Raport de activitate pentru anul 2009
Metode funcţionale, deterministe şi stochastice
în dinamica fluidelor
ID 404/2007-2010
Director de proiect: prof.dr. Cătălin Lefter

Obiective

- Metode numerice pentru ecuaţia de difuzie rapidă în medii poroase (cazul singular): studiul suprafeţei libere dintre domeniul saturat şi nesaturat.
- Caracterizarea operatorului Kolmogorov pentru ecuaţia neliniară a difuziei.
- Studiul unor metode şi tehnici de stabilizare cu controale frontieră sau controale distribuite în subdomenii, cu aplicaţii la ecuaţiile dinamicii fluidelor, ecuaţii Schrödinger etc.

Rezultate obţinute

1. Viorel Barbu, Giuseppe Da Prato, Luciano Tubaro, *Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert space*, **The Annals of Probability**, 2009, Vol. 37, No. 4, 1427-1458
2. V.Barbu, G.Da Prato, M. Roeckner, *Finite time extinction for solutions to fast diffusion stochastic porous media equations*, **Comptes Rendus Math.**,vol. 347, no. 1-2, pages 81-84.
3. Angelo Favini, Gabriela Marinoschi, *Periodic behavior for a degenerate fast diffusion equation*, **J. Math. Anal. Appl.** 351 (2009) 509-521.
4. Jean Michel Coron, Catalin Lefter, Andreea Grigoriu, Gabriel Turinici, *Quantum control design by Lyapunov trajectory tracking for dipole and polarizability coupling*, **New Journal of Physics,** acceptată, va apare.
5. C. Ciutureanu, *Comsol modelling for a water infiltration model in an unsaturated mediu,* **Analele Ştiinţifice ale Universităţii Ovidius, Constanţa** (în curs de publicare)
6. A. Favini, G. Marinoschi, Identification of the time derivative coefficient in a fast diffusion degenerate equation, trimisă la **JOTA**, în proces de review.

Obiectivele propuse pentru această fază au fost îndeplinite. Rezultatele obţinute se regăsesc într-un număr de lucrări ştiinţifice ce sunt acceptate spre publicare sau sunt în curs de apariţie.

Rezultatele obţinute în cadrul acestui proiect, în direcţii de cercetare de actualitate pe plan mondial, sunt de nivel ştiinţific înalt şi prezintă interes deosebit, din punct de vedere teoretic şi practic. Acestea au fost prezentate în cadrul unor conferinţe şi seminarii ştiinţifice. A fost organizat de asemenea un workshop:

Mathematical approaches in optimization, modellisation and control, 7 mai 2009, Iaşi

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Urmează sinteza lucrărilor.

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KOLMOGOROV EQUATION ASSOCIATED TO THE STOCHASTIC REFLECTION PROBLEM ON A SMOOTH CONVEX SET OF A HILBERT SPACE

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We consider the stochastic reflection problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set K with nonempty interior and regular boundary Σ in a Hilbert space H. We prove the existence and uniqueness of a smooth solution for the corresponding elliptic infinite-dimensional Kolmogorov equation with Neumann boundary condition on Σ.

1. Introduction. Let us consider a stochastic differential inclusion in a Hilbert space H,

$$\begin{cases} dX(t) + \big(AX(t) + N_K(X(t))\big)\,dt \ni dW(t), \\ X(0) = x. \end{cases} \tag{1.1}$$

Here $A : D(A) \subset H \to H$ is a self-adjoint operator, $K = \{x \in H : g(x) \le 1\}$, where $g : H \to \mathbb{R}$ is convex and of class C^∞, $N_K(x)$ is the normal cone to K at x and $W(t)$ is a cylindrical Wiener process in H (see Hypothesis 1.1 for more precise assumptions). Obviously the expression in (1.1) is formal and its precise meaning should be defined.

When H is finite-dimensional a solution to (1.1) is a pair of continuous adapted processes (X, η) such that X is K-valued, η is of bounded variation with $d\eta$ concentrated on the set of times where $X(t) \in \Sigma$ (the boundary of K) and

$$X(t) + \int_0^t AX(s)\,ds + \eta(t) = x + W(t), \qquad t \ge 0,\ \mathbb{P}\text{-a.s.},$$

$$\int_0^T \big(d\eta(t), X(t) - z(t)\big) \ge 0, \qquad \mathbb{P}\text{-a.s.},$$

for all $z \in C([0, T]; K)$. The existence and uniqueness of a solution (X, η) to latter equation was first proven by Cépa in [5]. (See also [3] for a slightly different formulation.)

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Therefore, under the assumptions of [3] or [5], one can construct a transition semigroup in $C(K)$ by the usual formula

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \qquad t \geq 0,\ \varphi \in C(K).$$

The infinitesimal generator L of P_t is the Kolmogorov operator

$$L\varphi = \tfrac{1}{2}\Delta\varphi + \langle Ax, D\varphi\rangle$$

equipped with a Neumann condition at the boundary Σ of K. (See, e.g., [3], where the more general case of oblique derivative boundary conditions were also considered.)

Let us go now to the infinite-dimensional situation. In this context (1.1) was first studied by Nualart and Pardoux [18], when $H = L^2(0,1)$, A is the Laplace operator with Dirichlet or Neumann boundary conditions and K is the convex set of all nonnegative functions of $L^2(0,1)$; see also [13].

The Kolmogorov operator in this situation was described by Zambotti [21], in the space $L^2(H,\nu)$ where ν is the law of the $3D$-Bessel Bridge which coincides with the unique invariant measure of (1.1). Zambotti was able to show that the Dirichlet form

$$a(u,v) = \int_K \langle Du, Dv\rangle \, d\nu$$

is closable by proving a suitable integration by parts formula and to construct the corresponding Markov semigroup.

Except the situation mentioned above, no existence and uniqueness results for (1.1) are known for the infinite-dimensional equation (1.1). Also it was so far open the characterization of the of the domain of the corresponding Kolmogorov operator.

In this paper we shall consider a regular convex set K with nonempty interior and, though this does not cover the case considered by [21], we are able, however, to get sharp informations on the Kolmogorov generator for a quite general class of convex sets K. In this way, though we are not able to approach directly the stochastic variational problem (1.1), we can instead find a regular solution of the corresponding infinite-dimensional Kolmogorov equation equipped with the Neumann boundary condition,

$$\begin{cases} \lambda\varphi - \frac{1}{2}\operatorname{Tr}[D^2\varphi] - \langle x, AD\varphi\rangle = f, & x \in K, \\ \langle D\varphi, N_K(x)\rangle = 0, & \forall x \in \Sigma, \end{cases} \tag{1.2}$$

where $\lambda > 0$ and $f \in L^2(K,\nu)$.

In this way we obtain a Markov semigroup P_t which by the results of [16] provides a process corresponding to a martingale solution of (1.1) (see also the forthcoming paper [1]).

A basic tool we are using is a co-area formula from Malliavin; see [17] valid for g of class C^∞. Moreover, in the Appendix we present a direct proof of this formula when g is C^2 and fulfills some additional conditions which are covered in several situations, for instance when K is a ball; in that case the co-area formula was proved (1979) by Hertle [14].

Let us explain the content of this paper. As we said, we take a convex set of the form $K = \{x \in H : g(x) \le 1\}$ where $g : H \to \mathbb{R}$ is of class C^∞ and with second order derivative $D^2 g$ positive definite. Then we consider the probability measure ν given for any Borel set I of K by

$$\nu(I) = \frac{\mu(I)}{\mu(K)},$$

where μ is the Gaussian measure (corresponding to the linear problem without reflection) of mean 0 and covariance $Q = \frac{1}{2}A^{-1}$.

In Section 2, by exploiting a basic infinite-dimensional co-area formula, see [17], we are able to prove an integration by parts formula for ν. This allows us to show in Section 3 that the Dirichlet form

$$a(u, v) = \int_K \langle Du, Dv\rangle \, d\nu$$

is closable (see also [1] for a different approach). In this way, by the usual variational theory, we can define its generator N and construct the corresponding Markov transition semigroup P_t, which is reversible since N is self adjoint.

