(\lambda x, \lambda y) \overset{def}{=} \frac{|\lambda x - \lambda y|}{1+|\lambda x - \lambda y|} = \frac{|\lambda||x-y|}{1+|\lambda||x-y|} \neq \underset{\underset{|\lambda|\frac{|x-y|}{1+|x-y|}}{\parallel}}{|\lambda| d(x,y)} \text{ pentru } \begin{matrix} \lambda = 2 \\ x = 3 \\ y = 1 \end{matrix}$$

$$\left.\begin{matrix} \frac{2\cdot 1}{1+2\cdot 1} = \frac{2}{3} \neq 1 \\ 2\cdot\frac{1}{1+1} = \frac{2}{2} = 1 \end{matrix}\right\} \Rightarrow \text{nu respectă omogenitatea.}$$

③ i) $d_1 : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}_+$, $d_1(x,y) = \max\limits_{i=\overline{1,k}} |x_i - y_i|$ este o distanță pe $\mathbb{R}^k$.

D_1) $|x_i - y_i| \ge 0 \Rightarrow \max\limits_{i=\overline{1,k}} |x_i - y_i| \ge 0$, $\forall x, y \in \mathbb{R}^3$

$$d_1(x,y) = 0 \Rightarrow \max_{i=\overline{1,k}} |x_i - y_i| = 0 \Rightarrow$$

$$0 \le |x_j - y_j| \le \max_{i=\overline{1,k}} |x_i - y_i| = 0 \Rightarrow |x_j - y_j| = 0 \Rightarrow x_j = y_j, \forall j = \overline{1,k} \Rightarrow x = y.$$

D2) $d(x,y) = d(y,x)$

$|x_i - y_i| = |y_i - x_i| \Leftrightarrow \max\limits_{i=\overline{1,k}} |x_i - y_i| = \max\limits_{i=\overline{1,k}} |y_i - x_i| \ , \forall\ x, y \in \mathbb{R}^K$

$x = (x_1, x_2, \dots, x_K)$

$y = (y_1, y_2, \dots, y_K).$

vrem să arătăm că $d(x,y) \leq d(x,z) + d(z,y)$

D3) $\Leftrightarrow \max\limits_{i=\overline{1,k}} |x_i - y_i| \leq \max\limits_{i=\overline{1,k}} |x_i - z_i| + \max\limits_{i=\overline{1,k}} |z_i - y_i|$

$|x_i - y_i| = |x_i - z_i + z_i - y_i|$

$\Rightarrow |x_i - y_i| \leq |x_i - z_i| + |z_i - y_i| \leq d_1(x,z) + d_1(z,y) \ , \forall\ i = \overline{1,k}$

$|x_i - z_i| \leq \max\limits_{j=\overline{1,k}} |x_j - z_j| = d_1(x,z)$

$|z_i - y_i| \leq \max\limits_{j=\overline{1,k}} |z_j - y_j| = d_1(z,y)$

$\max\limits_{i=\overline{1,k}} |x_i - y_i| \leq d_1(x,z) + d_1(z,y)$

$\Leftrightarrow d_1(x,y) \leq d_1(x,y) + d_1(z,y)$

din D1), D2), D3) $\Rightarrow d_1$ - dist. pe $\mathbb{R}^K$.

④ (x_i, d_i), $i = \overline{1,n}$ $\quad X = X_1 \times X_2 \times \dots \times X_n$

$d(x,y) = \sqrt{\sum\limits_{i=1}^{n} d_i^2(x_i,y_i)} \ , \forall\ x=(x_1,\dots x_n),\ y = (y_1,\dots,y_n) \in X.$

D1) $d_i(x_i,y_i)$ - metrică , $\forall i \in \overline{1,n} \Rightarrow d_i^2(x_i,y_i) \geq 0 \Rightarrow d(x,y) \geq 0$

pp. că $d(x,y) = 0$

$\sqrt{\sum\limits_{i=1}^{n} d_i^2(x_i,y_i)} = 0 \ \uparrow^2 \Rightarrow \sum\limits_{i=1}^{n} d_i^2(x_i,y_i) = 0$

$d_i(x_i,y_i) = 0, \forall i = \overline{1,n}$

$\Big\} \Rightarrow d_1^2(x_i,y_i) = 0$

$d_i(x_i,y_i) = 0$

$\Rightarrow x_i = y_i, \forall i = \overline{1,n} \Rightarrow x = y.$

D2) $d(x,y) = d(y,x)\ \forall\ x,y \in X$

$d_1(x_i,y_i) = d_1(y_i,x_i) \Leftrightarrow d_i^2(x_i,y_i) = d_i^2(y_i,x_i)$

$\Leftrightarrow \sum\limits_{i=1}^{n} d_i^2(x_i,y_i) = \sum\limits_{i=1}^{n} d_i^2(y_i;x_i) \Leftrightarrow \sqrt{\sum\limits_{i=1}^{n} d_i^2(x_i,y_i)} = \sqrt{\sum\limits_{i=1}^{n} d_i^2(x_i,y_i)}$

$\Rightarrow d(x,y) = d(y,x)\ \forall\ x,y \in X.$

D3) $d(x,y) \le d(x,z) + d(z,y)$

$d_i(x_i,y_i) \le d_i(x_i,z_i) + d(z_i,y_i)$

not $a_i = d_i(x_i,z_i) \geq 0$ $\quad\Rightarrow\quad d_i(x_i,y_i) \le a_i + b_i \ \uparrow^2$

$b_i = d_i(z_i,y_i) \geq 0$ $\quad\quad d_i^2(x_i,y_i) \le (a_i+b_i)^2$

$$\sqrt{\sum_{i=1}^{n} d_i^2(x_i,y_i)} \le \underbrace{\sqrt{\sum_{i=1}^{n}(a_i+b_i)^2} \le \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}}_{(*)} \quad (*)$$

$$(*)^2 \Rightarrow \sum_{i=1}^{n}(a_i+b_i)^2 \le \sum_{i=1}^{n} a_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2} + \sum_{i=1}^{n} b_i^2$$

$$\Leftrightarrow \cancel{\sum_{i=1}^{n} a_i^2} + 2\sum_{i=1}^{n} a_i b_i + \cancel{\sum_{i=1}^{n} b_i^2} \le \cancel{\sum_{i=1}^{n} a_i^2} + 2\sqrt{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2} + \cancel{\sum_{i=1}^{n} b_i^2}$$

$$\sum_{i=1}^{n} a_i b_i \le \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}$$ (ineg. Cauchy-Buniakowski Schwartz)

(A) $\Rightarrow$ are loc (*).

$$\sqrt{\sum_{i=1}^{n} d_i^2(x_i,y_i)} \le \sqrt{\sum_{i=1}^{n} d_i^2(x_i,z_i)} + \sqrt{\sum_{i=1}^{n} d_i^2(z_i,y_i)}$$

$\Leftrightarrow d(x,y) \le d(x,z) + d(z,y)$

Din D1), D2) și D3) $\Rightarrow$ d - distanță.

Seminar 4

ex 1 fie $d:\mathbb{R}\times\mathbb{R} \to \mathbb{R}_+$, $d(x,y) = |\text{arctg}\, x - \text{arctg}\, y|$

i) d - metrică

1) $d(x,y) \geq 0$ (A) $d(x,y) = 0 \Rightarrow \text{arctg}\, x - \text{arctg}\, y = 0 \overset{\text{ardg inj.}}{\Leftrightarrow} \text{arctg}\, x = \text{arctg}\, y$

$\xrightarrow{\text{arctg inj.}} y = x.$

$d(x,y) = d(y,x) \Rightarrow |\text{arctg}\, x - \text{arctg}\, y| = |\text{arctg}\, y - \text{arctg}\, x|$ (A)

$d(x,y) \leq d(x,z) + d(z,y) \Rightarrow |\text{arctg}\, x - \text{arctg}\, y| =$

$= |\text{arctg}\, x - \text{arctg}\, z + \text{arctg}\, z - \text{arctg}\, y| \leq |\text{arctg}\, x - \text{arctg}\, z| + |\text{arctg}\, z - \text{arctg}\, y|$

$d(x,y)$ - metrică

$d(n+p, n) \leq \frac{1}{n}$, $\forall n, p \in \mathbb{N}$, $n \neq 0$

$|\text{arctg}(n+p) - \text{arctg}\, n| = \left|\text{arctg} \frac{n+p-n}{1+(n+p)n}\right| =$

$\left(\text{arctg}\, x - \text{arctg}\, y = \text{arctg} \frac{x-y}{1+xy}\right)$

~~$= \left|\text{arctg} \frac{p}{1+n^2+np}\right|$~~ $= \text{arctg} \frac{p}{1+n^2+np} \leq \frac{p}{1+n^2+np} \leq \frac{p}{1+np+n^2} \leq \frac{\cancel{p}}{n\cancel{p}} = \frac{1}{n}$

$\text{arctg}\, x \leq x$, $\forall x \in (0, +\infty)$

obs. că $\frac{p}{1+n^2+np} > 0$ $\forall n, p \in \mathbb{N}$.

$0 \leq d(n+p, n) \leq \frac{1}{n}$ (Crit. Cleștelui)

$\underset{n\to\infty}{\longrightarrow} 0 \underset{n\to\infty}{\longleftarrow}$

ex.2 $d: C^1[a,b] \times C^1([a,b]) \to \mathbb{R}_+$, $d(f,g) = \max\limits_{x\in[a,b]} |f(x) - g(x)| + \max\limits_{x\in[a,b]} |f'(x) + g'(x)|$

a) d - metrică 1. $d(f,g) \geq 0$, $\forall f,g \in C^1([a,b])$.

$d(f,g) = 0 \Leftrightarrow \max\limits_{x\in[a,b]} |f-g| + \max\limits_{x\in[a,b]} |f'(x) - g'(x)| = 0 \Rightarrow \begin{cases} \max\limits_{x\in[a,b]} |f-g| = 0 \\ \max |f' - g'| = 0 \end{cases}$

$\max\limits_{x\in[a,b]} |f(x) - g(x)| = 0 \Leftrightarrow |f(y) - g(y)| \leq \max\limits_{x\in[a,b]} |f(x) - g(x)| = 0$

$\Leftrightarrow |f(y) - g(y)| = 0 \Rightarrow f(y) = g(y)$, $\forall y \in [a,b]$. $\Rightarrow f = g \Rightarrow$

$d(f,g) = 0 \Leftrightarrow f = g.$

$d(f,g) = d(g,f) \Rightarrow \max\limits_{x\in[a,b]} |(f-g)(x)| + \max\limits_{x\in[a,b]} |(f'+g')(x)| =$

$= \max\limits_{x\in[a,b]} |(g-f)(x)| + \max\limits_{x\in[a,b]} |(g'-f')(x)|,$

~~$\max\limits_{x\in[a,b]} |(f-g)(x)| + \max\limits_{x\in}$~~

$d(f,g) \le d(f,h) + d(h,g)$, $\forall f,g,h \in C'([a,b])$

$\max\limits_{x\in[a,b]} |(f-g)(x)| + \max\limits_{x\in[a,b]} |(f'-g')(x)| =$

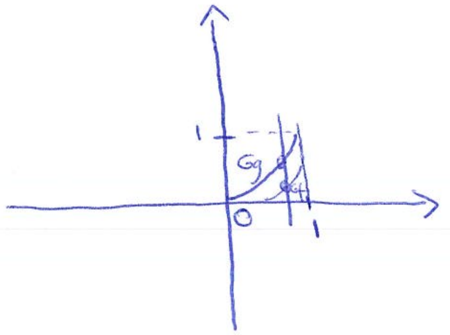
$= |f(x)-g(x)| = |f(x)-g(x)-h(x)+h(x)| \le |f(x)-h(x)| + |h(x)-g(x)| \le$

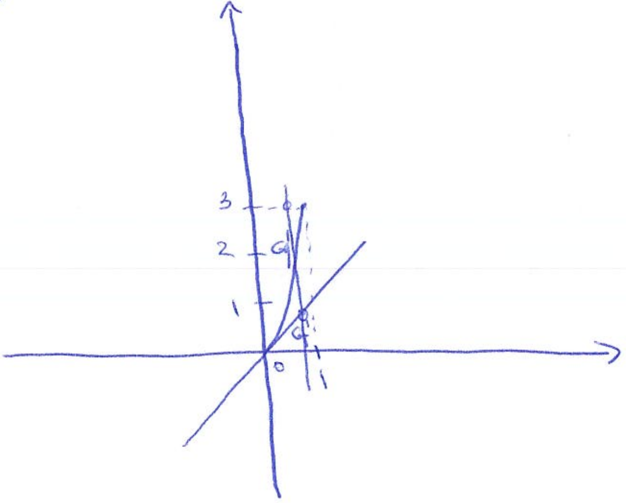
$\le \max\limits_{y\in[a,b]} |f(y)-h(y)| + \max\limits_{y\in[a,b]} |h(y)-g(y)|$, $\forall x \in [a,b] \Rightarrow$

$\Rightarrow \max\limits_{x\in[a,b]} |f(x)-g(x)| \le \max\limits_{x\in[a,b]} |(f-h)(x)| + \max\limits_{x\in[a,b]} |(h-g)(x)|$

$d(f,g) \le d(f,h) + d(h,g)$

b) $d(f,g)$, $f,g : [0,1] \to \mathbb{R}$, $f(x) = x^2$
$g(x) = x^3$





$f'(x) = 2x$
$g'(x) = 3x^2$
$\max\limits_{x\in[0,1]} |f'(x)-g'(x)| = |f'(1)-g'(1)|$

$d(f,g) = \max\limits_{x\in[0,1]} |x^2-x^3| + \max |2x-3x^2|$

$h : [0,1] \to \mathbb{R}_+$, $h(x) = |x^2-x^3| = x^2-x^3$, $\forall x \in [0,1]$.

h - derivabilă, $h'(x) = 2x-3x^2$, $h'(x) = a \Leftrightarrow 2x-3x^2 = 0$

$\Leftrightarrow x(2-3x) = 0 \Leftrightarrow \begin{cases} x = 0 \\ \vee \\ x = \frac{2}{3} \end{cases}$

x	0		$\frac{2}{3}$		1
$h'(x)$	0	$+ \; +$	0	$- \; - \; - \; -$	
$h(x)$		$\nearrow$		$\searrow$	

$x = \frac{2}{3}$ pct. de max. global pe $[0,1]$

valoarea maximă : $h(\frac{2}{3}) = \frac{4}{27}$.

~~$h(x) = |2x - 3x^2| = x|2-3x|$ pe $[0,1]$~~

~~$h(x) = 0 \Leftrightarrow \begin{cases} x = 0 \\ \vee \\ x = \frac{2}{3} \end{cases} \Rightarrow h$ - derivabilă în $[0,1] \setminus \{\frac{2}{3}\}$.~~

~~$h'(x) = |2-3x| - 3x \frac{2-3x}{|2-3x|}$~~

~~$h'(x) = 0 \Leftrightarrow |2-$~~

$h(x) = 2x - 3x^2$

$h'(x) = 2 - 6x$, $h'(x) = 0 \Leftrightarrow 2 - 6x = 0 \Leftrightarrow x = \frac{1}{3}$

$h'(\frac{2}{3}) = -2 < 0$.

x	0		$\frac{1}{3}$		1
$h'(x)$	0	$+ \; +$	0	$- \; - \; -$	
$h(x)$	0	$\nearrow$	$\frac{1}{3}$	$\searrow$	-1

$\Rightarrow \max_{x \in [0,1]} |h(x)| = 1$.

ex 3. fie $C([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f$ este cont pe $[a,b] \}$.

$\| f \| = \sqrt{\int_a^b f^2(x) dx}$

i) poz. def

$\sqrt{\int_a^b f^2(x) dx} \geq 0 \Rightarrow \| f \| \geq 0$

$\sqrt{\int_a^b f^2(x)} = 0 \quad \uparrow^2 \quad \Rightarrow \left. \begin{array}{l} \int_a^b f^2(x) = 0 \\ f^2 \geq 0 \end{array} \right\} \Rightarrow f^2(x) = 0, \forall x \in [a,b)$

$\Rightarrow f(x) = 0, \forall x \in [a,b]$

ii) omogenitate

$\|\alpha f\| = |\alpha| \cdot \|f\|$

$$\|\alpha f\| = \sqrt{\int_a^b (\alpha f(x))^2\, dx} = \sqrt{\int_a^b \alpha^2 f^2(x)\, dx}$$

$$= |\alpha| \cdot \underbrace{\sqrt{\int_a^b f^2(x)\, dx}}_{\|f\|} \Rightarrow \|\alpha f\| = |\alpha| \cdot \|f\|$$

iii) $\|f+g\| \leq \|f\| + \|g\| \Leftrightarrow$

$$\Leftrightarrow \sqrt{\int_a^b (f(x)g(x))^2\, dx} \leq \sqrt{\int_a^b f^2(x)dx} + \sqrt{\int_a^b g^2(x)\, dx} \quad \uparrow^2$$

$$\Leftrightarrow \int_a^b (f(x)g(x))^2 dx \leq \int_a^b f^2(x)dx + \int_a^b g^2(x)\, dx + 2\sqrt{\int_a^b f^2(x)\, dx}\sqrt{\int_a^b g^2(x)\, dx}$$

$$\Leftrightarrow \cancel{\int_a^b f^2(x)\, dx} + \cancel{\int_a^b g^2(x)dx} + 2\int_a^b f(x)\, g(x)\, dx \leq$$

$$\leq \cancel{\int_a^b f^2(x)\, dx} + \cancel{\int_a^b g^2(x)\, dx} + 2\sqrt{\int_a^b f^2(x)dx}\sqrt{\int_a^b g^2(x)dx} \quad \Big|:2$$

$$\Leftrightarrow \underbrace{\int_a^b f(x)g(x)\, dx \leq}_{} \sqrt{\int_a^b f^2(x)dx}\sqrt{\int_a^b g^2(x)dx}$$

(A) inegalitatea lui Schwarz.

ii) $d: C([a,b]) \times C([a,b]) \to \mathbb{R}$

$d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|$

$[a,b] = [\frac{1}{e}, e]$

$f(x) = x$

$g(x) = \ln x$

$d(x, \ln x) = \max_{x \in [\frac{1}{e}, e]} |x - \ln x|$

$h(x) = x - \ln x$

$h'(x) = 1 - \frac{1}{x} \Rightarrow x = 1$

$h'(x) = 0$

$h(\frac{1}{e}) = \frac{1}{e} - \ln(e^{-1}) = \frac{1}{e} + 1$

x	$\frac{1}{e}$		1		e
$h'(x)$		$-\ -$	0	$+\ +$	
$h(x)$	$\frac{1}{e}+1$	↘	1	↗	$e-1$