In Section 4 we study the Kolmogorov equation (1.2) by the classical method of penalization

$$\lambda\varphi_\varepsilon - \frac{1}{2}\operatorname{Tr}[D^2\varphi_\varepsilon] + \langle x, AD\varphi_\varepsilon\rangle + \frac{1}{\varepsilon}\langle x - \Pi_K(x), D\varphi_\varepsilon\rangle = f, \qquad x \in H, \tag{1.3}$$

where $\Pi_K(x)$ is the projection of x on K. We show that $\{\varphi_\varepsilon\}$ strongly converges to the solution $\varphi = (\lambda I - N)^{-1} f$ of (1.2) and that

$$D(N) \subset \biggl\{\varphi \in W^{2,2}(K, \nu) : \int_K |A^{1/2} D\varphi|^2 \, d\nu < +\infty \\ \text{and } \langle D\varphi, N_K(x)\rangle = 0 \text{ on } \Sigma\biggr\}. \tag{1.4}$$

These results seem to be new in infinite dimensions; see [2, 3, 7] for the finite-dimensional case.

Finally, Section 5 is devoted to equations of the form

$$\begin{cases} dX(t) + \bigl(AX(t) + F(X(t)) + N_K(X(t))\bigr)\,dt \ni dW(t), \\ X(0) = x, \end{cases} \tag{1.5}$$

where $F : H \to H$ is a nonlinear perturbation of A.

In Section 5.1 we assume that $F = DV$ where $V : H \to \mathbb{R}$ is a regular potential. This case is an easy generalization of the previous one (i.e., when $F = 0$), namely measure ν is replaced by the following one:

$$\zeta(dx) = \frac{e^{-2V(x)}}{\int_K e^{-2V(y)}\nu(dy)} \nu(dx).$$

This extension is briefly described in that section.

In Section 5.2 the case of a bounded Borel function F, not necessarily of potential type, is considered. Here we can solve the Kolmogorov equation

$$(1.6)\qquad \begin{cases} \lambda\varphi - \frac{1}{2}\operatorname{Tr}[D^2\varphi] + \langle x, AD\varphi\rangle - \langle F(x), D\varphi\rangle = f, & x \in K, \\ \langle D\varphi, N_K(x)\rangle = 0, & \forall x \in \Sigma \end{cases}$$

by a straightforward perturbation argument, taking avantage of the inclusion (1.4). In this way we obtain a solution $\varphi \in D(N)$ of (1.6) only for λ sufficiently large. Also, obviously, measure ν is not invariant for the corresponding semigroup Q_t. However, using the fact that operator Q_t is compact in $L^2(K, \nu)$, we can show the existence of an invariant measure ζ for Q_t so that the extension of Q_t to $L^1(K, \zeta)$ is the natural transition semigroup associated with (1.5). Notice, however, that this semigroup is not reversible (when F is not of potential type).

We conclude this section by precising assumptions and notation which will be used throughout in what follows.

Assumptions. We are given a real separable Hilbert space H (with scalar product $\langle\cdot,\cdot\rangle$ and norm denoted by $|\cdot|$). Concerning A, K and W we shall assume that:

HYPOTHESIS 1.1. (i) $A : D(A) \subset H \to H$ is a linear self-adjoint operator on H such that $\langle Ax, x\rangle \geq \delta|x|^2$, $\forall x \in D(A)$ for some $\delta > 0$. Moreover, A^{-1} is of trace class.

(ii) There exists a convex C^∞ function $g : H \to \mathbb{R}$ with $D^2 g$ positively defined, that is, $\langle D^2 g(x)h, h\rangle \geq \gamma|h|^2$, $\forall h \in H$ where $\gamma > 0$, such that

$$K = \{x \in H : g(x) \leq 1\}, \qquad \Sigma = \{x \in H : g(x) = 1\}.$$

(iii) W is a cylindrical Wiener process on H of the form

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k, \qquad t \geq 0,$$

where $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions on a filtered probability spaces $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ (see, e.g., [8]) and $\{e_k\}$ is an orthonormal basis in H which will be taken as a system of eigen-functions for A for simplicity, that is,

$$Ae_k = \alpha_k e_k \qquad \forall k \in \mathbb{N},$$

where $\alpha_k \geq \delta$.

We notice that the interior $\mathring{K}$ is nonempty since $D^2 g$ is positive definite.

Notation. We denote by $\mathcal{B}(H)$ [resp. $\mathcal{B}(K)$] the σ-field of all Borel subsets of H (resp. K) and by $\mathcal{P}(H)$ [resp. $\mathcal{P}(K)$] the set of all probability measures on

$(H, \mathcal{B}(H))$ [resp. $(K, \mathcal{B}(K))$].

Everywhere in the following $D\varphi$ is the derivative of a function $\varphi : H \to \mathbb{R}$. By $D^2\varphi : H \to L(H, H)$ we shall denote the second derivative of φ. We shall denote also by $C_b(H)$ and $C_b^k(H)$, $k \in \mathbb{N}$, the spaces of all continuous and bounded functions on H and, respectively, of k-times differentiable functions with continuous and bounded derivatives. The space $C^k(K)$, $k \in \mathbb{N}$, is defined as the space of restrictions of functions of $C_b^k(H)$ to the subset K.

The boundary of K will be denoted by Σ. $N_K(x)$ is the normal cone to K at x, that is,

$$N_K(x) = \{z \in H : \langle z, y - x\rangle \le 0, \forall y \in K\}.$$

Moreover, we shall denote by $d_K(x)$ the distance of x from K and by I_K the indicator function of K,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if } x \notin K. \end{cases}$$

For any $\varepsilon > 0$, U_ε will represent the Moreau approximation of I_K given by

$$U_\varepsilon(x) = \inf\left\{ I_K(y) + \frac{1}{2\varepsilon}|x - y|^2, y \in H \right\} = \frac{1}{2\varepsilon} d_K(x)^2, \qquad x \in H.$$

It is well known that

$$DU_\varepsilon(x) = \frac{1}{\varepsilon}\big(x - \Pi_K(x)\big), \qquad x \in H, \varepsilon > 0,$$

where $\Pi_K(x)$ is the projection of x over K. In particular, we have

$$D(d_K^2(x)) = x - \Pi_K(x) \qquad \forall x \in K^c, \tag{1.7}$$

(K^c is the complement of K) which implies

$$Dd_K(x) = \frac{x - \Pi_K(x)}{d_K(x)} \qquad \forall x \in K^c. \tag{1.8}$$

We denote by $\mathbf{n}(\Pi_K(x))$ the exterior normal at $\Pi_K(x)$,

$$\mathbf{n}(\Pi_K(x)) = \frac{x - \Pi_K(x)}{d_K(x)} \qquad \forall x \in K^c.$$

From (1.8) we deduce that

$$D\big(x - \Pi_K(x)\big) = Dd_K(x) \otimes Dd_K(x) + d_K(x) D^2 d_K(x) \qquad \forall x \in K^c. \tag{1.9}$$

Finally, μ will represent the Gaussian measure in H with mean 0 and covariance operator

$$Q := \tfrac{1}{2}A^{-1}.$$

Since A is strictly positive μ is nondegenerate and full. We set

$$\lambda_k = \frac{1}{2\alpha_k} \qquad \forall k \in \mathbb{N},$$

so that

$$Qe_k = \lambda_k e_k \qquad \forall k \in \mathbb{N}.$$

We denote by $\mathcal{E}_A(H)$ the space of all real and imaginary parts of exponential functions $e^{\mathrm{i}\langle h,x\rangle}, h \in D(A)$. Then the operator $D\colon \mathcal{E}_A(H) \subset L^2(H,\mu) \to L^2(H,\mu;H)$ is closable in $L^2(H,\mu)$ and the domain of its closure is denoted by $W^{1,2}(H,\mu)$ (the Sobolev space).

The following integration by parts formula for the measure μ is well known (see, e.g., [9]). For any $\varphi, \psi \in W^{1,2}(H,\mu)$ and $z \in H$,

$$\int_H \langle D\varphi, Q^{1/2}z\rangle \psi \, d\mu = -\int_H \langle D\psi, Q^{1/2}z\rangle \varphi \, d\mu + \int_H W_z \varphi\psi \, d\mu, \tag{1.10}$$

where W_z represents the *white noise* function,

$$W_z(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \langle x, e_k\rangle \langle z, e_k\rangle \qquad \forall z \text{ and } \mu\text{-a.e. } x \in H. \tag{1.11}$$

We recall that W_z is a Gaussian random variable in $L^2(H,\mu)$ with mean 0 and covariance $|z|^2$.

2. The measure μ conditioned to K.

We denote by ν the Gaussian measure μ conditioned to K, that is,

$$\nu(I) = \frac{\mu(K \cap I)}{\mu(K)} \qquad \forall I \in \mathcal{B}(H).$$

Since μ is full and $\mathring{K}$ is nonempty, this definition is meaningful. We notice that, thanks to Hypothesis 1.1(ii) the surface measure μ_Σ is well defined (see [17]).

We want now to prove an integration by parts formula with respect to measure ν which generalizes (1.10). For this it is convenient to introduce a sequence of approximating measures $\{\nu_\varepsilon\}_{\varepsilon>0}$ defined by,

$$\nu_\varepsilon(dx) = \rho_\varepsilon(x)\mu(dx), \qquad x \in H, \tag{2.1}$$

where,

$$\rho_\varepsilon(x) = Z_\varepsilon^{-1} e^{-1/\varepsilon d_K^2(x)} \tag{2.2}$$

and

$$Z_\varepsilon = \int_H e^{-1/\varepsilon d_K^2(y)} \mu(dy). \tag{2.3}$$

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Probability Theory

Finite time extinction for solutions to fast diffusion stochastic porous media equations

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Abstract

We prove that the solutions to fast diffusion stochastic porous media equations have finite time extinction with strictly positive probability. ***To cite this article: V. Barbu et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Extinction en temps fini pour les solutions des équations des milieu poreux avec diffusion rapide. Nous prouvons l'extinction avec une probabilité strictement positive pour les solutions des équations des milieux poreux avec diffusion rapide. ***Pour citer cet article : V. Barbu et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Consider the stochastic porous media equation

$$\begin{cases} \mathrm{d}X(t) - \rho\Delta(|X|^{\alpha}(t)\,\mathrm{sign}\,X(t))\,\mathrm{d}t - \Delta(\tilde{\Psi}(X(t))\,\mathrm{d}t = \sigma(X(t))\,\mathrm{d}W(t), \quad \text{in } (0,\infty)\times\mathcal{O}, \\ X = 0 \quad \text{on } (0,\infty)\times\partial\mathcal{O}, \quad X(0,x) = x \quad \text{on } \mathcal{O}, \end{cases} \tag{1}$$

where $\rho > 0$, $\alpha \in (0,1)$, $\tilde{\Psi}$ is a continuous monotonically nondecreasing function of linear growth and $\sigma(X)\,\mathrm{d}W = \sum_{k=1}^{\infty} \mu_k X e_k\,\mathrm{d}\beta_k$, $t \geqslant 0$, where $\{\beta_k\}$ is a sequence of independent real Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and $\{e_k\}$ is an orthonormal basis in $L^2(\mathcal{O})$ which for convenience will be taken as the eigenfunction system for the Laplace operator with Dirichlet boundary conditions, i.e., $-\Delta e_k = \lambda_k e_k$ in $\mathcal{O}$, $e_k = 0$ on $\partial\mathcal{O}$, where $\mathcal{O}$ is an open and bounded subset of $\mathbb{R}^d$, with smooth boundary $\partial\mathcal{O}$. We shall assume that $\sum_{k=1}^{\infty} \mu_k^2 \lambda_k^2 < \infty$. Eq. (1) for $0 < \alpha < 1$ is relevant in the mathematical modelling of the dynamics of an ideal gas in

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a porous medium and, in particular, in a plasma fast diffusion model (for $\alpha = 1/2$) (see e.g. [4]). The existence and uniqueness of a strong solution in the sense to be defined below was studied in [1–3,5] for more general nonlinear stochastic equations of the form (1). In [3] (see also [1]) it was also proven that for $\alpha = 0$ and $d = 1$ the solution $X = X(t,x)$ to (1) has the finite extinction property: $\mathbb{P}(\tau \leqslant n) \geqslant 1 - \frac{|x|_{-1}}{\rho\gamma}(\int_0^n \mathrm{e}^{-C_N s}\,\mathrm{d}s)^{-1}$ for $|x|_{-1} < C_N^{-1}\rho\gamma$ where $\tau = \inf\{t \geqslant 0\colon |X(t,x)|_{-1} = 0\} = \sup\{t \geqslant 0\colon |X(t,x)|_{-1} > 0\}$ and C_N, γ are constants related to the Wiener process W and respectively to the domain $\mathcal{O} \subset \mathbb{R}^1$.

The following notations will be used in the sequel. $H = L^2(\mathcal{O})$, $p \geqslant 1$, with the norm denoted by $|\cdot|_2$ and scalar product $\langle \cdot,\cdot \rangle$. $H^{-1}(\mathcal{O})$ is the dual of the Sobolev space $H_0^1(\mathcal{O})$ and is endowed with the scalar product $\langle u,v\rangle_{-1} = \langle u, (-\Delta)^{-1}v\rangle$, where Δ is the Laplace operator with domain $H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. All processes $X = X(t)$ arising here are adapted with respect to the filtration $\{\mathcal{F}_t\}$. For a Banach space E, $L_W^p(0,T;E)$ denotes the space of all adapted processes in $L^p(0,T;E)$. We shall use standard notation for Sobolev spaces and spaces of integrable functions on $\mathcal{O}$.

2. The main result

Definition 2.1. Let $x \in H$. An H-valued continuous $(\mathcal{F}_t)$-adapted process $X = X(t,x)$ is called a solution to (1) on $[0,T]$ if $X \in L^p(\Omega \times (0,T) \times \mathcal{O}) \cap L^2(0,T;L^2(\Omega,H))$, $p \geqslant 2$, such that $\mathbb{P}$-a.s. $\forall j \in \mathbb{N}$, $t \in [0,T]$,

$$\begin{aligned}\big\langle X(t,x), e_j\big\rangle &= \langle x, e_j\rangle + \int_0^t\int_{\mathcal{O}} \big(\rho\big|X(s,x)(\xi)\big|^\alpha \operatorname{sign} X(s,x)(\xi) + \tilde{\Psi}\big(X(s,x)(\xi)\big)\big)\Delta e_j(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\ &\quad + \sum_{k=1}^\infty \mu_k \int_0^t \big\langle X(s,x)e_k, e_j\big\rangle\,\mathrm{d}\beta_k(s). \end{aligned} \tag{2}$$

For $x \in L^p(\mathcal{O})$, $p \geqslant 4$ and $d = 1,2,3$ there is a unique solution $X \in L_W^\infty(0,T;L^p(\Omega,H))$ to (1) in the sense of Definition 2.1. Moreover, if $x \geqslant 0$ a.e. in $\mathcal{O}$ then $X \geqslant 0$ a.e. in $\Omega \times [0,T] \times \mathcal{O}$.

By the proof of [3, Theorem 2.2] and [3, Proposition 3.4] we also know that for $\lambda \to 0$,

$$\begin{cases} X_\lambda \to X \text{ strongly both in } L^2(0,T;L^2(\Omega,L^2(\mathcal{O}))) \text{ and in } L^2(\Omega;C([0,T];H)), \\ \text{weakly in } L^p(\Omega\times(0,T)\times\mathcal{O}), \text{ and weak* in } L^\infty(0,T;L^p(\Omega;L^p(\mathcal{O}))), \end{cases} \tag{3}$$

where X_λ, $\lambda > 0$, is the solution to approximating equation

$$\begin{cases} \mathrm{d}X_\lambda(t) - \Delta(\Psi_\lambda(X_\lambda(t)) + \lambda X_\lambda(t) + \tilde{\Psi}(X_\lambda(t)))\,\mathrm{d}t = \sigma(X_\lambda(t))\,\mathrm{d}W(t), \\ \Psi_\lambda(X_\lambda) + \lambda X_\lambda + \tilde{\Psi}(X_\lambda) = 0 \quad \text{on } \partial\mathcal{O}, \quad X_\lambda(0,x) = x, \\ \Psi_\lambda(x) = \frac{1}{\lambda}(x - (1+\lambda\Psi_0)^{-1}(x)) = \Psi_0((1+\lambda\Psi_0)^{-1}(x)), \quad \Psi_0(x) = \rho|x|^\alpha \operatorname{sign} x. \end{cases} \tag{4}$$

Everywhere in the sequel $X = X(t,x)$ is the solution to (1) in the sense of Definition 2.1 where $x \in L^4(\mathcal{O})$. Below γ shall denote the minimal constant arising in the Sobolev embedding $L^{\alpha+1}(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ (see (7) below) and $C^* = \sum_{k=1}^\infty \mu_k^2 |e_k|^2_{H_0^1(\mathcal{O})} = \sum_{k=1}^\infty \mu_k^2\lambda_k^2$. Theorem 2.2 is the main result of the paper.