$\Rightarrow \max_{x \in [\frac{1}{e}, e]} |x - \ln x| = e - 1.$

ex 4 Arătați că:

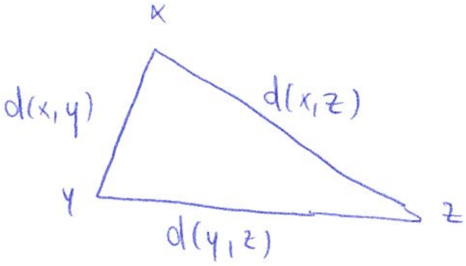
i) $|d(x,z) - d(y,z)| \leq d(x,y)$, $\forall x,y,\cancel{z} \in X$

$-d(x,y) \leq \underbrace{d(x,z) - d(y,z)} \leq d(x,y)$

$d(x,z) - d(y,z) \leq d(x,y)$

$d(x,z) \leq d(x,y) + d(y,z)$

adevărat (ineg Δ)



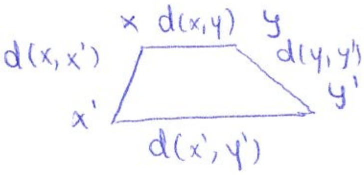
ii) $|d(x,x') - d(y,y')| \leq d(x,y) + d(x',y')$ $\Leftrightarrow$

$\Leftrightarrow$ $-d(x,y) - d(x',y') \leq d(x,x') - d(y,y') \leq d(x,y) + d(x',y')$

$d(x,x') - d(y,y') \leq d(x,y) + d(x',y')$

$d(x,x') \leq d(x,y) + d(y,y') + d(y,x')$ (ineg Δ generalizată)

(A)



$h\left(\frac{1}{e}\right) = \frac{1}{e} - \ln(e^{-1}) = \frac{1}{e} + 1$

Seminar 5

I. Fie $x, y \in \mathbb{R}^*$ oarecare. Verificați urm. echivalențe.

ⓘ $\langle x, y\rangle = 0 \Leftrightarrow \|x+y\| = \|x-y\|$

$\|\cdot\|$ - norma euclidiană

$\|x\|^2 = \langle x, x\rangle$

„$\Rightarrow$" pp. că $\langle x, y\rangle = 0$ este adevărată

vrem să dem $\|x+y\| = \|x-y\|$

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$
$$= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \quad (1)$

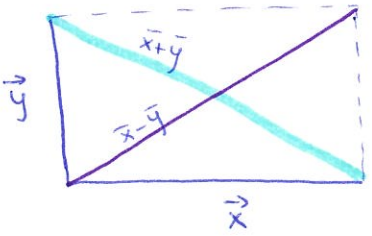
$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$

$\|x-y\|^2 = \|x\|^2 + \|y\|^2 \quad (2)$

din $(1)+(2) \Rightarrow \|x+y\|^2 = \|x-y\|^2$

$\Rightarrow \|x+y\| = \|x-y\|$

Semnificația geometrică



$\rightarrow$ lungimile diagonalelor unui dreptunghi sunt egale.

„$\Leftarrow$" pp. că $\|x+y\| = \|x-y\|$

vrem să arătăm $\langle x, y\rangle = 0$

$\|x+y\| - \|x-y\| \quad \uparrow^2$

$\|x+y\|^2 = \|x-y\|^2$

$\langle x+y, x+y\rangle = \langle x-y, x-y\rangle \Rightarrow$

$\Rightarrow \cancel{\langle x, x\rangle} + \langle x, y\rangle + \langle y, x\rangle + \cancel{\langle y, y\rangle} = \cancel{\langle x, x\rangle} + \langle -y, x\rangle + \langle x, -y\rangle + \cancel{\langle -y, -y}$

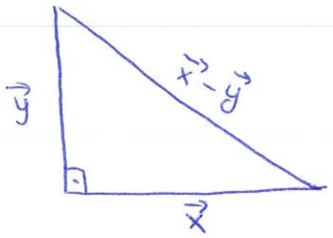
$4\langle x, y\rangle = 0 \Rightarrow \langle x, y\rangle = 0.$

(ii) $< x,y > = 0 \iff \|x+y\|^2 = \|x\|^2 + \|y\|^2$

„$\Rightarrow$" pp. $<x,y> = 0$ este adevărată, vrem să arătăm că $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

$$\|x+y\|^2 = <x+y, x+y> = <x,x> + <x,y> + <y,x> + <y,y>$$
$$= <x,x> + 2<x,y> + <y,y>$$
$$= \|x\|^2 + \|y\|^2.$$

$\Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$ (Teorema lui Pitagora)



$\|x-y\| = \|x+y\|$ (am dem. mai sus)
dacă $<x,y> = 0$

„$\Leftarrow$" pp. că $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ este adev.
vrem să arătăm că $<x,y> = 0$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$<x+y, x+y> = <x,x> + <y,y>$

$= \cancel{<x,x>} + 2<x,y> + \cancel{<y,y>} = \cancel{<x,x>} + \cancel{<y,y>}$

$\Rightarrow 2<x,y> = 0$

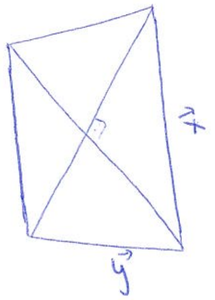
$\Rightarrow <x,y> = 0.$

(iii) $<x+y, x-y> = 0 \iff \|x\| = \|y\|$

„$\Rightarrow$" pp. că $<x+y, x-y> = 0$
vrem să arătăm că $\|x\| = \|y\|$

$<x+y, x-y> = 0 \Rightarrow <x,x> + <y,x> + <x,y> + <y,-y>$
$\Rightarrow <x,x> - <y,y> \iff <x,x> = <y,y>$
$\|x\|^2 = \|y\|^2$
$\Rightarrow \|x\| = \|y\|$

„$\Leftarrow$" pp. $\|x\| = \|y\|$ este adev.
vrem să arătăm că $<x+y, x-y> = 0$



$\|x\|^2 = \|y\|^2$

$<x,x> = <y,y>$

$<x,x> - <y,y> = 0$

$<x,x> + <x,y> - <x,y> - <y,y> = 0 \Rightarrow <x+y, x-y> = 0$

iv) $\langle x,y\rangle = 0 \iff \|x+ty\| \geq \|x\|, \ \forall t\in\mathbb{R}.$

"$\Rightarrow$" pp. $\langle x,y\rangle=0$ este adevărat

vrem să dem. că $\|x+ty\| \geq \|x\|, \ \forall t\in\mathbb{R}$

$$\|x+ty\| = \langle x+ty, x+ty\rangle = \langle x,x\rangle + \langle x,ty\rangle + \langle ty,x\rangle + \langle ty,ty\rangle$$

$$= \langle x,x\rangle + 2t\langle x,y\rangle + t^2\langle y,y\rangle$$

$$= \|x\|^2 + 2t\underbrace{\langle x,y\rangle}_{=0} + t^2\|y\|^2$$

$$\left.\begin{array}{l} t^2 \geq 0 \\ \|y\|^2 \geq 0 \end{array}\right\} \Rightarrow = \|x\|^2 + \|y\|^2 t^2 \geq \|x\|$$

$$\Rightarrow \|x+ty\|^2 \geq \|x\|^2$$

$$\Rightarrow \|x+ty\| \geq \|x\|$$

"$\Leftarrow$" pp. că $\|x+ty\| \geq \|x\|$

vrem să arătăm că $\langle x,y\rangle = 0$

$$\|x+ty\| \geq \|x\| \quad \uparrow^2$$

$$\langle x+ty, x+ty\rangle \geq \langle x,x\rangle$$

$$\cancel{\langle x,x\rangle} + \langle x,ty\rangle + \langle ty,x\rangle + \langle ty,ty\rangle \geq \cancel{\langle x,x\rangle}$$

$$t^2\langle y,y\rangle + 2t\langle x,y\rangle \geq 0$$

$$\left.\begin{array}{l} a = \langle y,y\rangle \\ b = 2\langle x,y\rangle \end{array}\right\} \Rightarrow \begin{array}{l} \Delta \leq 0 \\ \Delta = 4\langle x,y\rangle^2 = 0 \leq 0 \end{array}, \ \forall t\in\mathbb{R}$$

$$\Rightarrow 4\langle x,y\rangle^2 = 0$$

$$\langle x,y\rangle = 0$$

II. $\langle x,y\rangle = \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x-y\|^2\right)$

$$\frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x-y\|^2\right)$$

$$= \frac{1}{2}\left(\langle x,x\rangle + \langle y,y\rangle - \langle x-y, x-y\rangle\right)$$

$$= \frac{1}{2}\left[\langle x,x\rangle + \langle y,y\rangle - \left(\langle x,x\rangle + \langle x,-y\rangle + \langle -y,x\rangle + \langle -y,-y\rangle\right)\right]$$

$$= \frac{1}{2}\left(\cancel{\langle x,x\rangle} + \cancel{\langle y,y\rangle} - \cancel{\langle x,x\rangle} + \langle x,y\rangle + \langle y - x\rangle - \cancel{\langle y,y\rangle}\right)$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\langle x,y\rangle = \langle x,y\rangle$$

III.2. Arătați că mulțimile $A_\varepsilon = \{(x,y) \in \mathbb{R}^2, \max(|x|,|y| < \varepsilon)\}$ $\varepsilon > 0$ formează un sist. fundamental de vecinătate al originii în $\mathbb{R}^2$.

(i) $\mathcal{U}(x_0) \subset \mathcal{V}(x_0)$

(ii) $\forall V \in \mathcal{V}(x_0)$, $\exists V \in \mathcal{U}(x_0)$ a.î. $U \subseteq V$.

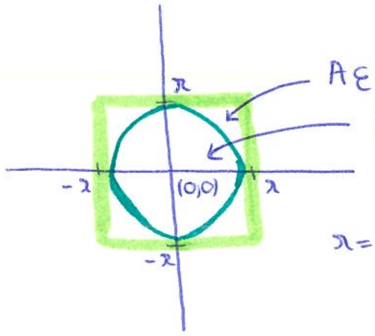
$\mathcal{U}(x_0)$ s.n. sist. fundamental de vecinătate pt x_0

obs: din i) $\forall U \in \mathcal{U}(x_0) \Rightarrow U \in \mathcal{V}(x_0) \Rightarrow U$ este vecinătatea lui x_0.

- O mulțime V este vecinătatea originii (0,0) dacă există o sferă deschisă în matricea euclidiană.

ⓘ Dem. A_ε este o vecinătate pt. originea $(0,0)$ $\forall \varepsilon > 0$

Asta înseamnă că $\exists r > 0$ a.î. $(S(0,0), r) \subset A_\varepsilon$.



$r = \varepsilon$

$(x,y) \in S((0,0), \varepsilon) \overset{def}{\Longrightarrow} \sqrt{x^2+y^2} < \varepsilon$

$x^2 \le x^2+y^2 \Rightarrow \sqrt{x^2} \le \sqrt{x^2+y^2} \Rightarrow \left.\begin{array}{l} |x| < \varepsilon \\ |y| < \varepsilon \end{array}\right\} \Rightarrow \max(|x|, |y|) < \varepsilon$

analog y

$\Rightarrow (x,y) \in A_\varepsilon$

$\Rightarrow S((0,0), \varepsilon) \subseteq A_\varepsilon \Rightarrow A_\varepsilon \in \mathcal{V}(0,0)$

ⓘⓘ fie $V \in \mathcal{V}(0,0) \Rightarrow \exists \underset{\underset{A_\varepsilon}{\cup}}{S((0,0), r)} \subset V$, $r > 0$

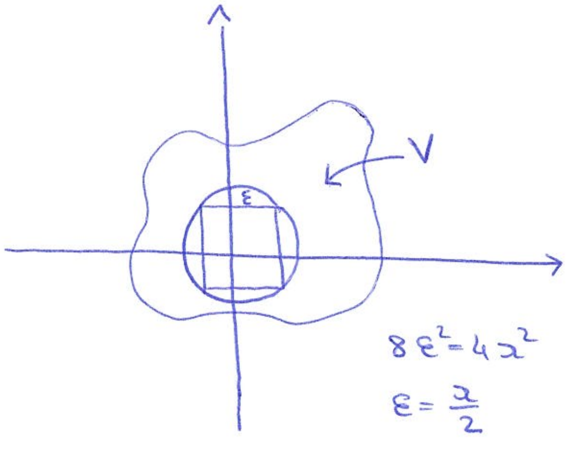
$A_{\frac{r}{2}} \subset S((0,0), r)$

$(x,y) \in A_{\frac{r}{2}} \Rightarrow \max(|x|, |y|) < \frac{r}{2}$

$\begin{cases} |x| < \frac{r}{2} \\ |y| < \frac{r}{2} \end{cases}$

vrem să arătăm că $(x,y) \in S((0,0), r)$

$\Leftrightarrow \sqrt{x^2+y^2} < r$



$8\varepsilon^2 = 4r^2$

$\varepsilon = \frac{r}{2}$

$$\begin{cases} |x| < \frac{r}{2} \quad ()^2 \\ |y| < \frac{r}{2} \quad ()^2 \end{cases} \Leftrightarrow \begin{cases} x^2 < \frac{r^2}{4} \\ y^2 < \frac{r^2}{4} \end{cases} \quad (+)$$

$$x^2+y^2 < \frac{2r^2}{4}$$

$$x^2+y^2 < \frac{r^2}{2}$$

$$\sqrt{x^2+y^2} < \frac{r}{\sqrt{2}} < r$$

$\Rightarrow (x,y) \in S((0,0),r)$

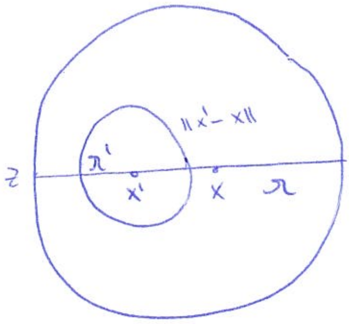
$A_\varepsilon \subset S((0,0),r) \subset V$

* Familia mulțimilor A_ε este un sistem fundamental de vecinătăți.

II.3. Fie $(X, \|\cdot\|)$ un spațiu normat. Arătați că $T(x',r') \subset T(x,r) \Leftrightarrow$ $\Leftrightarrow \|x-x'\| \leq r-r'$. Este adevărată afirmația în cazul spațiilor metrice abstracte?

* sfera închisă (sau discul) cu centru x și raza r este def ca

$$T(x,r) = \{ y \in X, \underset{\|x-y\| \leq r}{\underset{\|}{d(x,y)}} \leq r \} = \{ y \in X, \|y-x\| \leq r \}$$



$[x,x'] = \{ x + t(x'-x),\ t \in [0,1] \}$

$y = x + t_y(x'-x)$ ①

$z = t + t_z(x'-x)$ ② $t_z > 1$

1) $\Rightarrow y - x = t_y(x'-x)$

$\Rightarrow \|y-x\| = \|t_y(x'-x)\|$

$\Rightarrow r' + \|x'-x\| = t_y \cdot \|x'-x\| \quad | : \|x'-x\|$

$\Rightarrow \frac{r'}{\|x'-x\|} + 1 = t_y$

2) $z - x = t_z(x'-x)$

$\|z-x\| = \|t_z(x'-x)\|$

$r = \|t_z(x'-x)\| \Rightarrow t_z = \frac{r}{\|x'-x\|}$

$\Rightarrow t_y \leq t_z \Rightarrow \frac{r'}{\|x'-x\|} + 1 \leq \frac{r}{\|x'-x\|}$

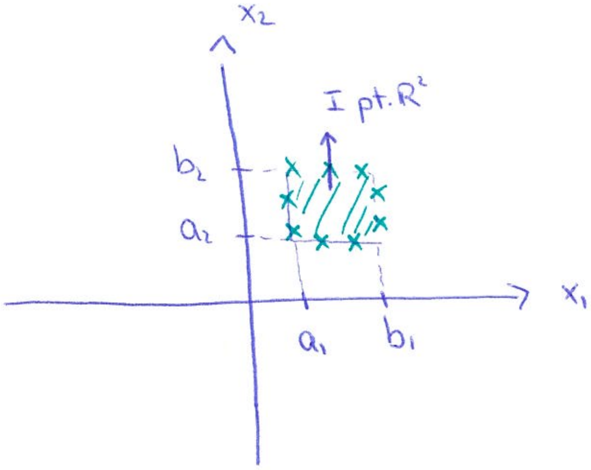
$\Rightarrow r' - r + \|x'-x\| \leq 0$

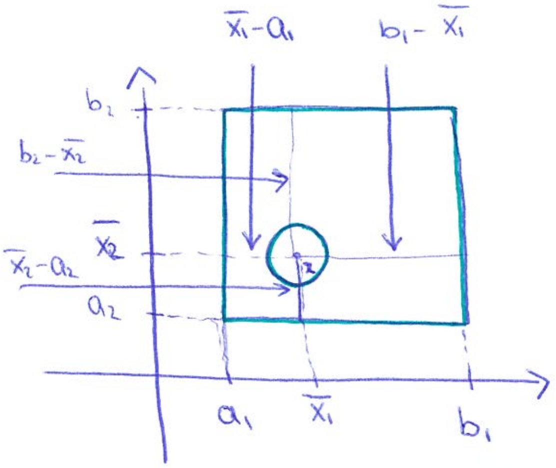
Seminar 6

ex.6 fișă Arătați că orice interval deschis din $(\mathbb{R}^k, \tilde{\mathcal{T}}_u)$ $(a_1,b_1)\times(a_2,b_2)\times\dots\times(a_k,b_k)$ este o mulțime deschisă.

$\mathcal{T}_u$ = topologia generată de metrica euclidiană

fie $x=(x_1,x_2,\dots,x_k)\in I \Leftrightarrow a_i<x_i<b_i$, $\forall i=\overline{1,k}$





alegem $r=\min\{\bar{x}_1-a_1, b_1-\bar{x}_1, \bar{x}_2-a_2, b_2-\bar{x}_2,\dots, \bar{x}_k-a_k, b_k-\bar{x}_k\}$

vrem să arătăm că $(S(\bar{x},r))\subset I$.

$S(\bar{x},r)=\{y\in\mathbb{R}^k \mid d(y,\bar{x})<r\}$

fie y arbitrar din $S(\bar{x},r)$;
vrem să arătăm că $y\in I$, adică $a_i<y_i<b_i$, $\forall i=\overline{1,k}$
unde $y=(y_1,y_2,\dots,y_k)$

Din faptul că lucrăm în $(\mathbb{R}^k,\mathcal{T}_u)$
avem: $d(y,\bar{x})=\sqrt{\sum_{i=k}^{k}(y_i-\bar{x}_i)^2}<r$

avem că: $\forall i_0=\overline{1,k}$, $|y_{i_0}-\bar{x}_{i_0}|\le\sqrt{\sum_{i=1}^{k}(y_i-\bar{x}_i)^2}<r$

$$|y_{i_0}-\bar{x}_{i_0}|<r=\min\{\bar{x}_1-a_1, b_1-\bar{x}_1, \bar{x}_2-a_2, b_2-\bar{x}_2,\dots, \underbrace{\bar{x}_{i_0}-a_{i_0}}, \underbrace{b_{i_0}-\bar{x}_{i_0}},\dots, \bar{x}_k-a_k, b_k-\bar{x}_k\}$$

$$\Rightarrow (\underset{(*)}{|y_{i_0}-\bar{x}_{i_0}|}<\bar{x}_{i_0}-a_{i_0})\wedge(\underset{(**)}{|y_{i_0}-\bar{x}_{i_0}|}<b_{i_0}-\bar{x}_{i_0})$$

din (*) => $a_{i0} - \bar{x}_{i0} < y_{i0} - \bar{x}_{i0} < \bar{x}_{i0} - a_{i0} \quad | + \bar{x}_{i0}$

<=> $a_{i0} < y_{i0} < 2\bar{x}_{i0} - a_{i0}$

=> $a_{i0} < y_{i0}, \forall i_0 \in \overline{1,k}$ (1)

din (**) $\bar{x}_{i0} - b_{i0} < y_{i0} - \bar{x}_{i0} < b_{i0} - \bar{x}_{i0} \quad | + \bar{x}_{i0}$

=> $y_{i0} < b_{i0} \quad \forall i_0 = \overline{1,k}$ (2)

din (1) + (2) => $a_{i0} < y_{i0} < b_{i0}, \forall i_0 = \overline{1,k} \Rightarrow y \in I$ } =>

cum am ales y arbitrar din $S(\bar{x}, r)$

=> $S(\bar{x}, r) \subset I, \forall \bar{x} \in I$ => I este mulţimea deschisă.

ex.4 fişă Fie (x, d) un sp. metric şi $A \subset X$.

Definim funcţia $d(x, A) = \inf_{y \in A} d(x, y)$

Arătaţi că $|d(x, A) - d(y, A)| \leq d(x, y) \quad \forall x, y \in X$.

$-d(x, y) \leq d(x, A) - d(y, A) \leq d(x, y)$

Mai întâi demonstrăm:

$-d(x, y) \leq d(x, A) - d(y, A)$ <=>

<=> $d(y, A) \leq d(x, A) + d(x, y)$ (1)

fie z arbitrar din A

Avem că : $d(y, z) \leq d(y, x) + d(x, z)$

$$\underset{d(y, A)}{\underset{\parallel}{\inf_{w \in A} d(y, w)}} \leq d(y, z) \leq d(y, x) + d(x, z)$$

$d(y, A) = \inf_{w \in A} d(y, w)$

=> $d(y, A) \leq d(y, x) + d(x, z), \forall z \in A \quad \| \inf$ după $z \in A$

$d(y,A) \le d(y,x) + \underset{\overset{\parallel}{d(x,A)}}{\inf_{z\in A} d(x,z)} \Rightarrow (1)$

fie z arbitrar din A

avem de demonstrat

$d(x,A) - d(y,A) \le d(x,y)$

$d(x,A) \le d(x,y) + d(y,A) \quad (2)$

$d(x,z) \le d(x,y) + d(y,z) \quad \parallel \inf z \in A$

$\downarrow$

$d(x,A) \le d(x,z)$

$\Rightarrow \quad d(x,A) \le d(A,y) + d(y,A) \Rightarrow (2)$

Din $(1) + (2) \Rightarrow$ ipoteza.

ex5 Arătați că dacă $A, D \subset (X, \|\cdot\|)$ și mulțimea D este deschisă, at. și mulțimea $A + D = \{a + d \mid a \in A, d \in D\}$ este deschisă.
fișa

fie $x \in A+D \Rightarrow (\exists a_x \in A) \wedge (\exists d_x \in D)$ a.î. $x = a_x + d_x$

$\left.\begin{array}{l} d_x \in D \\ D\text{ - deschisă} \end{array}\right\} \Rightarrow \exists r > 0$ a.î. $S(d_x, r) \subset D$.

vrem să arătăm că $S(x,r) \subset A+D$ (de unde va rezulta că $A+D \in \mathcal{V}(x), \forall x \in A+D$), deci

A+D este deschisă.

$\boxed{d(\alpha,\beta) = \|\alpha - \beta\|}$

fie $y \in S(x,r) \Rightarrow d(y,x) < r$

$d(y,x) = \|y - x\| < r \Leftrightarrow$

$\Leftrightarrow \|y - (a_x + d_x)\| < r \Leftrightarrow$

$\Leftrightarrow \|\underset{\overset{\parallel}{z}}{(y - a_x)} - d_x\| < r \quad \Rightarrow \|z - d_x\| < r$

$\Rightarrow z \in S(d_x, r) \subset D$

$\Rightarrow z \in D$

$\Rightarrow y - a_x \in D \Rightarrow y - a_x = d_y \in D$

$\Rightarrow y = a_x + d_y$, cu $a_x \in A$, $d_y \in D \Rightarrow y \in A + D$.

$\Rightarrow S(x, r) \subset (A+D)$

$\Rightarrow A + D$ mulțime deschisă

II. 1. ~~a)~~ / fișă

Orice sferă deschisă este mulțime închisă

$X = \mathbb{Z}$

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} \longleftarrow \text{metrica discretă}$$

Arătăm că d este o metrică :

$d(x,y) \geq 0 \quad \forall x,y \in \mathbb{Z}$ (A)

$d(x,y) = 0 \Leftrightarrow x = y$ (A)

$d(x,y) = d(y,x)$ (A)

$d(x,y) \leq d(x,z) + d(z,y)$, $\forall x, y, z \in X$.

caz I: $x = y \Rightarrow d(x,y) = 0 \leq d(x,z) + d(z,y)$ (A)

caz II: $x \neq y \Rightarrow d(x,y) = 1$ și $x \neq y. \Rightarrow$ sau $x \neq z$ sau $y \neq z$

$$\Rightarrow \begin{cases} d(x,z) = 1 \\ \text{sau} \\ d(y,z) = 1 \end{cases} \Rightarrow d(x,z) + d(y,z) \geq 1 = d(x,y)$$

din 1, 2, 3, 4 $\Rightarrow$ d metrică

fie $x \in \mathbb{Z}$

considerăm $S(x, r) = \{ y \in \mathbb{Z} \mid d(y,x) < r \}$

caz I: $0 < r \leq 1 \Rightarrow d(y,x) < r \leq 1 \Rightarrow d(y,x) = 0$

$\Rightarrow y = x \Rightarrow S(x, r) = \{x\}$
$\downarrow$
(este închisă pt. că este finită)

caz II: $r \geq 1$

$d(y,x) < r \Rightarrow d(y,x) = 0$ sau $d(y,x) = 1$.

$\Rightarrow S(x,r) = \mathbb{Z}$ - mulţime închisă

$\Rightarrow S(x,r)$ închisă

ii) Nicio sferă deschisă nu este mulţime închisă.

$X = \mathbb{R}$

$d(x,y) = |x-y|$

$S(x,r) = (x-r, x+r)$ - nu este mulţime închisă $\forall x \in \mathbb{R}, \forall r > 0$

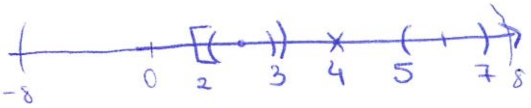
Seminar 7.

I.2. Aflați $\overset{\circ}{A}, \bar{A}, A'$ și $Fr(A)$.

Specificați pentru fiecare mulțime A dacă este simetrică, deschisă sau mărginită.

i) $A = [2,3) \cup \{4\} \cup (5,7)$

$x \in \overset{\circ}{A} \Leftrightarrow A \in \mathcal{V}(x) \Leftrightarrow \exists\, r > 0$ a.î. $S(x, r) \subset A$.



$S(x,r) = (x-r, x+r)$

pt. $x = 2$, $\forall r > 0$, $(+2-r, 2+r) \not\subset A$.

analog pt. $x = 4$

$(2,3) \cup (5,7) = \overset{\circ}{A} \Leftrightarrow$

$\Leftrightarrow \forall\, x \in (2,3) \cup (5,7)$, $x \in \overset{\circ}{A}$

$x \in (2,3)$

aleg $r = \frac{1}{2} \min(x-2, 3-x)$

$x \in \bar{A} \Leftrightarrow \forall r > 0$, $S(x,r) \cap A \neq \emptyset$

$A \subseteq \bar{A} \Rightarrow \bar{A} = [2,3] \cup \{4\} \cup [5,7]$

$x \in A' \Leftrightarrow \forall r > 0, (S(x,r) \setminus \{x\}) \cap A \neq \emptyset$

$A' = [2,3] \cup [5,7]$

$Fr\, A = \bar{A} \setminus \overset{\circ}{A} = ([2,3] \cup \{4\} \cup [5,7]) \setminus ((2,3) \cup (5,7))$

$= \{2, 3, 4, 5, 7\}$

A nu e nici deschisă, nici închisă

$\exists\, r > 0$ a.î. $A \subset S(0, r) \Rightarrow$ A mărginită

ex. $r = 8$

ii) $A = (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1)$; $(A = \mathbb{Q} \cap [0,1])$ analog

$\overset{\circ}{A} = \overset{\circ}{\overbrace{(R \setminus \mathbb{Q}) \cap [0,1)}} = \overset{\circ}{\widehat{R \setminus \mathbb{Q}}} \cap \overset{\circ}{\widehat{[0,1)}} = \emptyset \cap (0,1) = \emptyset$

$\bar{A} = \overline{(R \setminus \mathbb{Q}) \cap [0,1)} = \overline{(\mathbb{R} \setminus \mathbb{Q})} \cap \overline{[0,1)} = \mathbb{R} \cap [0,1] = [0,1]$.

$A' = \bar{A} = [0,1]$.

$Fr(A) = \bar{A} \setminus \mathring{A} = [0,1] \setminus \emptyset = [0,1]$.

- $A \neq \mathring{A} \Rightarrow A$ nu e deschisă

  $A \neq \bar{A} \Rightarrow A$ nu e închisă

$A \subset S(0,2) \Rightarrow A$ - mărginită

fie $x \in \mathbb{R}$, a.î. $\exists (x_n)_n \subset \mathbb{R} \setminus \mathbb{Q}$

şi $\exists (q_n)_n \subset \mathbb{Q}$ a.î.

$\lim\limits_{n \to \infty} x_n = x$ şi

$\lim\limits_{n \to \infty} q_n = x$

iii) $A = \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}^*}$

Fie $x = \frac{1}{n} \in A$, $r > 0$ arbitrare

$$\left.\begin{array}{l} (x-r, x+r) \subset A \\ \exists\, \alpha \in (x-r, x+r),\ \alpha \in \mathbb{R} \setminus \mathbb{Q} \end{array}\right\} \Rightarrow \alpha \notin A \Rightarrow A \notin \mathcal{V}(x), \forall x \in A \Rightarrow \mathring{A} = \emptyset.$$

$A \subset \bar{A}$.

obs $x \in \bar{A} \Leftrightarrow \exists (x_n)_{n \in \mathbb{N}} \subset A$ a.î. $\lim\limits_{n \to \infty} x_n = x$.

$$\left.\begin{array}{l} A \subset \bar{A} \\ 0 \in \bar{A} \text{ pt. că } \exists\, x_n = \frac{1}{n},\ n \in \mathbb{N}^* \text{ a.î. } x_n \in A, \forall n \end{array}\right\} \Rightarrow \bar{A} = A \cup \{0\}$$

şi $\lim\limits_{x \to \infty} x_n = 0$

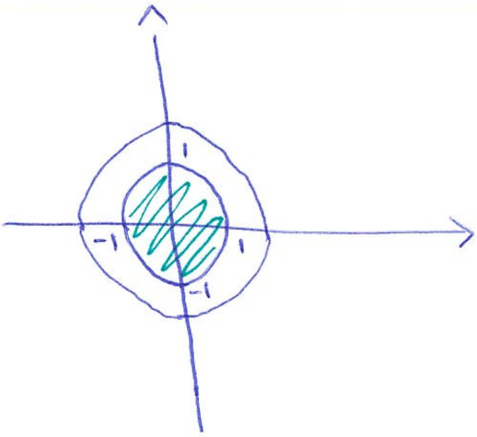
obs $x \in A' \Leftrightarrow \exists (x_n)_{x \in \mathbb{N}} \subset A$ a.î. $x_n \neq x, \forall n \in \mathbb{N}$ şi $\lim\limits_{n \to \infty} x_n = x$.

$Fr(A) = \bar{A} \setminus \mathring{A} = (A \cup \{0\}) \setminus 0 = A \cup \{0\}$

A nu e nici deschisă, nici închisă.

$A \subset S(0,2) \Rightarrow A$ mărginită

iv) $A - \{(x,y) \mid x^2 + y^2 = 1\}$



$\mathring{A} = \{(x,y) / x^2 + y^2 < 1\}$

$\bar{A} = \{(x,y) \mid x^2 + y^2 \leq 1\}$

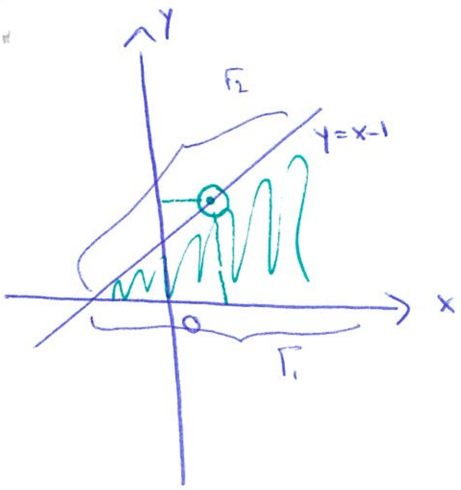
$A' = \{(x,y) \mid x^2 + y^2 \leq 1\}$

$Fr\, A = \bar{A} \setminus \mathring{A} = \{(x,y) \mid x^2 + y^2 = 1\}$

$A = \bar{A} \Rightarrow A$ închisă

$A \subset S((0,0), 2) \Rightarrow A$ - mărginită

v) $A = \{(x,y) \mid 0 < y < x+1\}$



$\mathring{A} = \{(x,y) \mid 0 < y < x+1\}$

$\bar{A} = \{(x,y) \mid 0 \leq y \leq x+1\}$

$A' = \{(x,y) \mid 0 \leq y \leq x+1\}$

$Fr\, A = \bar{A} \setminus \mathring{A} = \Gamma_1 \cup \Gamma_2$, unde $\Gamma_1 = \{(x,0) \mid x \geq -1\}$

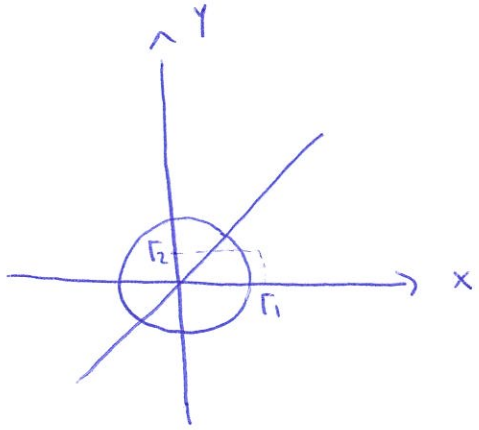
$\Gamma_2 = \{(x,y) \mid y = x+1 \wedge y \geq 0\}$.

$A = A^\circ \Rightarrow A$ deschisă

A - nemărginită pt. că $\exists\, (P_n)_{n\in\mathbb{N}}, (Q_n)_{n\in\mathbb{N}} \subset A$ a.î.

$\sqrt{n^2} = n \xrightarrow[n\to\infty]{} \infty \quad d(P_n, Q_n) \xrightarrow[n\to\infty]{} \infty$

viii) $\bar{A} = \{(x,y) \mid x^2+y^2 \leq 2, x \geq y\}$



$\mathring{A} = \{(x,y) \mid x^2+y^2 < 2, x > y\}$

$\bar{A} = \{(x,y) \mid x^2+y^2 \leq 2, x \geq y\}$

$A' = \{(x,y) \mid x^2+y^2 \leq 2, x \geq y\}$

$Fr\, A = \Gamma_1 \cup \Gamma_2$, unde

$\Gamma_1 = \{(x,y) \mid x^2+y^2 = 2 \wedge x \geq y\}$

$\Gamma_2 = \{(x,y) \mid y = x \wedge x \in [-\sqrt{2}, \sqrt{2}])$

I. 3.

$A = [0,1)$, $B = \left\{ \frac{n}{n+1} \sin \frac{n\pi}{3} + \left(1 + \frac{(-1)^n}{n}\right)^n \right\}_{n \geq 1}$

$\underline{C = A \times B \subset \mathbb{R}^2}$

$\mathring{A}, \bar{A}, \bar{A}, Fr\, A, \ldots$

$\mathring{A} = (0,1)$, $\bar{A} = [0,1]$, $A' = [0,1]$, $Fr\, A = \bar{A} \setminus A^\circ = \{0,1\}$.