Theorem 2.2. *Assume that $d = 1,2,3$ and that $0 < \alpha < 1$ if $d = 1,2$, $\frac{1}{5} \leqslant \alpha < 1$ if $d = 3$. Let $\tau := \inf\{t \geqslant 0\colon |X(t,x)|_{-1} = 0\}$. Then we have $|X(t,x)|_{-1} = 0$, for $t \geqslant \tau$, $\mathbb{P}$-a.s. Furthermore*

$$\mathbb{P}(\tau \leqslant t) \geqslant 1 - \frac{|x|_{-1}^{1-\alpha}}{(1-\alpha)\rho\gamma^{1+\alpha}}\left(\int_0^t \mathrm{e}^{-C^*(1-\alpha)s}\,\mathrm{d}s\right)^{-1}.$$

In particular, if $|x|_{-1}^{1-\alpha} < \rho\gamma^{1+\alpha}/C^*$, then $\mathbb{P}(\tau < \infty) > 0$, and if $C^* = 0$, then $\tau \leqslant |x|_{-1}^{1-\alpha}/((1-\alpha)\rho\gamma^{1+\alpha})$.

Remark 1. This result extends to $\mathcal{O} \subset \mathbb{R}^d$ with $d \geqslant 4$, if $\alpha \in [\frac{d-2}{d+2}, 1)$. However, we have to strengthen the assumption on μ_k, $k \in \mathbb{N}$, see [1, Section 4] and in particular [6, Remark 2.9(iii)] for a detailed discussion.

3. Proof of Theorem 2.2

We shall proceed as in the proof of [3, Theorem 4.2]. Consider the solution $X_\lambda \in L^2_W(0,T;L^2(\Omega;H^1_0(\mathcal{O})))$ to Eq. (4). Then by applying the classical Itô formula to the real valued semi-martingale $|X_\lambda(t)|^2_{-1}$, $t\in[0,T]$, and to the function $\varphi_\varepsilon(r)=(r+\varepsilon^2)^{(1-\alpha)/2}$, $r\in\mathbb{R}$, we find that

$$\begin{aligned}
&\mathrm{d}\varphi_\varepsilon\big(\big|X_\lambda(t)\big|^2_{-1}\big)+(1-\alpha)\big(\big|X_\lambda(t)\big|^2_{-1}+\varepsilon^2\big)^{-(1+\alpha)/2}\big\langle X_\lambda(t),\Psi_\lambda\big(X_\lambda(t)\big)+\lambda X_\lambda(t)+\tilde{\Psi}_\lambda\big(X_\lambda(t)\big)\big\rangle\,\mathrm{d}t\\
&\quad=\frac12\sum_{k=1}^\infty \mu_k^2(1-\alpha)\frac{|X_\lambda(t)e_k|^2_{-1}(|X_\lambda(t)|^2_{-1}+\varepsilon^2)-(1-\alpha)^2|\langle X_\lambda(t)e_k,X_\lambda(t)\rangle_{-1}|^2)}{(|X_\lambda(t)|^2_{-1}+\varepsilon^2)^{(3+\alpha)/2}}\,\mathrm{d}t\\
&\qquad+\big\langle\sigma\big(X_\lambda(t)\big)\,\mathrm{d}W(t),\varphi'_\varepsilon\big(\big|X_\lambda(t)\big|^2_{-1}\big)X_\lambda(t)\big\rangle_{-1}\\
&\quad\leqslant\frac12\sum_{k=1}^\infty\mu_k^2\frac{(1-\alpha)|X_\lambda(t)e_k|^2_{-1}}{(|X_\lambda(t)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,\mathrm{d}t+\big\langle\sigma\big(X_\lambda(t)\big)\,\mathrm{d}W(t),\varphi'_\varepsilon\big(\big|X_\lambda(t)\big|^2_{-1}\big)X_\lambda(t)\big\rangle_{-1}\\
&\quad\leqslant C^*\frac{(1-\alpha)|X_\lambda(t)e_k|^2_{-1}}{(|X_\lambda(t)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,\mathrm{d}t+\big\langle\sigma\big(X_\lambda(t)\big)\,\mathrm{d}W(t),\varphi'_\varepsilon\big(\big|X_\lambda(t)\big|^2_{-1}\big)X_\lambda(t)\big\rangle_{-1}.
\end{aligned}\tag{5}$$

Then letting $\lambda\to0$, by (3) we get that $\liminf_{\lambda\to0}\int_0^T\langle\Psi_\lambda(X_\lambda(t)),X_\lambda(t)\rangle\,\mathrm{d}t\geqslant\rho\int_0^T|X(t)|^{1+\alpha}_{L^{1+\alpha}(\mathcal{O})}\,\mathrm{d}t$, $\mathbb{P}$-a.s. and hence

$$\begin{aligned}
&\varphi_\varepsilon\big(\big|X(t)\big|^2_{-1}\big)+(1-\alpha)\rho\int_r^t\frac{|X(s)|^{\alpha+1}_{L^{\alpha+1}(\mathcal{O})}}{(|X(s)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,\mathrm{d}s\leqslant\varphi_\varepsilon\big(\big|X(r)\big|^2_{-1}\big)\\
&\quad+C^*\int_r^t\frac{(1-\alpha)|X(s)|^2_{-1}}{(|X(s)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,\mathrm{d}s+2\int_r^t\big\langle\sigma\big(X(s)\big)\,\mathrm{d}W(s),\varphi'_\varepsilon\big(\big|X(s)\big|^2_{-1}\big)X(s)\big\rangle_{-1},\quad\mathbb{P}\text{-a.s.},\ r<t.
\end{aligned}\tag{6}$$

Next by the Sobolev embedding theorem we have

$$|u|_{-1}\leqslant\gamma|u|_{L^{\alpha+1}(\mathcal{O})},\quad\forall u\in L^{\alpha+1}(\mathcal{O}),\quad\text{if } d>2 \text{ and } \alpha\geqslant\tfrac{d-2}{d+2}, \text{ and } \forall\alpha>0, \text{ if } d=1,2.\tag{7}$$

Then substituting (7) into (6) we get

$$\begin{aligned}
&\varphi_\varepsilon\big(\big|X(t)\big|^2_{-1}\big)+(1-\alpha)\rho\gamma^{1+\alpha}\int_r^t\frac{|X(s)|^{\alpha+1}_{-1}}{(|X(s)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,\mathrm{d}s\leqslant\varphi_\varepsilon\big(\big|X(r)\big|^2_{-1}\big)\\
&\quad+C^*\int_r^t\frac{(1-\alpha)|X(s)|^2_{-1}}{(|X(s)|^2_{-1}+\varepsilon^2)^{(1+\alpha)/2}}\,\mathrm{d}s+\int_r^t\big\langle\sigma\big(X(s)\big)\,\mathrm{d}W(s),\varphi'_\varepsilon\big(\big|X(s)\big|^2_{-1}\big)X(s)\big\rangle_{-1},\quad\mathbb{P}\text{-a.s.},\ r<t.
\end{aligned}\tag{8}$$

Now for $\epsilon\to0$ we have

$$\begin{aligned}
&\big|X(t)\big|^{1-\alpha}_{-1}+(1-\alpha)\rho\gamma^{1+\alpha}\int_r^t 1_{\{|X(s)|_{-1}>0\}}\,\mathrm{d}s\leqslant\big|X(r)\big|^{1-\alpha}_{-1}+C^*(1-\alpha)\int_r^t\big|X(s)\big|^{1-\alpha}_{-1}\,\mathrm{d}s\\
&\quad+(1-\alpha)\int_r^t\big\langle\sigma\big(X(s)\big)\,\mathrm{d}W(s),\big|X(s)\big|^{-(\alpha+1)}_{-1}X(s)\big\rangle_{-1},\quad\mathbb{P}\text{-a.s.},\ r<t.
\end{aligned}$$

Hence by Itô's product rule

$$\begin{aligned}
&\mathrm{e}^{-C^*(1-\alpha)t}\big|X(t)\big|^{1-\alpha}_{-1}+(1-\alpha)\rho\gamma^{1+\alpha}\int_r^t\mathrm{e}^{-C^*(1-\alpha)s}1_{\{|X(s)|_{-1}>0\}}\,\mathrm{d}s\\
&\quad\leqslant\mathrm{e}^{-C^*(1-\alpha)r}\big|X(r)\big|^{1-\alpha}_{-1}+(1-\alpha)\int_r^t\mathrm{e}^{-C^*(1-\alpha)s}\big\langle\sigma\big(X(s)\big)\,\mathrm{d}W(s),\big|X(s)\big|^{-(\alpha+1)}_{-1}X(s)\big\rangle_{-1},\quad\mathbb{P}\text{-a.s.},\ r<t.
\end{aligned}\tag{9}$$

From this it immediately follows that $e^{-C^*(1-\alpha)t}|X(t)|_{-1}^{1-\alpha}$, $t \geqslant 0$, is an $(\mathcal{F}_t)$-supermartingale, hence $|X(t)|_{-1} = 0$ for all $t \geqslant \tau$. So, (9) with $r = 0$ after taking expectation implies that $\int_0^t e^{-C^*(1-\alpha)s}\mathbb{P}(\tau > s)\,ds \leqslant |x|_{-1}^{1-\alpha}/((1-\alpha)\rho\gamma^{1+\alpha})$, $t \geqslant 0$. This implies that $\mathbb{P}(\tau > t) \leqslant |x|_{-1}^{1-\alpha}/((1-\alpha)\rho\gamma^{1+\alpha})(\int_0^t e^{-C^*(1-\alpha)s}\,ds)^{-1}$, $t \geqslant 0$, and the assertion follows.