$\mathring{B} = \emptyset$

R.A. $\exists\, x \in B^\circ \Rightarrow S(x,r) \subset B \Rightarrow (x-r, x+r) \subset B = \{b_n \mid n \in \mathbb{N}^*\}$

card $S(x,r) = c$

card $B = \aleph_0$

$\Rightarrow c \leq \aleph_0$ "F"

$(c > x)$

$B \subset \bar{B}$

$x \in \bar{B} \iff \exists (x_n)_{n\in\mathbb{N}} \subset B$ a.î. $\lim\limits_{n\to\infty} x_n = x$

$$\left(1+\frac{(-1)^n}{n}\right)^n$$

$$n=2k \Rightarrow \left(1+\frac{(-1)^{2k}}{2k}\right)^{2k} \xrightarrow[k\to\infty]{} e$$

$$n=2k+1 \Rightarrow \left(1+\frac{(-1)^{2k+1}}{2k+1}\right)^{2k+1} \xrightarrow[k\to\infty]{} e^{-1}$$

$$b_{6k} = \frac{6k}{6k+1}\sin\frac{6k\pi}{3} + \left(1+\frac{(-1)^{6k}}{6k}\right)^{6k}$$

$$= \frac{6k}{6k+1}\underbrace{\sin 2k\pi}_{0} + \left(1+\frac{1}{6k}\right)^{6k} = \left(1+\frac{1}{6k}\right)^{6k} \xrightarrow[k\to\infty]{} e.$$

$$b_{6k+1} = \frac{6k+1}{6k+2}\underset{\sin(2k\pi+\frac{\pi}{3})}{\underset{\parallel}{\sin\frac{(6k+1)\pi}{3}}} + \left(1+\frac{(-1)^{6k+1}}{6k+1}\right)^{6k+1} = \frac{6k+1}{6k+2}\cdot\frac{\sqrt{3}}{2} + \left(1-\frac{1}{6k+1}\right)^{6k+1}$$

$$\xrightarrow[k\to\infty]{} e^{-1}$$

$$b_{6k+2} = \frac{6k+2}{6k+3}\sin\frac{(k+2)\pi}{3} + \left(1+\frac{(-1)^{6k+2}}{6k+2}\right)^{6k+2}$$

$$= \frac{6k+2}{6k+3}\sin\left(2k\pi+\frac{2\pi}{3}\right) + \left(1+\frac{1}{6k+2}\right)^{6k+2}$$

$$= \frac{6k+2}{6k+3}\left(-\frac{\sqrt{3}}{2}\right) + \left(1+\frac{1}{6k+2}\right)^{6k+2} \xrightarrow[k\to\infty]{} -\frac{\sqrt{3}}{2}+e$$

$$b_{6k+3} = \frac{6k+3}{6k+3}\underset{0}{\underset{\parallel}{\sin\frac{(6k+3)\pi}{3}}} + \underset{e^{-1}}{\underset{\downarrow}{\left(\frac{1+(-1)^{6k+3}}{6k+3}\right)^{6k+3}}} \xrightarrow[k\to\infty]{} e^{-1}.$$

$$b_{6k+4} = \frac{6k+4}{6k+5}\underset{\sin\frac{4\pi}{3}=-\frac{\sqrt{3}}{2}}{\underset{\parallel}{\sin\frac{(6k+4)\pi}{3}}} + \left(1+\frac{1}{6k+4}\right)^{6k+4} \xrightarrow[k\to\infty]{} -\frac{\sqrt{3}}{2}+e$$

$$b_{6k+5} = \frac{6k+5}{6k+6}\underset{-\frac{\sqrt{3}}{2}}{\underset{\downarrow}{\sin\frac{(6k+5)\pi}{3}}} + \underset{e^{-1}}{\underset{\downarrow}{\left(1-\frac{1}{6k+5}\right)^{6k+5}}} \xrightarrow[k\to\infty]{} -\frac{\sqrt{3}}{2}+e^{-1}.$$

$$\Rightarrow \bar{B} = B \cup \left\{e, e^{-1}, \pm\frac{\sqrt{3}}{2}+e, \pm\frac{\sqrt{3}}{2}+e^{-1}\right\}$$

$B' = \{ e, e^{-1}, \pm\frac{\sqrt{3}}{2} + e, \pm \frac{\sqrt{3}}{2} + e^{-1} \}$

$Fr\, B = \overline{B} \setminus B^{o} = \overline{B}$

B nu e nici deschisă, nici închisă

$B \subset S((0,0), 3e) \Rightarrow B$ - mărginită

$C = A \times B \subset \mathbb{R}^2$

$\overset{o}{C} = \overset{o}{A} \times \overset{o}{B}$

$\overline{C} = \overline{A} \times \overline{B}$

$C' = A' \times B'$

$Fr\, C = \overline{C} \setminus \overset{o}{C}.$

ex. $D = \{ \underbrace{\sin\left(\frac{n\pi}{3}\right)}_{d_n}, n \geq 1 \}$

$D' = ?$

$n = 3k \Rightarrow d_{3k} = \sin\frac{3k\pi}{3} = \sin k\pi = 0 \xrightarrow[k\to\infty]{} 0$

$n = 3k+1 \Rightarrow d_{3k+1} = \sin\frac{(3k+1)\pi}{3} = (-1)^k \cdot \frac{\sqrt{3}}{2}$

$n = 3k+2 \Rightarrow d_{3k+2} = \sin\frac{(3k+2)\pi}{3} = (-1)^k\left(-\frac{\sqrt{3}}{2}\right)$

$\overline{D} = D \cup \{ 0, (-1)^k \cdot \left(\pm\frac{\sqrt{3}}{2}\right) \}$

Seminar 8

II. 4.

Fie (X,d) un sp. metric și $A \subset (X,d)$

Arătați că A este închisă $\Leftrightarrow$ $A' \subseteq A$.

„$\Rightarrow$" A - mulț. închisă $\Leftrightarrow$ $A = \bar{A}$.

$\left.\begin{array}{l} a \in A' \text{ d.n.d. } (\exists x_n)_n \subset A \text{ a.î. } x_n \neq a \text{ și } \lim\limits_{n\to\infty} x_n = a \end{array}\right\} \Rightarrow a \in A.$

A - închisă $\Rightarrow \lim\limits_{n\to\infty} x_n \in A$

„$\Leftarrow$" dacă $A' \subseteq A$

A este închisă

pp. (R.A) A nu este închisă

$\Rightarrow cA$ nu este deschisă $\Rightarrow \exists\, a \in A$ a.î. $cA \notin \mathcal{V}(a) \Rightarrow \forall r>0,\ S(a,r) \cap A \neq \emptyset$

obs. $M \in \mathcal{V}(a)$ dacă $\exists\, S(a,r) \subset M$.

$\Rightarrow S(a,r)\setminus\{a\} \cap A \neq \emptyset \Rightarrow a \in A' \subseteq A \Rightarrow a \in A$ - contradicție

II. 6.

Fie $A \subset (X,d)$, $A \neq \emptyset$. Arătați că A' este închisă.

pp. (R.A.) A' nu este închisă, CA' nu este deschisă

$\exists\, x_0 \in cA'$ a.î. $cA' \notin \mathcal{V}(x_0)$

$\Rightarrow \forall r>0,\ S(x_0,r) \cap A' \neq \emptyset$

$S(x_0,r)\setminus\{x_0\} \cap A' \neq \emptyset \Rightarrow x_0 \in (A')' \Rightarrow x_0 \in A' \Rightarrow$ contradicție

$\Rightarrow A'$ este o mulț. închisă

- exercițiu: Fie $(x_n) \subset \mathbb{R}$, $x_n = \dfrac{\ln n}{n^\alpha}$, $\alpha > 0$

$y_n = \dfrac{n^\alpha}{a^n}$, $\alpha > 0$, $a >$ [illegible]

$z_n = \dfrac{a^n}{n!}$, $a > 1$

$(w_n) \subset \mathbb{R}^3$, $w_n = (x_n, y_n, z_n)$

convergența lui w_n

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{\ln n}{n^{\alpha}} = 0$ (ln crește mai lent)

$\lim\limits_{n\to\infty} \frac{n^{\alpha}}{a^n} = 0$ (exponențiala crește mai repede)

$\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$

$\Rightarrow \lim\limits_{n\to\infty} w_n = 0+0+0=0$

$\Rightarrow (w_n)$ (C).

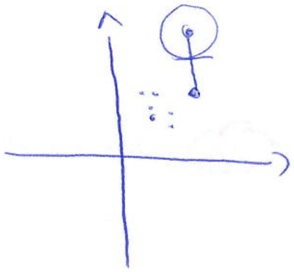
II.1 Arătați că într-un spațiu metric, o mulțime finită nu poate avea pct. de acumulare (deci mulț. este vidă).

Def: x s.n. pct. de acumulare pt. mulț. A dacă $\forall r>0$, $(S(x,r)\setminus\{x\})\cap A\neq\emptyset$.

A mulț. finită $\Rightarrow A=\{a_1, a_2, \dots, a_n\}$

caz I. $x\notin A$

fie $m=\min\{d(x,a_1), d(x,a_2), \dots, d(x,a_n)\}$ $\Rightarrow m>0$



fie $r=\frac{m}{2}$ și considerăm $S(x,r)$

vrem să arătam $S(x,r)\cap A\neq\emptyset$

fie $y\in S(x,r)$, arătăm că $y\notin A$.

$d(x,A)\leq d(x,y)+d(y,A)$

$d(y,A)\geq \underbrace{d(x,A)}_{m}-\underbrace{d(x,y)}_{<\frac{m}{2}} > \frac{m}{2}$

$m>0$

$\Rightarrow d>0 \Rightarrow y\notin A$

$\Rightarrow S(x,r)\cap A=\emptyset$

caz II $x\in A$

$B=A\setminus\{x\} \Rightarrow x\notin B$

$B=\{b_1, b_2, \dots b_n\}$

fie $m=\min d(x,b_i)$, $i=\overline{1,n}$

fie $r=\frac{m}{2}$ și considerăm $S(x,r)$

vrem să arătăm $S(x,r)\cap B=\emptyset$

fie $y\in S(x,r)$ arătăm $y\notin B$.

$d(x,B)\leq d(x,y)+d(y,B)$

$d(y,B)\geq \underbrace{d(x,B)}_{m}-\underbrace{d(x,y)}_{<\frac{m}{2}} > \frac{m}{2}$

$m>0$

$\Rightarrow d>0 \Rightarrow y\notin B$

$\Rightarrow S(x,r)\cap B=\emptyset$

II.5. Fie (x,d) un sp. metric şi $A \subset (x,d)$, $A \neq \emptyset$

i) $d(x,A) = 0 \Leftrightarrow x \in \bar{A}$

„$\Rightarrow$" $d(x,A)=0$

$x \in \bar{A} \Leftrightarrow \forall r>0,\ S(x,r) \cap A \neq \emptyset$

$d(x,A) = 0 \Rightarrow \inf_{y \in A} d(x,y) = 0$

$\forall \varepsilon > 0,\ \exists m,\ m = d(x,y_m),\ y_m \in A$ a.î.

$\varepsilon < 0 \leq d(x,y_m) < \varepsilon$

$\Rightarrow y_m \in S(x,\varepsilon)$

„$\Leftarrow$" $x \in \bar{A}$

$d(x,A)=0$

$x \in \bar{A} : \forall \varepsilon > 0,\ S(x,\varepsilon) \cap A \neq \emptyset$

$\varepsilon = \frac{1}{r}$

$S(x,\frac{1}{r}) \cap A \neq \emptyset,\ r \in \mathbb{N}^*$

$\Rightarrow r \geq 1,\ \exists y_m$ a.î. $y_m \in S(x,\frac{1}{r}) \cap A \Rightarrow \left.\begin{matrix} y_m \in A \\ y_m \in S(x,\frac{1}{m}) \end{matrix}\right\}$

$\Rightarrow y_m \in S(x,\frac{1}{n}) \Rightarrow d(y_m,x) < \frac{1}{n}$

$d(x,A) \leq d(y_m,x) < \frac{1}{n},\ \forall n \geq 1.$

$d(x,A) = \inf_{y \in A} d(x,y)$

ii) Dacă $S(A, r) = \{x \in X, d(x, A) < r\}$

$\Gamma(A, r) = \{x \in X, d(x, A) \le r\}, r > 0$

At. $\overline{A} = \bigcap_{r>0} S(A, r)$

I. $\overline{A} \subset \bigcap_{r>0} S(A, r)$

Fie $x \in \overline{A} \Rightarrow d(x, A) = 0 \Rightarrow \forall r > 0 \Rightarrow d(x, A) < r$.

$\Rightarrow x \in S(A, r), \forall r > 0 \Rightarrow x \in \bigcap_{r>0} S(A, r)$

$\bigcap_{r>0} S(A, r) \subset \overline{A}$

Fie $x \in \bigcap_{r>0} S(A, r) \Rightarrow \forall r > 0$, $0 \le d(x, A) < r$

$\searrow \quad \downarrow \quad \swarrow$

0 $\quad r \to 0$

$\Rightarrow d(x, A) = 0 \Rightarrow x \in \overline{A}$.

obs:

$f: X \to \mathbb{R}$, $f(x) = d(x, A) \Rightarrow f$ continuă pe X

$S(A, r) = \{x \in X, \varphi(x) < r\} = \{x \in X, f(x) \in (-\infty, r)\}$

$= \varphi^{-1}(-\infty, r)$ ← deschisă

Seminar 9

I. 1. Stabiliţi dacă urm. şiruri sunt convergente.

(i) $(x_n)_n \subset \mathbb{R}^3$, $x_n = \left(\frac{\sin n}{n}, n \sin \frac{1}{n}, \sqrt[n]{n+1} \right)$, $\forall n \geq 2$.

(ii) $(x_n)_n \subset \mathbb{R}^4$, $x_n = \left(\frac{1}{1^3+2} + \frac{1}{2^3+2} + \dots + \frac{1}{n^3+2}, \frac{1}{2} + \frac{-1}{3} + \dots + \frac{(-1)^{n+1}}{n}, \right.$

$\left. \left(\frac{1}{-3}\right)^n, n\left(\frac{1}{4}\right)^n \right)$, $\forall n \geq 1$.

(iii) $(x_n)_n \subset \mathbb{R}^2$, $x_n = \left(\sqrt[3]{n^2} \cdot \left(\sqrt[3]{n+1} - \sqrt[3]{n-1} \right), \frac{2^n+3^n}{3^n+4^n} \right)$, $\forall n \geq 1$.

i) $x_n^1 = \frac{\sin n}{n} = \underset{\hookrightarrow 0}{\frac{1}{n}} \cdot \sin n \hookrightarrow -1 \leq \sin n \leq 1$

$\lim\limits_{n \to \infty} x_n^1 = 0$

$$\boxed{\lim_{u \to 0} \frac{\sin u}{u} = 1.}$$

$x_n^2 = \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \frac{\sin u}{u}$

not $\frac{1}{n} = u$

$\lim\limits_{u \to 0} \frac{\sin u}{u} = 1 \Rightarrow \lim\limits_{n \to \infty} x_n^2 = 1$

$x_n^3 = \sqrt[n]{n+1}$

Criteriul rădăcinii

$x_n = \sqrt[n]{a_n}$, $a_n \geq 0$

Dacă $\exists \lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = \ell$, at. $\lim\limits_{n \to \infty} x_n = \ell$.

$\lim\limits_{n \to \infty} x_n^3 = \lim\limits_{n \to \infty} \frac{(n+1)+1}{n+1} = \lim\limits_{n \to \infty} \frac{n+2}{n+1} = 1$

x_n (c), $\lim\limits_{n \to \infty} x_n = (0, 1, 1) \in \mathbb{R}^3$.

ii) $x_n^1 = \sum_{k=1}^{n} \frac{1}{k^3+2}$ este şirul sumelor parţiale pt. seria $\sum_{n=1}^{\infty} \frac{1}{n^3+2}$

$\rightarrow$ este o serie cu termeni pozitivi

$k^3+2 > k^3 \Rightarrow \frac{1}{k^3+2} < \frac{1}{k^3}, \ \forall k \geq 1.$

Ştim că $\sum_{k=1}^{\infty} \frac{1}{k^3}$ este seria armonică generalizată cu $\alpha = 3 > 1$,

deci seria este convergentă.

Conform Crit. de comp de speţa I $\Rightarrow \sum_{k=1}^{n} \frac{1}{k^3+2}$ (C) $\Rightarrow x_n^1$ (C)

$\lim_{n\to\infty} x_n^1 = \sum_{n=1}^{\infty} \frac{1}{k^3+2}$

$x_n^2 = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$ şirul sumelor parţiale al seriei $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

$\rightarrow$ serie cu termeni alternanţi $\Rightarrow$ conform Crit. lui Leibniz este convergent

$x_n^3 = \left(-\frac{1}{3}\right)^n, \ q = -\frac{1}{3} \in (-1,1)$

$$\lim_{n\to\infty} q^n = \begin{cases} \infty, & q > 1 \\ 1, & q = 1 \\ 0, & q \in (-1,1) \\ \nexists, & q \leq -1 \end{cases}$$

$\Rightarrow \lim_{n\to\infty} x_n^3 = 0$

$x_n^4 = n\left(\frac{1}{4}\right)^n = \frac{n}{4^n} \Rightarrow \lim_{n\to\infty} x_n^4 = 0$

Deci, x_n este convergent în $\mathbb{R}^4$

iii) $x_n^1 = \sqrt[3]{n^2}\left(\sqrt[3]{n+1} - \sqrt[3]{n-1}\right) \xrightarrow[n\to\infty]{} \infty(\infty - \infty)$

$$\sqrt[3]{a} - \sqrt[3]{b} = \frac{a-b}{\sqrt[3]{a^2} + \sqrt[3]{a \cdot b} + \sqrt[3]{b^2}}$$

$$x_n^1 = \sqrt[3]{n^2} \cdot \frac{(n+1)-(n-1)}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)(n-1)} + \sqrt[3]{(n-1)^2}}$$

$$x_n^1 = \sqrt[3]{n^2} \cdot \frac{2}{\sqrt[3]{(n+1)^2} + \sqrt[3]{n^2-1} + \sqrt[3]{(n-1)^2}}$$

$$x_n^1 = \frac{2n^{2/3}}{n^{2/3}\left[\sqrt[3]{\left(1+\underset{\to 0}{\frac{1}{n}}\right)^2} + \sqrt[3]{1-\underset{\to 0}{\frac{1}{n}}} + \sqrt[3]{\left(1-\underset{\to 0}{\frac{1}{n}}\right)^2}\right]}$$

$$\lim_{n\to\infty} x_n^1 = \frac{2}{1+1+1} = \frac{2}{3}$$

$$x_n^2 = \frac{2^n+3^n}{3^n+4^n} = \frac{3^n\left(\frac{2^n}{3^n}+1\right)}{4^n\left(\frac{3^n}{4^n}+1\right)} = \left(\frac{3}{4}\right)^n \cdot \frac{\overset{\nearrow 0}{\left(\frac{2}{3}\right)^n}+1}{\left(\frac{3}{4}\right)^n+1}$$

I.2. Fie şirul $(x_n)_n \subset \mathbb{R}^3$, $x_n = \left(\left(1+\frac{1}{n}\right)^n, \sqrt[n]{n}, \frac{1}{n}\sin n \right)$, $\forall n \geq 2$.

Det. $\lambda \in \mathbb{R}$ a.î. $x = \lim\limits_{n\to\infty} x_n$ să fie la dist. minimă în $\mathbb{R}^3$

faţă de punctul $a = (1, e^{-\lambda}, \pi)$

$x = (e, 1, 0)$

$d(x,a) = \sqrt{(e-1)^2 + (1-e^{-\lambda})^2 + (0-\pi)^2}$

$f(\lambda) = \sqrt{(e-1)^2 + (1-e^{-\lambda})^2 + \pi^2}$

const. în rap cu λ

minimul se atinge când $1-e^{-\lambda} = 0 \Rightarrow \lambda = 0$

3. Studiaţi convergenţa şi în caz afirmativ, calculaţi limita şirului

$(x_n)_n \subset \mathbb{R}^4$

$$x_n = \left(\left(\cos\frac{a}{n^2}\right)^{n^2}, \sqrt{n}\left(\sqrt{n+1}-\sqrt{n-1}\right), \frac{n}{\sqrt[n]{n!}}; n\left(2^{\arcsin\frac{\pi}{3^n}}-1\right)\right), \forall n \geq 2,$$

$a \in \mathbb{R}$ fiind arbitrar, fixat.