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Periodic behavior for a degenerate fast diffusion equation ☆

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ABSTRACT

This work deals with the study of periodic solutions to a degenerate fast diffusion equation. The existence of the periodic solution to an intermediate problem restraint to a period T is proved first and then the result is extended by the data periodicity to all time real space. The approach involves an appropriate approximating problem whose periodic solution is proved via a fixed point theorem. Next, a passing to the limit procedure leads to the existence of the solution to the original problem on a time period. Finally, the behavior at large time of the solution to a Cauchy problem with periodic data is characterized.

1. Statement of the problem

Periodic problems for possibly degenerate equations of the type

$$\frac{d}{dt}\big(My(t)\big) + Ly(t) = f(t), \quad 0 \leqslant t \leqslant 1, \tag{1.1}$$

with the periodic condition

$$(My)(0) = (My)(1) \tag{1.2}$$

have been studied in the paper [2], for L and M two closed linear operators from a complex Banach space into itself, under the assumptions that the domain $D(L)$ of L is continuously embedded in $D(M)$ and L has a bounded inverse. Assuming suitable hypotheses on the modified resolvent $(\lambda M + L)^{-1}$, it has been proved that problem (1.1)–(1.2) admits one 1-periodic solution. Some examples of applications to partial differential equations and ordinary differential equations have been given. The latter case has been studied in the paper [3], too.

In this paper we shall approach a concrete PDE problem (1.1)–(1.2) where L is a nonlinear multivalued operator.

We consider Ω an open bounded subset of $\mathbf{R}^N$ ($N \in \mathbf{N}^* = \{1, 2, \ldots\}$), with the boundary $\Gamma := \partial\Omega$ of class C^1 and denote the space variable by $x := (x_1, \ldots, x_N) \in \Omega$ and the time by $t \in \mathbf{R}$. We are concerned with the study of periodic solutions to a nonlinear model consisting in a degenerate diffusion equation with homogeneous Dirichlet boundary conditions

$$\frac{\partial(m(x)u)}{\partial t} - \Delta\beta^*(u) \ni f \quad \text{in } \Omega \times \mathbf{R},$$
$$u(x,t) = 0 \quad \text{on } \Gamma \times \mathbf{R},$$

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$$u(x,t) = u(x,t+T) \quad \text{in } \Omega \times \mathbf{R}, \tag{1.3}$$

under the assumption of the T-periodicity of the function f,

$$f(x,t) = f(x,t+T) \quad \text{for } (x,t) \in \Omega \times \mathbf{R},\ 0 < T < \infty. \tag{1.4}$$

In this problem $\beta^* : (-\infty, u_s] \to \mathbf{R}$ is a multivalued function defined as

$$\beta^*(r) := \begin{cases} \int_0^r \beta(\xi)\,d\xi, & \text{if } r < u_s, \\ [K_s^*, +\infty), & \text{if } r = u_s, \end{cases} \tag{1.5}$$

where $\beta : (-\infty, u_s) \to \mathbf{R}$ is a positive differentiable, monotonically increasing function, which blows up at $r = u_s$, but having the integral finite at this point. Namely we set

$$\beta(r) \geqslant \rho > 0, \quad \text{for each } r < u_s, \qquad \beta(r) := \rho \quad \text{for } r \leqslant 0, \tag{1.6}$$

$$\lim_{r \nearrow u_s} \beta(r) = +\infty \quad \text{and} \quad \lim_{r \nearrow u_s} \int_0^r \beta(\xi)\,d\xi = K_s^*. \tag{1.7}$$

Consequently, β^* has the properties

$$\big(\beta^*(r) - \beta^*(\bar{r})\big)(r - \bar{r}) \geqslant \rho(r - \bar{r})^2, \quad \text{for every } r, \bar{r} \in (-\infty, u_s], \tag{1.8}$$

$$\lim_{r \to -\infty} \beta^*(r) = -\infty, \tag{1.9}$$

$$\lim_{r \nearrow u_s} \beta^*(r) = K_s^*. \tag{1.10}$$

In the above relationships ρ, u_s and K_s^* are positive known constants and the hypotheses (1.7) reveal the character of fast diffusion (see [1,4]).

We also notice that $(\beta^*)^{-1} : \mathbf{R} \to (-\infty, u_s]$ is single-valued, monotonically increasing on $(-\infty, K_s^*)$ and constant for $r \in [K_s^*, +\infty)$, i.e., $(\beta^*)^{-1}(r) = u_s$. Also, it follows that $(\beta^*)^{-1}$ is Lipschitz with the constant $\frac{1}{\rho}$.

We still assume that

$$m \in C^1(\overline{\Omega}), \qquad 0 \leqslant m(x) \leqslant 1, \quad x \in \overline{\Omega}. \tag{1.11}$$

More exactly, we consider that the degeneration of the equation may occur on $\overline{\Omega_0}$, where Ω_0 is an open bounded subset of Ω, strictly contained in Ω. The upper bound of m can be taken any positive constant, but by rescaling, we may consider it equal to 1, without any loss of generality.

The model (1.3) with initial data ($m(x)u(x,0) = v_0(x)$ instead of the periodic condition) was studied in [4] where it was proved that it has a unique weak solution in appropriate functional spaces. In fact, the model was introduced in [7] and it describes for example the water infiltration in a unsaturated porous medium in which saturation can occur. This event is mathematically modeled by both the blow-up of the function β at u_s and the multivalued function β^*. The function $m(x)$ characterizes the space variable porosity of the nonhomogeneous medium, while the vanishing of m indicates the existence of impermeable intrusions in the soil.

A study of the periodic solutions to fast diffusion equations with $m(x) = 1$ was done in [8] for the case with a nonlinear convection, in connection with some results given in [6].

The paper is organized as follows: first we shall prove that the problem

$$\begin{aligned} &\frac{\partial(m(x)u)}{\partial t} - \Delta\beta^*(u) \ni f \quad \text{in } \Omega \times \mathbf{R}, \\ &u(x,t) = 0 \quad \text{on } \Gamma \times \mathbf{R}, \\ &m(x)\big(u(x,t) - u(x,t+T)\big) = 0 \quad \text{in } \Omega \times \mathbf{R} \end{aligned} \tag{1.12}$$

has a unique solution.

In order to prove the existence for problem (1.12) we shall establish the existence for the solution to the problem on a time period

$$\begin{aligned} &\frac{\partial(m(x)u)}{\partial t} - \Delta\beta^*(u) \ni f \quad \text{in } Q := \Omega \times (0,T), \\ &u(x,t) = 0 \text{ on } \Sigma := \Gamma \times (0,T), \\ &m(x)\big(u(x,0) - u(x,T)\big) = 0 \quad \text{in } \Omega. \end{aligned} \tag{1.13}$$

This will be done by a fixed point argument in Section 2. The result obtained for (1.13) will be extended by periodicity to all $t \in \mathbf{R}$ and the longtime behavior of the solution corresponding to a periodic f and a certain initial datum v_0 will be established in connection with the periodic solution to (1.12).

Finally, we shall show that the existence of the unique periodic solution to (1.12) implies the existence of the unique periodic solution to (1.3).

Functional framework and preliminaries. For approaching the problems previously specified we shall consider the Hilbert space $V = H_0^1(\Omega)$ with the usual Hilbertian norm and its dual $V' = H^{-1}(\Omega)$, endowed with the scalar product $(u,\bar{u})_{V'} := \langle u,\psi\rangle_{V',V}$, where $\psi \in V$ satisfies $-\Delta\psi = \bar{u}$, $\psi = 0$ on Γ, and $\langle u,\psi\rangle_{V',V}$ is the pairing between V' and V.

For simplicity, we shall denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and the norm in $L^2(\Omega)$, respectively.