$x_n = (x_n^1, x_n^2, x_n^3, x_n^4)$

$x_n^1 = \left(\cos\frac{a}{n^2}\right)^{n^2}$

$$\lim_{n\to\infty} x_n^1 = \lim_{n\to\infty} \left(\cos\frac{a}{n^2}\right)^{n^2} \overset{1^\infty}{=} \lim_{n\to\infty} \left(1+\cos\frac{a}{n^2}-1\right)^{n^2}$$

$$= \lim_{n\to\infty} \Bigg(\underbrace{\left(1+\left(\cos\frac{a}{n^2}-1\right)\right)^{\frac{1}{\cos\frac{a}{n}-1}}}_{e} \Bigg)^{n^2\left(\cos\frac{a}{n^2}-1\right)}$$

$$= e^{\lim\limits_{n\to\infty} n^2\left(\cos\frac{a}{n^2}-1\right)}$$

$$\lim_{n\to\infty} n^2\left(\cos\frac{a}{n^2}-1\right) \overset{\infty\cdot 0}{=} \lim_{n\to\infty} \frac{\cos\frac{a}{n^2}-1}{\frac{1}{n^2}} = \left[\frac{0}{0}\right].$$

Asociem funcția $f(x) = \frac{\cos a \cdot x - 1}{x}$

$$\lim_{x\to 0} \frac{\cos ax - 1}{x} \overset{\frac{0}{0}}{\underset{LH}{=}} \quad x \to \frac{1}{a^2}$$

$$= \lim_{x\to 0} \frac{(\cos ax - 1)'}{x'} = \lim_{x\to 0} \frac{-\sin ax\,(ax)' - 0}{1}$$

$$= \lim_{x\to 0} -a\sin ax = 0$$

$$\Rightarrow \lim_{n\to\infty} x_n^1 = e^0 = 1$$

$$\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n-1}\right) = \lim_{n\to\infty} \overset{\sqrt{n^2+n}+\sqrt{n^2-n}}{\left(\sqrt{n^2+n} - \sqrt{n^2-n}\right)} \overset{\infty-\infty}{=}$$

$$= \lim_{n\to\infty} \frac{n^2+n-n^2+n}{\sqrt{n^2+n}+\sqrt{n^2-n}} = \lim_{n\to\infty} \frac{2n}{n\left(\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}\right)} = \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{2} = 1$$

$$\lim_{n\to\infty} x_n^3 = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}}$$

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{(\cancel{n+1})^{n+1}}{\underset{n+1}{\cancel{(n+1)!}}} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = 1^\infty$$

$$\Rightarrow \lim_{n\to\infty} x_n^3 = e.$$

Limite fundamentale

$$\lim_{x_n\to 0} \frac{a^{x_n} - 1}{x_n} = \ln a, \ \forall a > 0 \quad x_n \to 0$$

$$\lim_{u_n\to 0} \frac{\arcsin u_n}{u_n} = 1$$

$$\lim_{n\to\infty} x_n^4 = \lim_{n\to\infty} n\left(2^{\overbrace{\arcsin\frac{\pi}{3n}}^{u_n}} - 1\right)$$

$$= \lim_{n\to\infty} \frac{n\left(2^{\arcsin\frac{\pi}{3n}} - 1\right)}{\arcsin\frac{\pi}{3n}} \cdot \arcsin\frac{\pi}{3n} \quad \searrow \ln 2$$

$$= \ln 2 \lim_{n\to\infty} n\arcsin\frac{\pi}{3n}$$

$$= \ln 2 \lim_{n\to\infty} \frac{n\arcsin\frac{\pi}{3n}}{\frac{\pi}{3n}} \cdot \frac{\pi}{3n} = \ln 2 \lim_{n\to\infty} \cancel{n} \cdot \frac{\pi}{3\cancel{n}} = \ln 2 \cdot \frac{\pi}{3}$$

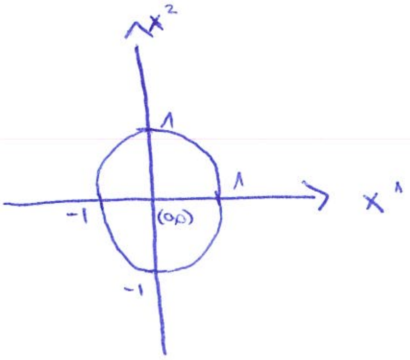
$$\Rightarrow \lim_{n\to\infty} x_n^4 = \ln 2 \cdot \frac{\pi}{3}$$

$$\Rightarrow (x_n)(C), \quad \lim_{n\to\infty} x_n = \left(1, 1, e, \ln 2 \cdot \frac{\pi}{3}\right)$$

4. Fie șirul $(x_n)_n \subset \mathbb{R}^2$, $x_n = \left(2n\sin\frac{1}{n}, \frac{\ln n}{n}\right)$, $\forall n \geq 2$.

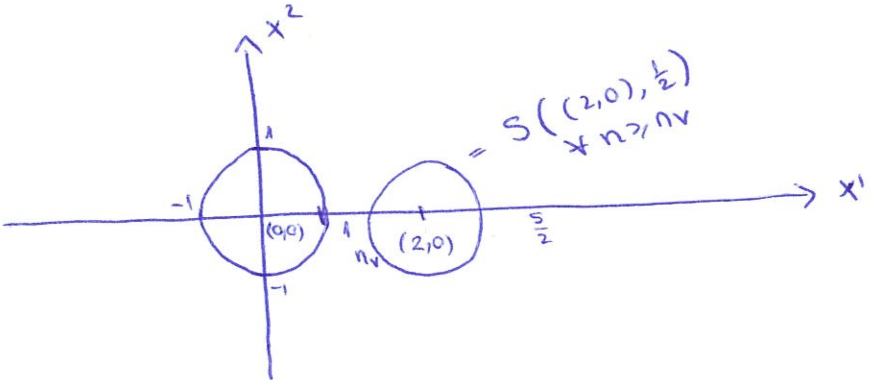
Arătați că sfera $S((0,0),1) \subset \mathbb{R}^2$ poate conține doar un număr finit de termeni ai șirului.

$x_n = (x_n^1, x_n^2)$



$$\lim_{n\to\infty} x_n^1 = \lim_{n\to\infty} 2n\sin\frac{1}{n} = \lim_{n\to\infty} 2n\underbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}_{1}\cdot\frac{1}{n}$$

$$\left.\begin{array}{l}\lim\limits_{n\to\infty} 2\cancel{n}\cdot\frac{1}{\cancel{n}} = 2 \\ \lim\limits_{n\to\infty} x_n^2 = \lim\limits_{n\to\infty} \frac{\ln n}{n} = 0\end{array}\right\} \Rightarrow \lim_{n\to\infty} x_n = (2,0).$$



Pentru $V = S((2,0),\frac{1}{2})$, $\exists n_V \in \mathbb{N}$ a.î. $x_n \in V$, $\forall n \geq n_V$

$S((2,0),\frac{1}{2}) \cap S((0,0),1) = \emptyset \Rightarrow \forall n \geq n_V$, $x_n \notin S((0,0),1)$

* Deci, în $S((0,0),1)$ se pot găsi doar primii $n_1 - 1$ termeni ai șirului
$\Rightarrow$ un nr. finit de termeni ai șirului.

II.9 Arătați că orice mulțime mărginită și infinită din $\mathbb{R}^*$ ~~are~~ cel puțin un punct de acumulare.

(arătăm A' este nevidă)

Cum A = infinită, putem alege un șir $x_n \subset A$ a.î. $x_i \neq x_j$, $\forall i,j \in \mathbb{N}$.

Cum A = mărginită $\Rightarrow (x_n)$ mărginit $\overset{\text{conform TC}}{\Longrightarrow}$ $\exists$ un subșir al lui (x_n) convergent

Notăm $(x_{n_k}) \Rightarrow \exists \lim_{k \to \infty} x_{n_k} = \ell$, $x_{n_k} \neq \ell$ $(x_i \neq x_j)$

$\ell \in A' \Leftrightarrow \exists (x_n) \subset A$ cu $x_n \neq \ell, \forall n \in \mathbb{N}$ a.î. $\lim_{n \to \infty} x_n = \ell$.

$x_{n_{k^\lambda}} = \ell$

$x_{n_k} \neq \ell \; \forall \; k \neq \lambda$

- $\Rightarrow$ definiția.

Seminar 10

5. Fie șirul $(x_n)_n \subset \mathbb{R}^3$, $x_n = \left((-1)^n \frac{n+1}{n+2} , 3^n \left(2^{\frac{1}{n}} - 1 \right) , \cos \frac{n\pi}{2} \right), \forall n \geq 1$.

Precizați dacă șirul este convergent. Studiați apoi mărginirea șirului.

Fie $a_n = (-1)^n \frac{n+1}{n+2}$

$$a_{2k} = (-1)^{2k} \cdot \frac{2k+1}{2k+2} = \frac{2k+1}{2k+2}$$

$$\lim_{k \to \infty} a_{2k} = \lim_{k \to \infty} \frac{2k+1}{2k+2} = 1$$

$$a_{2k+1} = (-1)^{2k+1} \cdot \frac{2k+2}{2k+3} = -\frac{2k+2}{2k+3} = -1$$

$\Rightarrow$ (a_n) divergent

$|a_n| = \left| (-1)^n \frac{n+1}{n+2} \right| = \left| \frac{n+1}{n+2} \right| = \frac{n+1}{n+2}$ $\Rightarrow$ $|a_n| < 1, \forall n \geq 1$

$\Rightarrow -1 < a_n < 1$

$\Rightarrow$ (a_n) mărginit

$$\boxed{\lim_{u_n \to 0} \frac{a^{u_n} - 1}{u_n} = \ln a, \forall a > 0}$$

Fie $b_n = 3^n \left(2^{\frac{1}{n}} - 1 \right) = 3^n \cdot \frac{2^{\frac{1}{n}} - 1}{\frac{1}{n}} \cdot \frac{1}{n}$

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} 3^n \underset{\to \ln 2}{\boxed{\frac{\left(2^{\frac{1}{n}} - 1\right)}{\frac{1}{n}}}} \cdot \frac{1}{n} = \lim_{n \to \infty} \frac{3^n \ln 2}{n} = +\infty$$

Deoarece $\lim\limits_{n \to \infty} b_n = +\infty \Rightarrow (b_n)$ mărginit

Fie $c_n = \cos \frac{n\pi}{2}$

$$\left. \begin{array}{l} c_{4k} = \cos \frac{4k\pi}{2} = \cos 2k\pi = \cos 0 = 1 \longrightarrow 1 \\ c_{4k+1} \Rightarrow \cos (4k+1) = \cos \left(\frac{4k\pi}{2} + \frac{\pi}{2} \right) = \cos \left(2k\pi + \frac{\pi}{2} \right) = 0 \longrightarrow 0 \end{array} \right\} \Rightarrow (c_n) \text{ divergent}$$

$c_n \in [-1, 1] \Rightarrow (c_n)$ mărginit

Concluzie (x_n) - nemărginit
- divergent

6. Fie $A, D \subset \mathbb{R}^k$. Arătați că dacă mulț. D este deschisă, atunci $D+A = D+\bar{A}$. Arătați că ipoteza asupra mulț. D este esențială.

$A \subset \bar{A} \Rightarrow D+A \subset D+\bar{A}$

arătăm că $\underline{D+\bar{A} \subseteq D+A}$

Fie $y \in D+\bar{A} \Rightarrow y = d+a$, $d \in D$ și $a \in \bar{A}$

Pp. (RA) $\exists y \in D+\bar{A}$ a.î. $y \notin D+A$.

$y = d+a$, $d \in D$ și $a \in \bar{A}$

$y \notin D+A$

$\forall \bar{d} \in D$ și $\bar{a} \in A \Rightarrow \bar{d}+\bar{a} \neq y$

$y = d+a \Rightarrow y-d = a$, $a \in \bar{A}$

$\Longrightarrow \forall V \in \mathcal{V}(a) \Rightarrow V \cap A \neq \emptyset$

$\exists (a_n)_n \subseteq A$ a.î. $\lim\limits_{n\to\infty} a_n = a$.

9. i) Fie $(x_n), (y_n) \in (X,d)$ a.î. $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$.

Arătați că $\lim\limits_{n\to\infty} d(x_n,y_n) = d(x,y)$.

Deoarece $\lim\limits_{n\to\infty} x_n = x$ și $\lim\limits_{n\to\infty} y_n = y$ avem că $\lim\limits_{n\to\infty} d(x_n, x) = 0$ și

$\lim\limits_{n\to\infty} d(y_n,y) = 0$.

Știm că $\forall a,b,a',b' \in X$ are loc:

$|d(a,b) - d(a',b')| \le d(a,a') + d(b,b')$

$\Rightarrow$ pt. $\left.\begin{array}{l} a = x_n \\ b = y_n \\ a' = x \\ b' = y \end{array}\right\} \Rightarrow 0 \le |d(x_n,y_n) - d(x,y)| \le d(x_n,x) + d(y_n,y)$

crit cleștelui

0

$\Rightarrow \lim\limits_{n\to\infty} |d(x_n,y_n) - d(x,y)| = 0$.

$\Rightarrow \lim\limits_{n\to\infty} d(x_n,y_n) = d(x,y)$.

ii) $(x_n), (y_n)$ șiruri Cauchy, at. $d(x_n,y_n)$ este tot un șir Cauchy.

$\forall \varepsilon > 0, \exists n_x$ a.î. $\forall n,m \ge n_x$, $d(x_n, x_m) < \frac{\varepsilon}{2}$

$\forall \varepsilon > 0, \exists n_y$ a.î. $\forall n,m \ge n_y$, $d(y_n, y_m) < \frac{\varepsilon}{2}$

$|d(x_n,y_n) - d(x_m,y_m)| \le d(x_n,x_m) + d(y_n,y_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\forall n,m \ge \max\{n_x, n_y\}$

7. Fie mulțimea $A = \{(x,y) : x^2+y^2 \le 9, y \ge x+1\}$. Arătați cu ajutorul caracterizării cu șiruri că mult. A este închisă. Este mulț. $B = \{(x,y);\ x^2+y^2 \le 9, y > x+1\}$ închisă?

Fie $(x_n,y_n)_n \subset A$ un șir arbitrar, $(x_n,y_n) \to (x_0,y_0)$. Ar. că $(x_0,y_0) \in A$.

$(x_n,y_n) \in A, \forall n \Rightarrow x_n^2+y_n^2 \le 9, \forall n \Rightarrow \lim\limits_{n\to\infty}(x_n^2+y_n^2) \le 9 \Rightarrow \boxed{x_0^2+y_0^2 \le 9} \Rightarrow$

$y \ge x+1 \quad \Rightarrow \lim\limits_{n\to\infty}(y_n) = y_0 \ge \boxed{\lim\limits_{n\to\infty}(x_n+1) = x_0+1}$

$\lim\limits_{n\to\infty} x_n = x_0$

$\lim\limits_{n\to\infty} y_n = y_0$

$\Rightarrow (x_0,y_0) \in A \Rightarrow A$ închisă.

$x_n = 0$, $\forall n \geq 1$ și $y_n = 1 + \frac{1}{n}$, $\forall n \geq 1$

Ar. că $\underline{(x_n, y_n) \in B}$ ($(x_n, y_n) = (0, 1+\frac{1}{n}), n \geq 1$)

$x_n^2 + y_n^2 = 0^2 + \underline{(1+\frac{1}{n})^2} = 1 + \frac{2}{n} + \frac{1}{n^2}$, $\forall n \geq 1$

$\leq 0 + 2^2 = 4 \leq 9$.

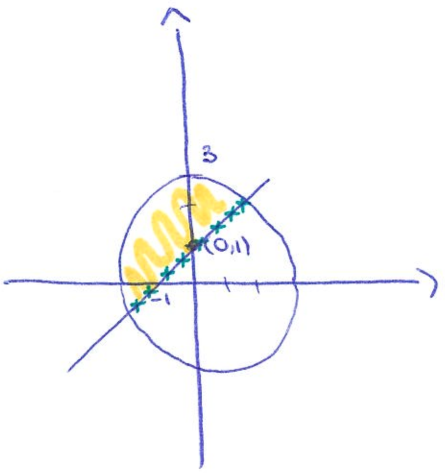
$\Rightarrow (x_n, y_n) \in B$.

$1 + \frac{1}{n} \leq 2$, $\forall n \geq 1$

$y_n = 1 + \frac{1}{n} > 0 + 1 = x_n + 1$, $\forall n \geq 1$.

$\lim\limits_{n \to \infty} (x_n, y_n) = (0, 1) \in B$?

$1 > 0 + 1 = 1$ (F)   ($y > x+1$)



8. Cercetați dacă mulț. $A = \{(x,y), (x^2+y^2)^2 = 2(x^2-y^2)\} \subset \mathbb{R}^2$ este mărginită. Este închisă?

$A \subset \mathbb{R}^2$ mărginită $\Rightarrow \exists r > 0$ a.î. $A \subseteq S((0,0), r)$

$(x^2+y^2)^2 = 2(x^2-y^2) \leq 2x^2 \leq 2(x^2+y^2)$

$y^2 \geq 0 \Rightarrow -2y^2 \leq 0$

$(x^2+y^2)^2 \leq 2(x^2+y^2) \mid : (x^2+y^2)$, $(x,y) \neq (0,0)$

$x^2 + y^2 \leq 2$

$\sqrt{x^2+y^2} \leq \sqrt{2} \Rightarrow \|(x,y) - (0,0)\| \leq \sqrt{2} \Rightarrow (x,y) \in S((0,0), \sqrt{2})$.

$\Rightarrow$ $A \in S((0,0), \sqrt{2}) \Rightarrow A$ mărginită

Fie $(x_n, y_n)_n \subset A$ arbitrar $(x_n, y_n) \xrightarrow[n\to\infty]{} (x_0, y_0)$. Ar. că $(x_0, y_0) \in A$

$(x_n, y_n) \in A \Rightarrow (x_n^2 + y_n^2)^2 = 2(x_n^2 - y_n^2)$, $\forall n$

Trecem la limită $(x_0^2 + y_0^2)^2 = 2(x_0^2 - y_0^2) \Rightarrow$ $(x_0, y_0) \in A \Rightarrow A$ închisă.

8. Arătați că $(0,1]$ privit ca subspațiu $(\mathbb{R}, d_u)$ nu este complet.

Arătăm că $\exists$ un șir Cauchy în $(0,1]$ care nu este convergent în $(0,1]$.