Definition 1.1. Let

$$m \in C^1(\overline{\Omega}), \qquad f \in L^\infty(0,T;V'). \tag{1.14}$$

We call a *solution* to (1.13) a function u which satisfies

$$\begin{aligned} &u \in L^2(0,T;V), \qquad u \leqslant u_s, \quad \text{a.e. } (x,t) \in Q, \\ &mu \in C\big([0,T];L^2(\Omega)\big) \cap W^{1,2}(0,T;V'), \\ &\zeta \in L^2(0,T;V), \qquad \zeta(x,t) \in \beta^*\big(u(x,t)\big), \quad \text{a.e. } (x,t) \in Q, \end{aligned} \tag{1.15}$$

the condition $m(x)(u(x,0) - u(x,T)) = 0$ in Ω and the equation

$$\int_0^T \left\langle \frac{d(m(x)u)}{dt}(t),\phi(t)\right\rangle_{V',V} dt + \int_Q \nabla\zeta(x,t)\cdot\nabla\phi(x,t)\,dx\,dt = \int_0^T \big\langle f(t),\phi(t)\big\rangle_{V',V}\,dt, \quad \text{a.e. } t \in (0,T), \tag{1.16}$$

for each $\phi \in L^2(0,T;V)$, where $\zeta(x,t) \in \beta^*(u(x,t))$, a.e. $(x,t) \in Q$.

On the domain

$$D(A) := \big\{u \in L^2(\Omega);\ \text{there exists } \eta \in V,\ \text{such that } \eta(x) \in \beta^*\big(u(x)\big),\ \text{a.e. } x \in \Omega\big\}$$

we define the multivalued operator $A : D(A) \subset V' \to V'$ by

$$\langle Au,\psi\rangle_{V',V} := \int_\Omega \nabla\eta\cdot\nabla\psi\,dx, \quad \text{for each } \psi \in V,\ \text{where } \eta(x) \in \beta^*\big(u(x)\big),\ \text{a.e. } x \in \Omega.$$

We remark that $u \in D(A)$ implies $u \in V$, due to the Lipschitz property of the inverse of β^*.

Next, we introduce the multiplication operator $M : D(A) \to L^2(\Omega)$, $Mu := mu$, whose inverse is multivalued. Thus, we can write the abstract problem

$$\frac{d}{dt}\big(Mu(t)\big) + Au(t) \ni f(t), \quad \text{a.e. } t \in (0,T), \tag{1.17}$$

$$M\big(u(0) - u(T)\big) = 0 \tag{1.18}$$

and notice that the solution to (1.17)–(1.18) is a solution to (1.13) in the sense of Definition 1.1.

Denoting $v(x,t) := m(x)u(x,t)$ we can rewrite (1.17)–(1.18) in terms of v as,

$$\begin{aligned} &\frac{dv}{dt} + A_M v \ni f, \quad \text{a.e. } t \in (0,T), \\ &v(0) = v(T), \end{aligned} \tag{1.19}$$

where $A_M = AM^{-1}$ and

$$D(A_M) := \left\{v \in L^2(\Omega);\ \frac{v}{m} \in L^2(\Omega),\ \exists\zeta \in V,\ \zeta(x) \in \beta^*\left(\frac{v}{m}(x)\right),\ \text{a.e. } x \in \Omega\right\}.$$

We easily see that $v \in D(A_M)$ if and only if $u = \frac{v}{m} \in D(A)$.

For a later use we define $j : \mathbf{R} \to (-\infty,+\infty]$ by

$$j(r) := \begin{cases} \int_0^r \beta^*(\xi)\,d\xi, & r \leqslant u_s, \\ +\infty, & r > u_s, \end{cases} \tag{1.20}$$

where the left limit of β^* at u_s was specified in (1.10). The function j is proper, convex, lower semicontinuous and

$$\partial j(r) = \begin{cases} \beta^*(r), & r < u_s, \\ [K_s^*,+\infty), & r = u_s, \\ \emptyset, & r > u_s \end{cases} \tag{1.21}$$

(see [7, p. 166]).

Also, we recall a result proved in [4] (see Theorem 3.2) related to the problem

$$\begin{aligned} &\frac{\partial(m(x)u)}{\partial t} - \Delta\beta^*(u) \ni f \quad \text{in } Q, \\ &u(x,t) = 0 \quad \text{on } \Sigma, \\ &m(x)u(x,0) = v_0 \quad \text{in } \Omega. \end{aligned} \tag{1.22}$$

Theorem 1.2. *Let*

$$m \in C^1(\overline{\Omega}), \qquad f \in L^2(0,T;V'), \qquad \frac{v_0}{m} \in L^2(\Omega), \qquad \frac{v_0}{m} \leqslant u_s, \quad \text{a.e. } x \in \Omega.$$

Then, the Cauchy problem (1.22) *has a unique solution u, such that*

$$\begin{aligned} &mu \in C\big([0,T];L^2(\Omega)\big) \cap W^{1,2}(0,T;V'), \\ &\beta^*(u) \in L^2(0,T;V), \\ &u \in L^2(0,T;V), \qquad u \leqslant u_s, \quad \text{a.e. } (x,t) \in Q. \end{aligned}$$

2. Existence on the time period $(0,T)$

In this section we shall study the existence of the solution to the problem (1.13) defined on the time period $(0,T)$. To this end we shall establish first an existence result for the approximate problem obtained by replacing m by

$$m_\varepsilon(x) := m(x) + \varepsilon, \quad \text{where } \varepsilon \leqslant m_\varepsilon(x) \leqslant 1 + \varepsilon$$

and β^* by the single-valued function $\beta^*_\varepsilon : \mathbf{R} \to \mathbf{R}$,

$$\beta^*_\varepsilon(r) := \begin{cases} \beta^*(r), & \text{if } r < u_s - \varepsilon, \\ \beta^*(u_s - \varepsilon) + \frac{K_s^* - \beta^*(u_s-\varepsilon)}{\varepsilon}[r - (u_s - \varepsilon)], & \text{if } r \geqslant u_s - \varepsilon, \end{cases} \tag{2.1}$$

for each positive ε. The function β^*_ε is continuous and monotonically increasing on $\mathbf{R}$, differentiable on $\mathbf{R} \setminus \{u_s - \varepsilon\}$, but with lateral finite derivatives at $u = u_s - \varepsilon$, satisfies (1.8) for any $r, \bar{r} \in \mathbf{R}$ and

$$\lim_{r\to-\infty} \beta^*_\varepsilon(r) = -\infty, \qquad \lim_{r\to+\infty} \beta^*_\varepsilon(r) = +\infty.$$

We denote by β_ε the derivative of β^*_ε defined as

$$\beta_\varepsilon(r) := \begin{cases} \beta(r), & \text{if } r < u_s - \varepsilon, \\ \frac{K_s^* - \beta^*(u_s-\varepsilon)}{\varepsilon}, & \text{if } r \geqslant u_s - \varepsilon \end{cases} \tag{2.2}$$

and remark that $\beta_\varepsilon(r) \geqslant \rho$ for any $r \in \mathbf{R}$.

Then we introduce $A_\varepsilon : D(A_\varepsilon) \subset V' \to V'$ by

$$\begin{aligned} &\langle A_\varepsilon u, \psi \rangle_{V',V} := \int_\Omega \nabla\beta^*_\varepsilon(u) \cdot \nabla\psi \, dx, \quad \text{for every } \psi \in V, \\ &D(A_\varepsilon) := \big\{u \in L^2(\Omega);\ \beta^*_\varepsilon(u) \in V\big\} \end{aligned}$$

and consider the periodic approximating problem

$$\frac{d(m_\varepsilon u_\varepsilon)}{dt} + A_\varepsilon u_\varepsilon = f, \quad \text{a.e. } t \in (0,T), \tag{2.3}$$

$$m_\varepsilon\big(u_\varepsilon(0) - u_\varepsilon(T)\big) = 0 \tag{2.4}$$

which is equivalent with

$$\begin{aligned} &\frac{dv_\varepsilon}{dt} + B_\varepsilon v_\varepsilon = f, \quad \text{a.e. } t \in (0,T), \\ &v_\varepsilon(0) = v_\varepsilon(T), \end{aligned} \tag{2.5}$$

by the function replacement

$$v_\varepsilon = m_\varepsilon u_\varepsilon. \tag{2.6}$$

Here, $B_\varepsilon v_\varepsilon = A_\varepsilon(\frac{v_\varepsilon}{m_\varepsilon})$. We are going to prove the following existence result.

Quantum control design by Lyapunov trajectory tracking for dipole and polarizability coupling

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Abstract. We analyse in this paper the Lyapunov trajectory tracking of the Schrödinger equation for a coupling control operator containing both a linear (dipole) and a quadratic (polarizability) term. We show numerically that the contribution of the quadratic part cannot be exploited by standard trajectory tracking tools and propose two improvements: discontinuous feedback and periodic (time-dependent) feedback. For both cases we present theoretical results and support them by numerical illustrations.