Fie $x_n = \frac{1}{n} \in (0,1], \forall n \geq 1$

Ar. că $\forall \varepsilon > 0$, $\exists n_\varepsilon$ a.î. $n, m \geq n_\varepsilon$, $|x_n - x_m| < \varepsilon$

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

$\forall n, m \geq \overline{n_\varepsilon}$.

$\frac{1}{n} \xrightarrow[n\to\infty]{} 0$

$\forall \varepsilon > 0, \exists \overline{n_\varepsilon} \in \mathbb{N}$ a.î.

$\forall n \geq n_\varepsilon$, $\left|\underset{\frac{1}{n}}{\underset{\shortparallel}{\frac{1}{n} - 0}}\right| < \frac{\varepsilon}{2}$

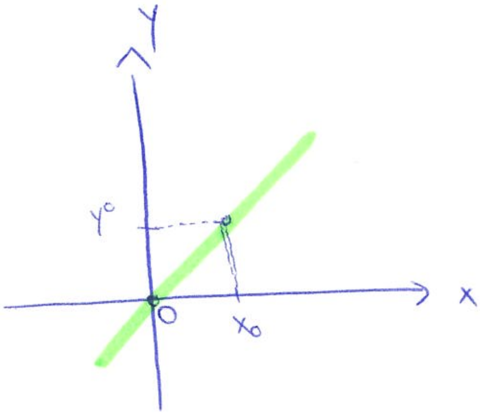
$\forall m > n_\varepsilon \Rightarrow \frac{1}{m} < \frac{\varepsilon}{2}$

$\lim\limits_{n\to\infty} x_n = 0 \notin (0,1] \Rightarrow (x_n)$ este Cauchy în $(0,1]$, dar nu este convergent în $(0,1]$.

Seminar 11

1. Cercetați existența lim. urm. și, în caz afirmativ calculați lim coresp.

1) $f: D \to \mathbb{R}$, $D = \mathbb{R}^2 \setminus \{(x,x) \mid x \in \mathbb{R}\}$ $D' = \mathbb{R}^2$



$$f(x,y) = \frac{x+y}{x-y} = \frac{2x_0}{x_0-x_0}$$

$$(x_n,y_n) = \left(\frac{1}{n} + x_0, x_0\right) \xrightarrow{n\to\infty} (x_0,x_0)$$

$$(\bar{x}_n,\bar{y}_n) = \left(-\frac{1}{n} + x_0, x_0\right) \xrightarrow{n\to\infty} (x_0,x_0)$$

$$f(x_n,y_n) = \frac{x_n+y_n}{x_n-y_n} = \frac{\frac{1}{n}+x_0+x_0}{\frac{1}{n}+x_0-x_0}$$

$$= \frac{\frac{1}{n}+2x_0}{\frac{1}{n}} = 1+2nx_0 \xrightarrow{n\to\infty} sgn(x_0)\cdot\infty$$

$$f(\bar{x}_n,\bar{y}_n) = \frac{-\frac{1}{n}+x_0+x_0}{-\frac{1}{n}+x_0-x_0} = \frac{-\frac{1}{n}+2x_0}{-\frac{1}{n}} = 1-2nx_0 \xrightarrow{n\to\infty} sgn(x_0)\cdot -\infty$$

~am obț. lim. dif $\Rightarrow \nexists \lim\limits_{(x,y)\to(x_0,x_0)} \frac{x+y}{x-y}$

studiem limita în $(0,0)$

$(x_n,y_n) = \left(\frac{1}{n}, 0\right)$

$(\bar{x}_n,\bar{y}_n) = \left(\frac{1}{n}, \frac{1}{2n}\right)$, $\forall n$

obs. că $(x_n,y_n) \xrightarrow{n\to\infty} (0,0)$

$(\bar{x}_n,\bar{y}_n) \xrightarrow{n\to\infty} (0,0)$

Dar, $f(x_n,y_n) = \frac{\frac{1}{n}+0}{\frac{1}{n}-0} = 1$

$f(\bar{x}_n,\bar{y}_n) = \frac{\frac{1}{n}+\frac{1}{2n}}{\frac{1}{n}-\frac{1}{2n}} \cdot \frac{\frac{3}{2n}}{\frac{1}{2n}} = 3$

deoarece $1 \neq 3 \Rightarrow$ $\nexists \lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x-y}$.

2) $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} =$$

$$f(r\cos\theta, r\sin\theta) = \frac{(r\cos\theta)^2(r\sin\theta)}{(r\cos\theta)^4+(r\sin\theta)^2}$$

$$= \frac{r^2\cos^2\theta \cdot r\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \frac{r^3(\cos^2\theta\sin\theta)}{r^2(r^2\cos^4\theta+\sin^2\theta)}$$

$$= \frac{r(\cos^2\theta\ \sin\theta)}{r^2\cos^4\theta+\sin^2\theta} \xrightarrow[r\to 0]{} \frac{0}{\sin^2\theta} = 0$$

$$(x_n, y_n) = \left(\frac{1}{n}, 0\right) \Rightarrow f\left(\frac{1}{n}, 0\right) = \frac{\left(\frac{1}{n}\right)^2 \cdot 0}{\left(\frac{1}{n}\right)^4 + 0^2} = 0$$

$$(x_n', y_n') = \left(\frac{1}{n}, \frac{1}{n^2}\right) \Rightarrow f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \frac{\left(\frac{1}{n}\right)^2 \frac{1}{n^2}}{\left(\frac{1}{n}\right)^4 + \left(\frac{1}{n}\right)^2} = \frac{n^4}{n^2+1} = \frac{1}{2}$$

$$\left.\begin{array}{l} \lim\limits_{n\to\infty} f\left(\frac{1}{n}, 0\right) = 0 \\ \lim\limits_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \frac{1}{2} \end{array}\right\} \Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}.$$

3) $$f(x,y) = \frac{x^2y}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \frac{0}{0}$$

$$0 \le \left|\frac{x^2y}{x^2+y^2}\right| = \underset{\downarrow \atop \text{mărg.}}{\boxed{\frac{x^2}{x^2+y^2}}}\, |y| \longrightarrow 0 \quad \text{când } (x,y)\to(0,0)$$

$$0 \le \frac{x^2}{x^2+y^2} \le 1$$

$$0 \le \left|\frac{x^2y}{x^2+y^2}\right| \le |y|$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0 \quad (\text{Crit. Cleştelui})$$

4) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$

$$f(x,y) = \frac{x^2+y^2}{x^2-y^2}$$

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta - r^2\sin\theta} = \frac{-1}{\cos^2\theta - \sin^2\theta}$$

$$(x_n, y_n) = \left(\frac{1}{n}, 0\right) \to (0,0)$$

$$(x_n', y_n') = \left(0, \frac{1}{n}\right) \to (0,0)$$

$$\left.\begin{array}{l} f\left(\frac{1}{n}, 0\right) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1 \\ \\ f\left(0, \frac{1}{n}\right) = \frac{\frac{1}{n^2}}{-\frac{1}{n^2}} = -1 \end{array}\right\} \Rightarrow \lim_{n\to\infty} f(x_n, y_n) \neq \lim_{n\to\infty} f(x_n', y_n')$$

$$1 \neq -1.$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$$

* $\lim\limits_{x\to a} f(x) = \begin{cases} f(a) - \text{dacă e cont.} \\ \lim\limits_{y\to 0} f(y+a) \end{cases}$

2. $f(x,y) = \frac{x+y}{x-y}$, $y = x - a$

$D_f = \mathbb{R}^2 \setminus \{(x,x) \mid x \in \mathbb{R}\}$

$D'_f = \mathbb{R}^2$

Calculați: $\lim\limits_{(x,y)\to(1,1)} \frac{x+y}{x-y}$

$$\lim_{(x,y)\to(1,1)} \frac{x+y}{x-y} = \frac{2}{0}$$

Fie $(x_n, y_n) = \left(1+\frac{1}{n}, 1\right) \to 1$

$(x_n', y_n') = \left(1-\frac{1}{n}, 1\right) \to 1$

$$f(x_n, y_n) = \left(\frac{1+\frac{1}{n}+1}{1+\frac{1}{n}-1}\right) = \frac{2+\frac{1}{n}}{\frac{1}{n}} \to +\infty$$

$$f(x_n', y_n') = \frac{\cancel{1}-\frac{1}{n}+\cancel{1}}{\cancel{1}-\frac{1}{n}\cancel{-1}} \xrightarrow[n\to\infty]{} -\infty$$

Not. $(\bar{x}, \bar{y}) = (x,y) - (1,1)$

$$\lim_{(\bar{x},\bar{y})\to(0,0)} \frac{\bar{x}+1+\bar{y}+1}{\bar{x}+\cancel{1}-\bar{y}-\cancel{1}} = \frac{\bar{x}+\bar{y}+2}{\bar{x}-\bar{y}}$$

3. Fie $f: \mathbb{R}^2 \setminus \{(0,0)\} \to [-1,1]$

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$$

$$\boxed{\lim_{n\to 0} \frac{\sin n}{n} = 1}$$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1 \text{ (lim fundamentală)}$$

4. $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{\sin(x^3+y^3)}{x^3+y^3}}_{\searrow 1} \cdot (x^3+y^3) \cdot \frac{1}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

$$0 \le \left|\frac{x^3+y^3}{x^2+y^2}\right| = \frac{|x^3+y^3|}{x^2+y^2} \le \frac{|x^3|+|y^3|}{x^2+y^2} = \frac{x^2|x|+|y^2||y|}{x^2+y^2} \leftarrow \text{mărginită}$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

5. Cercetați existența lim. urm. și în caz afirmativ aflați lim lui f.

1) $$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}, \quad f(x,y) = \frac{x^4+y^4}{x^2+y^2}$$

$$f(r\cos\theta, r\sin\theta) = \frac{(r\cos\theta)^2+(r\sin\theta)^2}{(r\cos\theta)^2-(r\sin\theta)^2} = \frac{r^4\cos^4\theta + r^4\sin^4\theta}{r^2\cos\theta - r^2\sin\theta}$$

$$= \frac{r^2(\cos^2\theta+\sin^2\theta)}{1} \xrightarrow[r\to 0]{} 0 \ \forall \theta$$

$$0 \le \left|\frac{x^4+y^4}{x^2+y^2}\right| = \frac{|x^4|}{|x^2+y^2|} + \frac{|y^2|}{|x^2+y^2|} \le \frac{x^2}{x^2+y^2} \cdot \underset{\overset{\downarrow}{0}}{x^2} + \frac{y^2}{x^2+y^2} \cdot \underset{\overset{\downarrow}{0}}{y^2}$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2+2x}{y^2-2x} \leftarrow f(x,y)$

$$f(x\cos\theta, x\sin\theta) = \frac{x^2\cos^2\theta - 2x\sin\theta}{x^2\cos^2\theta - 2x\sin\theta}$$

Fie $(x_n, y_n) = \left(\frac{1}{n}, 0\right) \xrightarrow[n\to\infty]{} (0,0)$

$(x_n', y_n') = \left(0, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$

$$\left.\begin{array}{l} f(x_n, y_n) = \dfrac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1 \\ \\ f(x_n', y_n') = \dfrac{2\cdot\frac{1}{n}}{2\left(-\frac{1}{n}\right)} = -1 \end{array}\right\} \Rightarrow -1 \neq 1 \Rightarrow \nexists \text{ lim.}$$

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^3}{x^4+y^4}$ $\qquad f(x,y) = \dfrac{x^2y^3}{x^4+y^4}$

$$0 \le \left|\frac{x^2y^3}{x^4+y^4}\right| \le \frac{|x^2||y^3|}{x^4+y^4} = \frac{x^2y^2}{x^4+y^4}\underset{\downarrow 0}{|y|}$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

$x^4+y^4 \geq 2x^2y^2$

$x^4+y^4-2x^2y^2 \geq 0 \quad |: 2(x^4+y^4)$

$(x^2-y^2)^2 \geq 0 \quad (A)$

$\frac{1}{2} \geq \underset{\text{mărginită}}{\underbrace{\frac{x^2\cdot y^2}{x^4+y^4}}} \geq 0$

4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{(x-y)^2+x^2+y^2} \leftarrow f(x,y)$

fie $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$

$(x_n', y_n') = \left(0, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$

$$\left.\begin{array}{l} f(x_n, y_n) = \dfrac{\left(\frac{1}{n^2}\right)\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n}-\frac{1}{n}\right)^2+\frac{1}{n^2}+\frac{1}{n^2}} = \dfrac{\frac{1}{n^2}}{\frac{2}{n^2}} = \dfrac{1}{2} \to \dfrac{1}{2} \\[2ex] f(x_n', y_n') = \dfrac{0\left(\frac{1}{n}\right)^2}{\left(0-\frac{1}{n}\right)^2+\left(0^2+\frac{1}{n}\right)^2} = 0 \longrightarrow 0 \end{array}\right\} \Rightarrow \frac{1}{2} \neq 0$$

7. $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$

$$f(x,y) = \frac{x^2 y}{x^4 + y^4}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^4} = \frac{0}{0}$$

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos^2\theta \cdot r\sin\theta}{r^4\cos^4\theta + r^4\sin^4\theta} = \frac{r^3\cos^2\theta\sin\theta}{r^4(\cos^4\theta+\sin^4\theta)} = \frac{\cos^2\theta\sin\theta}{\underset{\downarrow\infty}{r(\cos^4\theta+\sin^4\theta)}}$$

Construim 2 şiruri:

$((x_n,y_n)_n, (\overline{x_n},\overline{y_n}))_n \subset \mathbb{R}^2 \setminus \{(0,0)\}$

$(x_n,y_n) = \left(\frac{1}{n}, 0\right) \xrightarrow{n\to\infty} (0,0)$, $\forall n$

$(\overline{x_n},\overline{y_n}) = \left(\frac{1}{n}, \frac{1}{n^2}\right) \xrightarrow{n\to\infty} (0,0)$

$$f(x_n,y_n) = \frac{\frac{1}{n^2}\cdot 0}{\frac{1}{n^4}+0} = 0$$

$$f(\overline{x_n},\overline{y_n}) = \frac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^4}+\frac{1}{n^4}} = \frac{\frac{1}{n^4}}{\frac{2}{n^4}} = \frac{1}{2}$$

$$\left.\begin{aligned}\lim_{n\to\infty} f(x_n,y_n) &= 0\\ \lim_{n\to\infty} f(\overline{x_n},\overline{y_n}) &= \frac{1}{2}\end{aligned}\right\} \Rightarrow 0 \neq \frac{1}{2} \Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^4}$$

5. Fie $f: \mathbb{R}^2 \setminus \{(0,0)\} \to [-1,1]$

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^3+y^3} \cdot \frac{x^3+y^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

$$0 \le \left|\frac{x^3+y^3}{x^2+y^2}\right| \le \frac{|x+y|\,|x^2-xy+y^2|}{x^2+y^2} \le (|x|+|y|) \cdot \frac{x^2+y^2+|xy|}{x^2+y^2}^{\le\frac{1}{2}}$$

$$= (|x|+|y|)\left(1+\frac{|xy|}{x^2+y^2}\right) \le (|x|+|y|)\left(1+\frac{1}{2}\right) = \frac{3}{2}(|x|+|y|)$$

$$0 \le \left|\frac{x^3+y^3}{x^2+y^2}\right| \le \frac{3}{2}(|x|+|y|)$$

---

6. Fie $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$

Su ex. 4

$$f(x,y) = \frac{x^2y}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \frac{0}{0}$$

$$0 \le \left|\frac{x^2y}{x^2+y^2}\right| = \underbrace{\boxed{\frac{x^2}{x^2+y^2}}}_{\text{mărginită}}|y| \to 0 \text{ când } (x,y)\to(0,0)$$

$$0 \le \frac{x^2}{x^2+y^2} \le 1$$

$$\underset{\searrow}{0} \le \underset{\downarrow}{\left|\frac{x^2y}{x^2+y^2}\right|} \le \underset{\swarrow}{|y|}$$

0 crit. Cleștelui.