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1. Introduction

We consider in this work the evolution of a quantum system with wavefunction $\Psi(t)$ under the external influence of a laser field; the system satisfies the Time Dependent Schrödinger equation (TDSE)

$$i\frac{d}{dt}\Psi(t) = H(t)\Psi(t), \tag{1}$$

with $H(t)$ a Hermitian operator; the control is realized by selecting a convenient laser intensity $u(t)$. When the laser is shut off $H(t)$ is the internal Hamiltonian of the system, denoted H_0; when the laser is present $H(t)$ is the sum of H_0 and additional terms that describe the interaction of the system with the laser field. The first order term is the dipole coupling [30] of the form $u(t)H_1$; in the limit of small laser intensities this term may be enough to adequately describe the interaction.

However, situations exist where the dipole coupling does not have enough influence on the system to reach the control goal; the goal may become accessible only after adding a polarizability term $u^2(t)H_2$ in the expansion of $H(t)$ (see e.g. [13, 14] and related works); to make effective use of this term one needs higher laser intensities $u(t)$.

The focus of the paper is on practical procedures to find suitable control fields $u(t)$ for the Hamiltonian $H(t) = H_0+u(t)H_1+u^2(t)H_2$ by adapting feedback tracking control procedures to this setting. Here and in the following H_0, H_1 and H_2 are $n \times n$ Hermitian matrices with complex coefficients and the control is the laser intensity $u(t) \in \mathbb{R}$.

In what concerns the mere possibility to find a control, we recall that the controllability of the finite dimensional quantum system evolving with equation

$$i\frac{d}{dt}\Psi(t) = (H_0 + u(t)H_1 + u^2(t)H_2)\Psi(t), \tag{2}$$

eq_gen

can be studied via the general accessibility criteria [4, 32] based on Lie brackets; more specific results can be found in [34].

Let us consider for a moment the system with Hamiltonian $H_0 + u(t)H_1 + v(t)H_2$, $v(t)$ being a second control independent of $u(t)$. It can be shown [34] that this system is controllable under the same circumstances as $H_0 + u(t)H_1 + u^2(t)H_2$ i.e. all target states that can be reached with Hamiltonian $H_0 + u(t)H_1 + v(t)H_2$ can also be reached by $H_0 + u(t)H_1 + u^2(t)H_2$ (although obviously the second Hamiltonian is a particular case of the first for $v(t) = u^2(t)$). This rather counter-intuitive result suggests that $u^2(t)$ can be considered, for the purpose of theoretical controllability, as independent of $u(t)$; however, $u^2(t)$ having a particular functional dependence on $u(t)$ will play a role at the level of the numerical procedure to find the control: in general finding the control for $H_0 + u(t)H_1 + u^2(t)H_2$ is more difficult than for $H_0 + u(t)H_1 + v(t)H_2$.

The characterization of the controllability does not provide in general a simple and efficient way for open-loop trajectory generation. Optimal control techniques (cf., [23] and [30] and the references herein) provide a first set of methods. A different approach consists in using feedback to generate trajectories and open-loop steering control [5, 19, 22]. More recent results can be found in [27] for decoupling techniques, in [3, 15, 17, 23, 31, 35, 36] for Lyapunov-based techniques and in [1, 7, 28] for factorizations techniques of the unitary group.

In order to study feedback control of systems with Hamiltonian $H(t) = H_0 + u(t)H_1 + u^2(t)H_2$ we adapt the analysis [20, 24], initially proposed for bilinear quantum systems $H_0 + u(t)H_1$. In the previous work it has been shown that the success of the feedback control depends on whether there exists (non-zero) direct coupling, through H_1, between the target state and **all** other eigenstates. When H_1 has the same property for $H(t) = H_0 + u(t)H_1 + u^2(t)H_2$ we show that same feedback formulas hold. However we argued that the polarizability term $u^2(t)H_2$ is added when dipole $u(t)H_1$ is not enough to control the system; consequently the most interesting question is what happens when some of the (direct) coupling is realized by H_2 instead of H_1. We show that the previous feedback formulas do not hold any more and we propose two alternatives. Our method is valid to track any eigenstate trajectory of a Schrödinger equation (2) when the Hamiltonian includes a second order coupling operator.

The balance of the paper is as follows: in Section 2 we introduce the main notations and the Lyapunov tracking feedback for a particular case. Section 3 contains the presentation of two types of feedback: discontinuous and time-dependent (periodic) forcing, that can be applied for all types of second order coupling operators. Both sections present theoretical results on the convergence illustrated by numerical simulations. Concluding remarks are presented in Section 4.

2. Tracking feedback design

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2.1. Dynamics and global phase

gauge:ssec

We consider a n-level quantum system evolving under the equation (2). The wave function $\Psi = (\Psi_j)_{j=1}^n$ is a vector in $\mathbb{C}^n$, verifying $\sum_{j=1}^n |\Psi_j|^2 = 1$, thus it lives on the unit sphere $\mathbb{S}^{2n-1}$ of $\mathbb{C}^n$. Physically, Ψ and $e^{i\theta(t)}\Psi$ describe the same physical state for any global phase $\theta(t) \in \mathbb{R}$, i.e. Ψ_1 and Ψ_2 are identified when exists $\theta(t) \in \mathbb{R}$ such that $\Psi_1 = \exp(i\theta(t))\Psi_2$. To take into account such non trivial geometry we add a second control ω corresponding to $\dot{\theta}$ (see also [24]). Thus we consider the following control system

$$i\frac{d}{dt}\Psi(t) = (H_0 + u(t)H_1 + u^2(t)H_2 + \omega(t))\Psi(t), \tag{3}$$ dyne:eq

where $\omega \in \mathbb{R}$ is a new control playing the role of a gauge degree of freedom. We can choose it arbitrarily without changing the physical quantities attached to Ψ. With such additional fictitious control ω, we will assume in the sequel that the state space is $\mathbb{S}^{2n-1}$ and the dynamics given by (3) admits two independent controls u and ω.

2.2. Lyapunov control design

esign:ssec

Take a reference trajectory $t \mapsto (\Psi_r(t), u_r(t), \omega_r(t))$, i.e., a smooth solution of (3):

Comsol modelling for a water infiltration problem in an unsaturated medium

Cornelia-Andreea Ciutureanu

Keywords Boundary value problems for nonlinear parabolic PDE; Stability and convergence of numerical methods; Flows in porous media.

AMS Subject Classification 35K60; 65M12; 76S05.

Abstract

The paper deals with the COMSOL modelling of fluid diffusion in unsaturated porous media. A representative phenomenon in this class of problems is water infiltration in soils.

The model we are concerned of describes the water infiltration into an isotropic, nonhomogeneous, unsaturated porous medium with a variable porosity. It consists of a diffusion equation with a transport term in addition with a initial data and a Dirichlet boundary condition

$$m(x)\frac{\partial u}{\partial t} - \Delta\beta^*(u) + \frac{\partial K(u)}{\partial x_3} = F \quad \text{in} \quad Q := \Omega \times (0,T), \tag{1}$$

$$m(x)u(x,0) = \theta_0(x) \quad \text{in} \quad \Omega, \tag{2}$$

$$u(x,t) = g(x) < u_s \quad \text{on} \quad \Sigma := \Gamma \times (0,T). \tag{3}$$

The domain Ω is an open bounded subset of $\mathbf{R}^3$, with the boundary $\Gamma := \partial\Omega$ piecewise smooth. We denote the space variable by $x := (x_1, x_2, x_3) \in \Omega$ and the time by $t \in (0,T)$, with T finite. The model is written in dimensionless form. The porosity is denoted by m, the function u stands for the water saturation, while by u_s we shall denote its maximum value.

The volumetric water content is given by mu and θ_0 is the initial volumetric water content.

Hypothesis

In the unsaturated case the diffusivity $\beta : (-\infty, u_s) \to [\rho, +\infty)$ is a continuous and monotonically increasing function that satisfies the following hypotheses:

Identification of the time derivative coefficient in a fast diffusion degenerate equation

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Abstract In this paper we deal with the identification of the space variable time derivative coefficient u, in a degenerate fast diffusion differential inclusion. The function u is vanishing on a subset strictly included in the space domain Ω. This problem is approached as a control problem (P), with the control u. An approximating control problem (P_ε) is introduced and the existence of an optimal pair is proved. Under certain assumptions on the initial data the control is found in $W^{2,m}(\Omega)$ with $m > N$, in an implicit variational form. Next, it is shown that a sequence of optimal pairs $(u_\varepsilon^*, y_\varepsilon^*)$ of (P_ε) converges as ε goes to 0 to a pair (u^*, y^*) which realizes the minimum in (P), and y^* is the solution to the original state system.