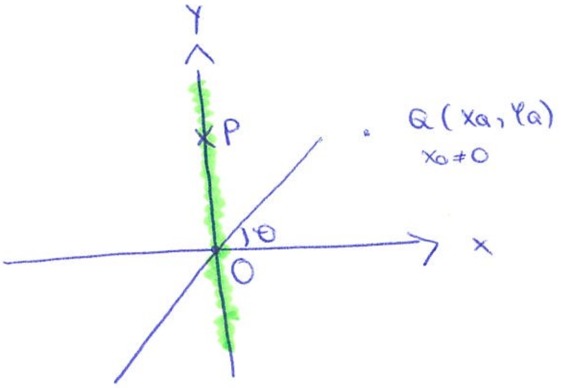
Seminar 12

① Fie $f(x,y) = \operatorname{arctg} \frac{y}{x^2}$

i) aflați mult. de def $D \subset \mathbb{R}^2$ a fct. f.

ii) det D'

iii) cercetați dacă există lim. fct. f în punctele din D'.



i) $\operatorname{arctg} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$x^2 \neq 0 \Rightarrow x = 0$

$D = \{ (x,y) \notin \mathbb{R}^2 \mid x \neq 0 \}$

ii) $D' = \mathbb{R}^2$

$\forall P \in OY$

$S(P,r) \setminus \{P\} \cap D \neq \emptyset$

iii) $\lim\limits_{(x,y)\to(x_0,y_0)} \operatorname{arctg} \frac{y}{x^2} = \operatorname{arctg} \frac{y_0}{x_0^2}$, $x_0 \neq 0$.

$$\lim_{\substack{(x,y)\to(0,y_0) \\ y_0 \neq 0}} \operatorname{arctg} \frac{y}{x^2} = \operatorname{arctg} \frac{y_0}{0^2} < \begin{matrix} \operatorname{arctg} \infty = \frac{\pi}{2}, \ y_0 > 0 \\ \operatorname{arctg} -\infty = -\frac{\pi}{2}, \ y_0 < 0. \end{matrix}$$

$(x,y) \to (0,0)$

$$f(r\cos\theta, r\sin\theta) = \operatorname{arctg} \frac{r\sin\theta}{r^2\cos^2\theta} = \operatorname{arctg}\left(\frac{1}{r} \cdot \frac{\sin\theta}{\cos^2\theta}\right)$$

$$\lim_{(x,y)\to(0,0)} \boxed{\frac{y}{x^2}} \overset{not}{=} h(x,y)$$

$$(x_n, y_n) = \left(\frac{1}{n}, 0\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$(\bar{x}_n, \bar{y}_n) = \left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$\left.\begin{matrix} h(x_n, y_n) = \frac{y_n}{x_n^2} = \frac{0}{\left(\frac{1}{n}\right)^2} = 0 \xrightarrow[n\to\infty]{\lim} 0 \\ h(\bar{x}_n, \bar{y}_n) = \frac{\frac{1}{n}}{\left(\frac{1}{n}\right)^2} = n \xrightarrow[n\to\infty]{\lim} +\infty \end{matrix}\right\} \Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{y}{x^2}$$

② Calculați $\lim_{x \to 0} f(x)$, $f : \mathbb{R}^* \to \mathbb{R}^3$

$$f(x) = \left(\frac{\sqrt{1+x} - 1}{\sqrt[3]{1+x} - 1} , \left(\frac{a^x + \ln^x}{2} \right)^{\frac{1}{x}} \cdot \frac{\sqrt[m]{1+\alpha x} - \sqrt[n]{1-\beta x}}{x} \right)$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{\sqrt[3]{1+x} - 1} = \left[\frac{0}{0}\right]. \to L'H : \lim_{x \to 0} \frac{(\sqrt{1+x} - 1)'}{(\sqrt[3]{1+x} - 1)'}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{\frac{1}{3}(x+1)^{\frac{1}{3}-1}} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{\frac{1}{3}(x+1)^{\frac{2}{3}}} = \frac{1}{2} \cdot \frac{3}{1} = \frac{3}{2}$$

$$\lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{\frac{1}{x}} = \lim_{x \to 0} \left(1 + \frac{a^x + b^x}{2} - 1 \right)^{\frac{1}{x}}$$

$$\lim_{x \to 0} \left(1 + \frac{a^x + b^x - 2}{2} \right)^{\frac{1}{x} \cdot \frac{2}{a^x + b^x - 2} \cdot \frac{1}{x}}$$

$$= e^{\lim_{x \to 0}} \frac{a^x + b^x - 2}{2} \cdot \frac{1}{x} = \frac{1}{2} \left(\frac{a^x - 1}{x} + \frac{b^x - 1}{x} \right)$$

$$= \frac{1}{2} \ln a + \ln b = \frac{1}{2} \ln ab \Rightarrow e^{\ln \sqrt{ab}} = \sqrt{ab}$$

3. $$\lim_{x \to 0} \frac{\sqrt[m]{1+ax} - \sqrt[n]{1+\beta x}}{x} \overset{L'H}{=} \lim_{x \to 0} \frac{(\sqrt[m]{1+\alpha x} - \sqrt[n]{1+\beta x})'}{(x)'}$$

$$= \lim_{x \to 0} \frac{(1+\alpha x)^{\frac{1}{n}} - (1+\beta x)^{\frac{1}{m}}}{(x)'} = \lim_{x \to a} \frac{1}{n} (1+\alpha x)^{\frac{1}{n}-1} \frac{1}{m} (1+\beta x)^{\frac{1}{m}-1}$$

$$= \lim_{x \to 0} \frac{1}{n} (1+ax)^{\frac{1-n}{n}} - \frac{1}{m} (1+\beta x)^{\frac{1-m}{m}}$$

$=$

④ Calculați existența limitei în $(0,0)$ pt funcția

$f: \mathbb{R}^2 \setminus \{0,0\} \to \mathbb{R}^2.$

$$f(x,y) = \left(\frac{xy}{\sqrt{x^2+y^2}} \sin \frac{1}{x^2+y^2} ; \frac{xy}{\sqrt{x^2+y^2}} \cos \frac{1}{x^2+y^2} \right)$$

$\underset{=}{\text{Not}}$: $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \sin \frac{1}{x^2+y^2}$

Știm că $\left| \sin \frac{1}{x^2+y^2} \right| \leq 1 \ \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

Calculăm

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \cdot \sin \frac{1}{x^2+y^2} = 0 \quad \in [-1, 1]$$

analog pt. cos $\Rightarrow \exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = (0,0)$

⑤ Fie $f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y) = (x+y, x-y)$. Ar. că f este Lipschitz.

Găsiți Inf $\mathcal{C}((0,0), r)$ prin f.

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

Lips: $d((x_1,y_1),(x_2,y_2)) \leq L\, d((x_1,y_1),(x_2,y_2)) \quad , \forall (x_1,y_1),(x_2,y_2)$

$$d(f(x_1,y_1), f(x_2,y_2)) = d((x_1+y_1, x_1-y_1), (x_2+y_2, x_2-y_2))$$

$$= \sqrt{((x_1+y_1)-(x_2+y_2))^2 + (x_1-y_1-(x_2-y_2))^2}$$

$$= ((x_1+y_1)-(x_2-y_2))^2 = (x_1-x_2+y_1-y_2)^2 \leq$$

~~$\leq 2\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \Rightarrow 2d((x_1,x_2),(y_1,y_2))^2$~~

$$\leq (x_1-x_2)^2 + (y_1-y_2)^2$$

$$((x_1-x_2)+(y_1-y_2))^2 \leq 2((x_1-x_2)^2 + (y_1+y_2)^2)$$

$$\sqrt{(x_1+y_1)^2-(x_2+y_2))^2 + (x_1-y_1-(x_2-y_2))^2} \leq 2\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

$\Rightarrow 2d((x_1,x_2),(y_1,y_2))^2$

$\Rightarrow L = 2$

$f(x,y) \in ?$

$(x,y) \in ((0,0),1)$

$x^2+y^2=1$

$$(x+y)^2+(x-y)^2 = x^2+y^2+2xy+x^2+y^2-2xy$$
$$= 2x^2+2y^2$$
$$= 2(x^2+y^2) = 2 = (\sqrt{2})^2$$

$f(x,y) \in \mathcal{B}((0,0),\sqrt{2})$.

Seminar 13

① Limitele iterate şi globale în $(0,0)$

$$f(x,y)=x\sin\frac{1}{y} \;,\; \{f(x,y)\in\mathbb{R}^2\;(x,y),\; y=0\}$$

L_1: $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} x\sin\frac{1}{y}\right)$

$$\lim_{y\to 0} x\sin\frac{1}{y} = x\lim_{y\to 0}\sin\frac{1}{y} - \sin\infty \Rightarrow L_1 \not\in \lim_{x\to\infty}\sin x$$

$$\forall \alpha\in[-1,1],\; \exists \sin \;\to\infty$$

$$\sin x_n \to \infty$$

$$x_n = \frac{3\pi}{2} + \underbrace{2n\pi}\to\infty$$

$$\sin x_n = \sin\frac{3\pi}{2} + 2n\pi$$

$$= \sin\frac{3\pi}{2} = -1 \to -1.$$

② $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{xy^2 + \sin(x^3+y^5)}{x^2+y^4}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

L_1: $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}\dfrac{xy^2+\sin(x^3+y^5)}{x^2+y^4}\right)$

$$\lim_{y\to 0}\frac{\overset{\nearrow 0}{x}y^2+\sin(x^3+y^5)}{x^2+y^4} = \frac{x\cdot 0+\sin(x^3+0)}{x^2+0} = \frac{\sin x^3}{2}$$

$$\lim_{x\to 0}\frac{\sin x^3}{\cancel{x^2}} = \lim_{x\to 0}\frac{\overset{\to 1}{\frac{\sin x^3}{x^3}}}{\cancel{x^2}}\cdot x^{\cancel{3}\,1} = \lim_{x\to 0} x = 0.$$

L_2: $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}\dfrac{xy^2+\sin(x^3+y^3)}{x^2+y^4}\right)$

$$\lim_{x\to 0}\frac{xy^2+\sin(x^3+y^3)}{x^2+y^4} = \frac{\sin y^5}{y^4}$$

$$\lim_{y\to 0}\frac{\sin y^5}{y^4} = \lim_{y\to 0}\frac{\sin y^5}{y^5}\cdot\frac{\cancel{y^5}^{\,1}}{\cancel{y^4}} = \lim 1\cdot y = 0$$

- Limita globală :

$$\lim_{(x,y)\to(0,0)} \frac{xy^2 + \sin(x^3+y^5)}{x^2+y^4}$$

$$(x_n, y_n) = \left(0, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$(x_n', y_n') = \left(\frac{1}{n^2}, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$\lim_{(x_n,y_n)\to(0,0)} \frac{0\cdot\frac{1}{n^2} + \sin\left(0+\frac{1}{n^5}\right)}{0+\frac{1}{n^4}} = \lim_{(x_n,y_n)\to(0,0)} \frac{\sin\frac{1}{n^5}}{\frac{1}{n^4}}$$

$$= \lim_{(x_n,y_n)\to(0,0)} \frac{\sin\frac{1}{n^5}}{\frac{1}{n^5}} \cdot \frac{\cancel{n^5}\,{}^{n}}{\cancel{n^4}} = \lim_{(x_n,y_n)\to(0,0)} 1\cdot n = 0.$$

$$\lim_{(x_n',y_n')\to(0,0)} \frac{\frac{1}{n^2}\cdot\frac{1}{n^2} + \sin\left(\frac{1}{n^6}+\frac{1}{n^5}\right)}{\frac{1}{n^4}+\frac{1}{n^4}} = \lim_{(x_n',y_n')\to(0,0)} \frac{\frac{1}{n^4} + \left(\sin\frac{n+1}{n^6}\right)}{2\,\frac{1}{n^4}}$$

$$= \lim_{(x_n',y_n')} \frac{\frac{1}{\cancel{n^4}}}{\frac{2}{\cancel{n^4}}} + \frac{\sin\frac{n+1}{n^6}}{\frac{n+1}{n^6}} \to 1 = \frac{1}{2}$$

③ $f : (X, d) \to (\mathbb{R}, d_n)$

Dacă pt. orice $\alpha \in \mathbb{R}$, mulț. $\{x \in X, f(x) > \lambda\}$ și $\{x \in X, f(x) < \lambda\}$ sunt deschise, at. f continuă pe Y.

Știm că f cont pe $X \Leftrightarrow \mathcal{D} \subset \mathbb{R}$ deschis avem că $f^{-1}(\mathcal{D})$ deschisă.

$f^{-1}(\mathcal{D}) = \{x \in X \mid f(x) \in \mathcal{D}\}$

$A = \{x \in X \mid f(x) > \lambda\} = \{x \in X \mid f(x) \in (\lambda, \infty)\} = f^{-1}(\lambda, +\infty).$

$B = \{x \in X, f(x) < \lambda\} = \{x \in X \mid f(x) \in (-\infty, \lambda) = f^{-1}(-\infty, \lambda),$

$\forall \lambda \in \mathbb{R}$ sunt deschiși

Vrem să arătăm că $\forall\ \mathcal{D} = \overset{S(x_0,\varepsilon)}{\overset{\shortparallel}{(x_0-\varepsilon, x_0+\varepsilon)}}$

$f^{-1}(\mathcal{D})$ deschis

$$f^{-1}(x_0-\varepsilon, x_0+\varepsilon) = \{x \in X \mid f(x) \in (x_0-\varepsilon, x_0+\varepsilon)\}$$

$$= \{x \in X \mid f(x) \in (-\infty, x_0+\varepsilon) \cap ((x_0-\varepsilon) - \infty)\}$$

$$= \{x \in X \mid f(x) \in (-\infty, x_0+\varepsilon)\} \cap \{x \in X \mid f(x) \in (x_0-\varepsilon, +\infty)\}$$

$$= |f^{-1}(-\infty, x_0+\varepsilon)\} \cap \{f(x-\varepsilon, +\infty)\}$$

$f^{-1}(-\infty, x_0+\varepsilon) = B$ cu $\lambda = x_0+\varepsilon \Rightarrow$ deschis

$f^{-1}(x_0-\varepsilon, +\infty) = A$ cu $\lambda = x_0-\varepsilon \Rightarrow$ deschis

4. Studiați continuitatea pe mulțimea de def. a fct. urm.:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} \dfrac{x^2 \sin y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Evident este continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ fiind compunere de fct. elem.

Studiem continuitatea în $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\frac{x^2 \sin y}{y} \cdot y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$$

$$0 < \left|\frac{x^2 y}{x^2+y^2}\right| = \left|\frac{x^2}{x^2+y^2}\right| |y| \leq |y|$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

$= \ell(0,0) \Rightarrow f$ este cont. în $(0,0)$

5. Studiați continuitatea pe mulț de def a funcțiilor urm:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} \dfrac{x \ln(1+y^2) - x \sin y}{\sqrt{x^2+y^2}}, & f(x,y) \neq (0,0) \\ 0, & f(x,y) = (0,0) \end{cases}$$

f cont pe $\mathbb{R}^2 \setminus \{0,0\}$.

studiem cont. în (0,0)

$$\lim_{(x,y)\to(0,0)} \frac{x\ln(1+y^2) - x\sin y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \left(\frac{x\ln(1+y^2)}{\sqrt{x^2+y^2}} - \frac{x\sin y}{\sqrt{x^2+y^2}} \right)$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\ln(1+y^2)}{\sqrt{x^2+y^2}} - \lim_{(x,y)\to(0,0)} \frac{x\sin y}{\sqrt{x^2+y^2}} \to 0.$$

$$\underset{\substack{\downarrow\\0}}{0} \le \underset{\substack{\downarrow\\0}}{\left| \frac{x\ln(1+y^2)}{\sqrt{x^2+y^2}} \right|} \le \underset{\substack{\downarrow\\0}}{\left| \frac{x}{\sqrt{x^2+y^2}} \right|} \underset{\substack{\downarrow\\ \ln 1}}{|\ln(1+y^2)|}$$

$= 0 - 0 = 0 \Rightarrow f(0,0) \Rightarrow f$ cont în (0,0)

⑥ Fie $f,g: (X,d_2) \to (Y,d_2)$ 2 aplicații continue.

Ar. că $A = \{x \in X \mid f(x) = g(x)\}$ este închisă.

METODA I:

A este închisă $\Leftrightarrow \forall (x_n)_n \subset A$ cu $x_n \to a \Rightarrow a \in A$.

Fie $(x_n)_n \in A$ a.î. $\exists \lim_{n\to\infty} x_n = a$

$f(x_n) = g(x_n), \forall n \in \mathbb{N}$

$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) \Leftrightarrow f(\lim_{n\to\infty} x_n) = g(\lim_{n\to\infty} x_n)$

$f(a) = g(a) \Rightarrow a \in A \Rightarrow$ A închisă

METODA II:

Știm că $|d_2(x,y) - d_2(x',y')| \le d_2(x,x') + d_2(y,y')$

Fie $x_n \to x'$, $y_n \to y'$

$$\underset{\searrow}{0} \le \underset{\substack{\downarrow\\0}}{d_2(x_n,y_n) - d_2(x',y')} \le \underset{\swarrow}{d_2(x_n,x') + d_2(y_n,y')}, \forall n \in \mathbb{N}$$

$\lim_{n\to\infty} d_2(x_n,y_n) = d_2(x',y') \quad \forall (x_n,y_n) \in (Y,Y)$

$\Rightarrow d_2$ cont. pe (Y,Y)

$A = \{x \in X \mid d_2(f(x), g(x)) = 0\}$

$A = \{x \in X \mid h(x) = 0\} = h^{-1}(\{0\}) \Rightarrow$ este cont

$\{0\}$ închisă $\Rightarrow$ A închisă.

Seminar 14

1. Fie (X,d) un sp. metric şi $A\subset X$. Definim fct. $f_A(x) = d(x,A) = \inf_{y\in A} d(x,y), \forall x\in X$.

Ar. că:

i) f_A este continuă pe X

pentru orice $x, x' \in X$ şi $y \in A$.

Are loc relaţia: $|d(x,y)-d(x',y')| \le d(x,x') + d(y,y')$

Vrem să arătăm că $|f_A(x) - f_A(y)| \le d(x,y), \forall x,y \in X$

$\Leftrightarrow -d(x,y) \le f_A(x) - f_A(y) \le d(x,y)$

$f_A(x) - f_A(y) \le d(x,y)$

$\Leftrightarrow f_A(x) \le d(x,y) + f_A(y)$

$\Leftrightarrow \inf_{a\in A} d(x,a) \le d(x,y) + \inf_{a\in A} d(y,a)$ (*)

Din ineg. triunghiulară ştim că $d(x,a) \le d(x,y) + d(y,a)\ \forall a \in A$

Trecem la inf după $a \in A$ (*)

$\Rightarrow f_A$ este Lischitz $\Rightarrow$ continuă

ii) $A, B \subset X$ închise, disjuncte

$D_A = \{x\in X, d(x,A) < d(x,B)\}$

$D_B = \{x \in X, d(x,A) > d(x,B)\}$

D_A-deschisă

D_B-deschisă

$D_A = \{x\in X, f_A(x) < f_B(x)\} = \{x\in X \mid f_A(x) - f_B(x) < 0\}$

$D_A = \{x\in X \mid f_A(x) - f_B(x) \in (-\infty, 0)\} = (f_A - f_B)^{-1}((-\infty,0))$

f_A, f_B continue $\Rightarrow f_A - f_B$ continuă, $(-\infty, 0)$-deschis $\Big\} \Rightarrow$

$\Rightarrow (f_A - f_B)^{-1}$

$(-\infty,0)$ e deschis

$\Rightarrow D_A$ deschis

$D_B = \{ x \in X, f_A(x) > f_B(x) \} = \{ x \in X \mid f_A(x) - f_B(x) > 0 \}$

$D_B = \{ x \in X, f_A(x) - f_B(x) \in (0, +\infty) \} = (f_A - f_B)^{-1}((0, +\infty))$ $\Big\} \Rightarrow$

f_A, f_B continue $\Rightarrow$ $f_A - f_B$ continuă, $(0, +\infty)$ deschis

$\Rightarrow (f_A - f_B)^{-1}((0,\infty))$ e deschisă $\Rightarrow D_B$ deschisă

Vrem să arătăm că $D_A \cap D_B = \emptyset$

$x \in D_A \Rightarrow d(x, A) < d(x, B)$

$x \in D_B \Rightarrow d(x, A) > d(x, B)$ $\Big\} \Rightarrow D_A \cap D_B = \emptyset$.

Vrem să arătăm că $A \subset D_A$, adică $\forall x \in A \Rightarrow x \in D_A$.

$x \in A$, at. $d(x, A) = 0$

Are loc $0 < d(x, B)$

Am arătat că $d(x, B) = 0 \Leftrightarrow x \in \overline{B}$

B este închis $\Rightarrow x \in B$

Dar ştim şi că $x \in A$, $A \cap B = \emptyset \Rightarrow x \notin B \Rightarrow d(x, B) > 0 = d(x, A)$

$\Rightarrow x \in D_A$

Vrem să arătăm că $B \subset D_B$, adică $\forall x \in B \Rightarrow x \in D_B$

$x \in B$, at $d(x, B) = 0$

Are loc $0 < d(x, A)$ deoarece dacă $d(x, A) = 0 \Rightarrow x \in A$.

Noi ştim că $x \in B$ şi $A \cap B = \emptyset$

$\Rightarrow x \notin A \Rightarrow d(x, A) > 0 = d(x, B) \Rightarrow x \in D_B$

② Funcțiile sunt contracții

i) $(\mathbb{R}, d)$, $d(x,y) = |x-y|$, $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{5} \operatorname{arctg} x$

$$d(f(x), f(y)) = |f(x) - f(y)| = \left|\frac{1}{5}\operatorname{arctg} x - \frac{1}{5}\operatorname{arctg} y\right|$$
$$= \frac{1}{5}|\operatorname{arctg} x - \operatorname{arctg} y|$$

$g: [a,b] \to \mathbb{R}$, cont. pe $[a,b]$, derivabilă pe (a,b), at.
$\exists\, c \in (a,b)$ a.î. $\frac{g(b)-g(a)}{b-a} = g'(c)$

$g = \operatorname{arctg}$
$a = x \quad b = y$

$$\frac{\operatorname{arctg} y - \operatorname{arctg} x}{y-x} = \operatorname{arctg}(c)'$$

$$\operatorname{arctg}'(c) = \frac{1}{1+c^2}$$

$$\left|\frac{\operatorname{arctg}(y) - \operatorname{arctg}(x)}{y-x}\right| = \left|\frac{1}{1+c^2}\right| \leq 1 \quad | \cdot |y-x|$$

$\Rightarrow |\operatorname{arctg}(y) - \operatorname{arctg}(x)| \leq |y-x|$

$\frac{1}{5}|\operatorname{arctg} x - \operatorname{arctg} y| \leq \frac{1}{5}|y-x| = \frac{1}{5} d(x,y)$

$\Rightarrow d(f(x), f(y)) \leq \frac{1}{5} d(x,y)$

$\Rightarrow$ f contracție cu coeficientul $\frac{1}{5}$.

Cum spațiul $\mathbb{R}$ este complet $\to$ conform teoremei de pct. fix al lui Banach

$\Rightarrow \exists!\, x^* \in \mathbb{R}$ a.î. $f(x^*) = x^* \to$ pct. fix.

$\frac{1}{5}(\operatorname{arctg} x^*) = x^* \Rightarrow x^* = 0$

fie $h(x) = \operatorname{arctg} x - 5x$

$$h'(x) = \underbrace{\frac{1}{1+x^2}}_{\leq 1} - 5 < 0 \quad \Rightarrow \begin{cases} h \searrow \text{ pe } \mathbb{R} \to h \text{ inj. pe } \mathbb{R} \\ h(0) = 0 \\ h(x) \neq 0, \forall x \neq 0 \end{cases}$$

ii) $(\mathbb{R}_+, d)$, $d(x,y) = |x-y|$

$f: \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \sqrt{x+1}$

$d(f(x), f(y)) = |f(x) - f(y)| = |\sqrt{x+1} - \sqrt{y+1}|$

$$\left|\frac{(\sqrt{x+1} - \sqrt{y+1})(\sqrt{x+1} + \sqrt{y+1})}{\sqrt{x+1} + \sqrt{y+1}}\right| = \left|\frac{x + \cancel{1} - y - \cancel{1}}{\sqrt{x+1} + \sqrt{y+1}}\right| = \underbrace{\frac{1}{\sqrt{x+1}+\sqrt{y+1}}}_{\le \frac{1}{2}} |x-y|$$

$\le \frac{1}{2} d(x,y) \Rightarrow f$ contractie

$$\left.\begin{array}{l} \mathbb{R}_+ = [0, +\infty) \\ \mathbb{R}_+ \subset \mathbb{R} - \text{complet} \\ \mathbb{R}_+ \text{ închis} \end{array}\right\} \Rightarrow \mathbb{R}_+ - \text{complet}.$$

$f(x^*) = x^* \Rightarrow \sqrt{x^*+1} = x^* \quad ()^2$

$x^* + 1 = (x^*)^2$

$x^{*2} - x^* - 1 = 0$

$\Delta = 1 + 4 = 5$

$x_1 = \frac{1+\sqrt{5}}{2} = \frac{3{,}23}{2} > 0$ $\Rightarrow$ $\underline{x^* = \frac{1+\sqrt{5}}{2}}$.

$x_2 = \frac{1-\sqrt{5}}{2} = \frac{-1{,}23}{2} < 0$

iii) $([\frac{\pi}{4}, \frac{\pi}{2}], d)$, $d(x,y) = |x-y|$

$f: [+\frac{\pi}{4}, \frac{\pi}{2}] \to [\frac{\pi}{4}, \frac{\pi}{2}]$

$f(x) = \sqrt{\sin x}$

$$d(f(x) - f(y)) < |\sqrt{\sin x} - \sqrt{\sin y}| = \left|\frac{(\sqrt{\sin x} - \sqrt{\sin y})(\sqrt{\sin x} + \sqrt{\sin y})}{\sqrt{\sin x} + \sqrt{\sin y}}\right|$$

$$= \left|\frac{\sin x - \sin y}{\sqrt{\sin x} + \sqrt{\sin y}}\right|$$

$x \in [\frac{\pi}{4}, \frac{\pi}{2}] \Rightarrow \sin x \in [\frac{\sqrt{2}}{2}, 1]$

$\Rightarrow \sqrt{\sin x} \in [\sqrt{\frac{\sqrt{2}}{2}}, 1] \Rightarrow \sqrt{\sin x} \geq \sqrt{\frac{\sqrt{2}}{2}}$ (+)

$\sqrt{\sin y} \geq \sqrt{\frac{\sqrt{2}}{2}}$

$$\sqrt{\sin x}+\sqrt{\sin y} \geq 2\sqrt{\frac{\sqrt{2}}{2}}$$

$$\Rightarrow \frac{1}{\sqrt{\sin x}+\sqrt{\sin y}} \leq \frac{1}{2\sqrt{\frac{\sqrt{2}}{2}}}$$

$$\left|\frac{\sin x-\sin y}{\sqrt{\sin x}+\sqrt{\sin y}}\right| = \frac{1}{2\sqrt{\frac{\sqrt{2}}{2}}}|\sin x-\sin y| \overset{\boxed{\text{T. Lagrange}}}{=} \frac{1}{2\sqrt{\frac{\sqrt{2}}{2}}}|\sin'(c)||x-y|$$

$$= \frac{1}{2\sqrt{\frac{\sqrt{2}}{2}}}|\cos c||x-y|$$

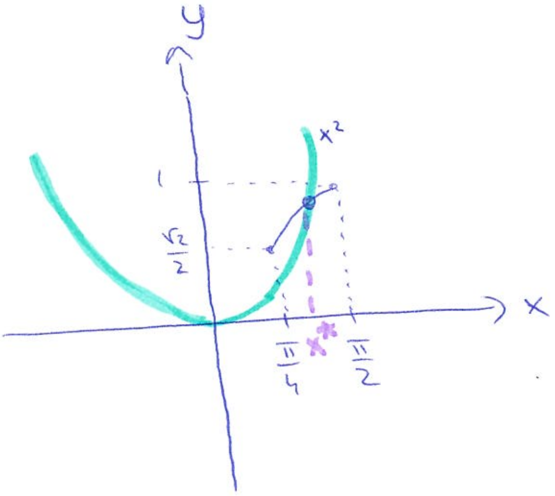
$$\Rightarrow \leq \frac{1}{2\sqrt{\frac{\sqrt{2}}{2}}}|x-y|$$

$\Rightarrow$ • f contracție, cu coef de contracție $\frac{1}{2\sqrt{\frac{\sqrt{2}}{2}}}$

$\sqrt{\sin x^*} = x^* \quad ()^2$

$\sin x^* = {x^*}^2$

$\Rightarrow \exists\, x^*$, dar nu îl putem afla.



Seminar 15

Model de examen

1. Studiați convergența șirului

$$x_n = \left(\frac{2^n+n^2}{3n+\ln n} \; ; \; \frac{5^{\operatorname{arctg}\frac{1}{n}}-1}{\frac{1}{n}} \; ; \; \sqrt[n]{n!} \right)$$

$$x_n^1 = \frac{2^n+n^2}{3n+\ln n}$$

$$\lim_{n\to\infty} x_n^1 = \lim_{n\to\infty} \frac{2^n+n^2}{3n+\ln n} = \lim_{n\to\infty} \frac{2^n\overset{\nearrow 1}{\left(1+\frac{n^2}{2^n}\right)}}{3n\underset{\searrow 1}{\left(1+\frac{\ln n}{3n}\right)}} = \lim_{n\to\infty} \frac{2^n}{3^n} = \infty$$

$$x_n^2 = \frac{5^{\operatorname{arctg}\frac{1}{n}}-1}{\frac{1}{n}}$$

$$\boxed{\begin{array}{l} \lim\limits_{u\to 0} \frac{\operatorname{arctg} u}{u} = 1 \\ \lim\limits_{u\to 0} \frac{a^u-1}{u} = \ln a \end{array}} \to x_n^2$$

Criteriul rădăcinii: $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ → x_n^3

$$\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} \underbrace{\frac{5^{\operatorname{arctg}\frac{1}{n}}-1}{\operatorname{arctg}\frac{1}{n}}}_{\searrow \ln 5} \cdot \underbrace{\frac{\operatorname{arctg}\frac{1}{n}}{\frac{1}{n}}}_{\searrow 1} = \ln 5$$

$$x_n^3 = \sqrt[n]{n!}$$

$$\lim_{n\to\infty} x_n^3 = \lim_{n\to\infty} \sqrt[n]{n!} \overset{CR}{=} \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n+1 = +\infty$$

=> Deci, șirul x_n este divergent.

2. Studiați continuitatea funcției:

$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{\sin\left(\sqrt{x^2+y^2}\right)}{\sqrt[4]{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Pe $\mathbb{R}^2 \setminus \{0,0\}$ fct. e cont., fiind compunere de fct. elem. continue

Studiem continuitatea în (0,0)

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(\sqrt{x^2+y^2}\right)}{\sqrt[4]{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \overset{\uparrow 1}{\boxed{\frac{\sin\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}}} \cdot \frac{\sqrt{x^2+y^2}}{\sqrt[4]{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^{\frac{1}{2}}}{(x^2+y^2)^{\frac{1}{4}}} = \lim_{(x\to y)\to(0,0)} (x^2+y^2)^{\frac{1}{4}} = 0 = f(0,0)$$

$\Rightarrow$ f este cont în 0

$\Rightarrow$ f cont. pe $\mathbb{R}^2$.

3. Fie $(X, \|\cdot\|)$ sp. liniar normat, $D \subset X$ o mulț. deschisă și $A \subset X$ o mulțime nevidă. Ar. că $A+D = \bar{A}+D$.

$A+D = \{a+d \mid a \in A, d \in D\}$

$\bar{A}+D = \{a'+d \mid a' \in \bar{A}, d \in D\}$

(dubla incluziune)

"$\subseteq$" $A+D \subseteq \underline{\bar{A}+D}$

Știm că $A \subseteq \bar{A}$, prin urmare $A+D \subseteq \bar{A}+D$.

"$\supseteq$" $\underline{\bar{A}+D \subseteq A+D}$

Fie $x \in \bar{A}+D \Rightarrow x = a'+d$, $a' \in \bar{A}$, $d \in D$

! $a' \in \bar{A} \Rightarrow \exists (a_n)_n \subset A$ a.î. $\lim\limits_{n\to\infty} a_n = a'$.

$$\left.\begin{array}{l} \lim\limits_{n\to\infty}(x-a_n) = \lim\limits_{n\to\infty} a'+d-a_n = d \\ d \in D,\ D \text{ deschis} \Rightarrow D \in \mathcal{V}(d) \end{array}\right\} \Rightarrow \exists N \in \mathbb{N} \text{ a.î. } n \geq N,\ x-a_n \in D.$$

În particular, $x - a_N \in D \Rightarrow x - a_N = d' \in D$.

$$\Rightarrow x = \underbrace{d'}_{D} + \underbrace{a_N}_{\in A} \in A+D.$$

I.4 (curs 7)

$$\overset{(\mathbb{R}^2, \|\cdot\|)}{d_1 \uparrow} \qquad \overset{(\mathbb{R}, |\cdot|)}{\uparrow d_2}$$

Cercetați dacă funcția $f:\mathbb{R}^2 \to \mathbb{R},\ f(x,y) = \sin\sqrt{x^2+y^2}$ este Lipschitz pe $\mathbb{R}^2$?

$f: (X, D_1) \to (Y, D_2)$ s.n. Lipschitz dacă $\exists\, L > 0$ a.î.

$$d_2(f(x), f(y)) \le L\, d_1(x,y) \quad \forall x,y \in D.$$

Vrem să arătăm că $d_2(f(x,y), f(\bar{x},\bar{y})) \le L\, d_1((x,y),(\bar{x},\bar{y}))$

$$d_2(f(x,y), f(\bar{x},\bar{y})) = \left|\sin\sqrt{x^2+y^2} - \sin\sqrt{\bar{x}^2-\bar{y}^2}\right|$$

Aplicăm T. lui Lagrange pt. fct. sin pe $[a,b]$.

$\exists\, c \in (a,b)$ a.î. $\dfrac{\sin b - \sin a}{b-a} = \sin' c$

$b = \sqrt{x^2+y^2}$

$a = \sqrt{\bar{x}^2+\bar{y}^2}$

$$\left|\frac{\sin\sqrt{x^2+y^2} - \sin\sqrt{\bar{x}^2+\bar{y}^2}}{\sqrt{x^2+y^2} - \sqrt{\bar{x}^2-\bar{y}^2}}\right| = |\cos c|$$

$$\left|\sin\sqrt{x^2+y^2} - \sin\sqrt{\bar{x}^2+\bar{y}^2}\right| = |\cos c| \cdot \left|\sqrt{x^2+y^2} - \sqrt{\bar{x}^2+\bar{y}^2}\right| \le 1 \cdot \Big|\,\|(x,y)\| - \|(\bar{x},\bar{y})\|\,\Big|$$

$$\overset{\text{curs 1}}{\le} \|(x,y) - (\bar{x},\bar{y})\|$$

$$= d_1((x,y),(\bar{x},\bar{y}))$$

$$\Rightarrow \left|\sin\sqrt{x^2+y^2} - \sin\sqrt{\bar{x}^2+\bar{y}^2}\right| \le d_1((x,y),(\bar{x},\bar{y}))$$

$$d_2(f(x,y), f(\bar{x},\bar{y})) \le d_1((x,y),(\bar{x},\bar{y}))$$

$\Rightarrow$ f Lipschitz $\Rightarrow$ constanta $= 1$.

I.6. $f:(X,d_1)\to(Y,d_2)$ s.n. uniform continuă dacă $\forall \varepsilon>0, \exists \delta_\varepsilon>0$

a.î. $\forall x,y\in X$, $d_1(x,y)<\delta \Rightarrow d_2(f(x),f(y))<\varepsilon$.

i) $f(x)=e^x$, $x\in\mathbb{R}$

$f:\mathbb{R}\to\mathbb{R}$

Fie $x_n=\ln n$

$y_n=\ln(n+1)$

$d_1(x_n,y_n)=|x_n-y_n|=|\ln n-\ln n+1|=|\ln\frac{n}{n+1}|\xrightarrow[n\to\infty]{}0$

$d_2(f(x_n),f(y_n))=|f(x_n)-f(y_n)|=|e^{\ln n}-e^{\ln(n+1)}|=|\cancel{n}-\cancel{n}-1|\xrightarrow[n\to\infty]{}1.$

Dacă f. ar fi uniform cont, at. pt. $\varepsilon=\frac{1}{2}$, $\exists\delta>0$ a.î.

$d_1(x,y)<\delta\Rightarrow d_2(f(x),f(y))<\frac{1}{2}$

$d_1(x_n,y_n)\xrightarrow[n\to\infty]{}0 \Rightarrow \exists N$ a.î. $\forall n\geq N$, $d_1(x_n,y_n)<\delta\Rightarrow$

$\Rightarrow d_2(f(x_n),f(y_n))<\frac{1}{2}$ FALS, deoarece $d_2(f(x_n),f(y_n))=1$.

$\Rightarrow$ f nu e unif. cont. pe $\mathbb{R}$.

obs Fie $A\subset\mathbb{R}$ o mulț. mărginită, at. $f:A\to\mathbb{R}$, $f(x)=e^x$ este uniform continuă pe A.

Funcția f este cont. pe $\mathbb{R}$, prin urmare f poate fi prelungită prin continuitate la aderența lui A ($\bar{A}$).

$e^x:A\to\mathbb{R} \rightsquigarrow e^x:\bar{A}\to\mathbb{R}$

$A=(0,1]$

$\exists\lim_{x\to0}e^x=e^0=1$

$e=\begin{cases}e^x, & x\in(0,1]\\ 1, & x=0.\end{cases}$

$\bar{A}\to$ mărginită, $\to$ închisă $\Big\}\Rightarrow$ compactă

Prelungirea fct. f prin continuitate definită pe aderența lui A, care e mulț. compactă, este uniform continuă pe aderența lui A. $\Rightarrow$ f u.c. și pe A.

iii) $f(x) = \ln x$, $x \in (0;\infty)$

$x_n = e^{-n}$

$y_n = e^{-(n+1)}$

$d_1(x_n, y_n) = |x_n - y_n| = |e^{-n} - e^{-n-1}| = |e^{-n}(1-e^{-1})| \xrightarrow[n\to\infty]{} 0$

$d_2(f(x_n), f(y_n)) = |f(x_n) - f(y_n)| = |\ln e^{-n} - \ln e^{-(n+1)}| = |-n+n+1| = 1.$

$\Rightarrow$ f nu este uniform continuă pe $(0;+\infty)$.

iv) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (3x, y-x)$

↑ funcția f este un operator liniar, prin urmare este Lipschitz.

știm că o fct. Lipschitz este uniform continuă, prin urmare f este uniform continuă pe $\mathbb{R}^2$.

II.1. Ar. că în $(\mathbb{R}, d_u)$, subspațiile $(0,1)$, $[0,1]$ nu sunt homeomorfe.

R.A. pp $(0,1)$ și $[0,1]$ sunt homeomorfe $\overset{def}{\Rightarrow} \exists$ o fct. f,

$f: [0,1] \to (0,1)$, f cont și f bij

$$\left.\begin{array}{l} [0,1] \to \text{închisă și mărg} \Rightarrow \text{compactă} \\ f \text{ cont} \end{array}\right\} \Rightarrow \left.\begin{array}{l} f([0,1]) \text{ compactă} \\ f \text{ surj.} \end{array}\right\} \Rightarrow \Bigg\} \Rightarrow$$

$\Rightarrow f([0,1]) = (0,1)$.

$(0,1)$ = compactă FALS.

$\Rightarrow$ nu sunt homeomorfe