An alternative approach of the control problem is done by considering two controls related between them by a certain elliptic problem. This approach leads to the determination of simpler conditions of optimality, but under an additional restriction upon the initial data of the direct problem.

Key Words Identification problems, fast diffusion differential inclusions, degenerate parabolic PDE, flows in porous media.

1 Statement of the problem

Let Ω be an open bounded subset of $\mathbf{R}^N$ $(N = 1,2,3)$, with the boundary $\Gamma := \partial\Omega$ sufficiently smooth, $(0,T)$ a finite time interval, and let us consider the diffusion degenerate problem

$$\frac{\partial}{\partial t}(u(x)y) - \Delta\beta^*(y) \ni f \text{ in } Q := (0,T)\times\Omega, \tag{1}$$

$$y(t,x) = 0 \text{ on } \Sigma := (0,T)\times\Gamma, \tag{2}$$

$$u(x)y(0,x) = \theta_0 \text{ in } \Omega, \tag{3}$$

where

$$0 \le u \le \phi_{\max},\ \ u = 0 \text{ on } \overline{\Omega_0},\ \ u > 0 \text{ on } \Omega\backslash\overline{\Omega_0}, \tag{4}$$

with $\overline{\Omega_0} \subset \Omega$, $\overline{\Omega_0} \cap \Gamma = \emptyset$.

In equation (1) $\beta^* : (-\infty, y_s] \to \mathbf{R}$ is multivalued, defined by

$$\beta^*(r) = \begin{cases} \int_0^r \beta(s)ds,\ r < y_s \\ [K_s^*, +\infty),\ r = y_s, \end{cases} \tag{5}$$

where $\beta \in C^3(-\infty, y_s)$, is monotonically increasing and has the properties

$$\beta(r) \ge \rho > 0, \text{ for any } r < y_s,\ \beta(r) = \rho \text{ for } r \le 0, \tag{6}$$

$$\lim_{r\nearrow y_s} \beta(r) = +\infty,\ \ \lim_{r\nearrow y_s}\int_0^r \beta(s)ds = K_s^*, \tag{7}$$

with y_s, ρ, K_s^*, $\phi_{\max}$ positive constants. We recall that assumptions (7) account for the fast diffusion character of equation (1) and refer the reader to the monograph [1] in which many details about this model are given.

Model (1)-(3) can arise in applications related to fluid diffusion in porous materials, in which u represents the space variable porosity of the material and y is the concentration of the fluid in pores. In these applications $\phi_{\max}$ can be taken equal to 1.

In [2] existence and uniqueness of the solution to (1)-(3) were proved for u, β^*, β defined by (4)-(7), under the assumptions $u \in C^1(\Omega)$, $f \in L^2(0,T;H^{-1}(\Omega))$ and

$$\theta_0 \in L^2(\Omega),\ \frac{\theta_0}{u} \in L^2(\Omega),\ \frac{\theta_0}{u} \leq y_s \text{ a.e. on } \Omega. \tag{8}$$

The present paper is devoted to an inverse problem, namely of identifying the time derivative coefficient u from known data upon the solution. This identification will be approached under stronger assumptions on the problem data, respectively,

$$f \in W^{1,2}(0,T;L^2(\Omega)), \tag{9}$$

$$\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega),\ \theta_0 \geq 0 \text{ for any } x \in \Omega,\ \theta_0 = 0 \text{ in } \overline{\Omega_0}, \tag{10}$$

and u will be found to be in a more restrictive class of functions, $W^{2,m}(\Omega)$, with $m > N$.

We introduce the function

$$\theta(t,x) = u(x)y(t,x) \tag{11}$$

(which has a certain physical meaning too, being for example the volumetric fluid content in a porous medium) and assuming that we have data $\theta_g(t,x)$ measured for θ in a domain $Q_g = (0,T_g) \times \Omega_g$, where $T_g \le T$ and $\Omega_g \sqsubseteq \Omega$, $\theta_g \in L^2(Q_g)$, we are going to approach the identification problem as an optimal control problem, by searching for a control u which satisfies the problem

$$\text{Minimize} \left\{ \frac{1}{2} \int_{Q_g} (u(x)y(t,x) - \theta_g(t,x))^2 dxdt + \frac{k_1}{m} \int_{\Omega} (u(x) - \Delta u(x))^m dx \right\} \tag{P}$$

subject to (1)-(3), for all $u \in U$, where

$$U = \left\{ u \in W^{2,m}(\Omega),\ \frac{\theta_0}{y_s}(x) \le u(x) \le \phi_{\max} \text{ on } \Omega,\ \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma} = 0,\ u = 0 \text{ on } \overline{\Omega_0} \right\}, \tag{12}$$

and m is a positive even integer,

$$m = 2k > N,\ k \ge 1. \tag{13}$$

In a rigorous way, in (P) we should have denoted y^u in order to specify that y depends on u, but for not complicating the notation we agree to skip this superscript. The positive constant k_1 is a weight that may be exploited in numerical simulations in order to confer a greater importance to a term in (P) against the other.

For $u \in W^{2,m}(\Omega)$ we denote

$$v := u - \Delta u$$

and so $v \in L^m(\Omega)$.

Conversely, according to a result due to Agmon & al. (see [3]), problem

$$\begin{aligned} u - \Delta u &= v \text{ in } \Omega, \\ \nabla u \cdot \nu &= 0 \text{ on } \Gamma \end{aligned} \tag{14}$$

with $v \in L^m(\Omega)$, has a unique solution $u \in W^{2,m}(\Omega)$, such that

$$\|u\|_{W^{2,m}(\Omega)} \leq C \|u\|_{L^m(\Omega)} \text{ for any } u \in L^m(\Omega). \tag{15}$$

We still recall by the Sobolev embedding theorem, that $W^{2,m}(\Omega)$ is compactly embedded in $C^1(\overline{\Omega})$ for $m > N$. Since $N \geq 1$, we choose $m \geq 2$, so that u also belongs to $H^2(\Omega)$. Thus, we get that $u \in H^2(\Omega) \cap C^1(\overline{\Omega})$.

The last term on the right-hand side in (P) was introduced to induce the regularity (15) required for u, especially that $u \in C^1(\overline{\Omega})$. This will be necessary to ensure that u is a multiplier in V' in the convergence result in Section 3.

The inequality $\frac{\theta_0}{y_s} \leq u$ is required in the proof of existence in the state system, while $u \leq \phi_{\max}$ is mostly related to the physical interpretation of u. From the mathematical point of view the latter is not absolutely necessary because an upper bound of u will follow from the boundedness of the norm in $W^{2,m}(\Omega)$. However, we shall keep it in order to give a more accurate upper bound of the control.

We mention that from the mathematical point of view, Robin or non-homogeneous Dirichlet boundary conditions might also be considered in U instead of the Neumann one. A homogeneous Dirichlet condition must be avoided because $\overline{\Omega_0}$ was considered to be strictly included in Ω.

We remark that for $N \leq 2$ it is sufficient to take $m = 2$ and to replace the last term in (P) by the norm of u in $H^2(\Omega)$ without involving problem (14).

Because the state system involves a multivalued operator and in the perspective of establishing a basis for numerical computations we shall introduce an appropriate approximating problem (P_ε) indexed on a small positive parameter ε, involving an approximating state system. The paper will concern the following aspects:

a) The existence, uniqueness and regularity of the approximating state system;

b) The proof of the existence of a solution to (P_ε);

c) The computation of the approximate optimality condition, after introducing and studying the system of first order variations and the dual system for the approximating problem;

d) The proof of the fact that (P_ε) approximates in some sense (P), i.e., that a sequence of optimal pairs $\{(u_\varepsilon^*, y_\varepsilon^*\}_\varepsilon$ for (P_ε) tends to a pair (u^*, y^*) which realizes the minimum in (P) and the state y^* corresponding to u^* is the solution to (1)-(3); for this last point we shall use weaker assumptions than those considered in [2].

2 The approximating problem

Let ε be positive, small and consider a smooth approximation of the multivalued function β^*. For the purposes of this work it is enough to replace it by a three times differentiable function $\beta_\varepsilon^* : \mathbf{R} \to \mathbf{R}$, taken for example as

$$\beta_\varepsilon^*(r) = \begin{cases} \beta^*(r), & r < y_s - \varepsilon \\ \beta_{reg}^*(r), & y_s - \varepsilon \leq r \leq y_s \\ \beta^*(y_s - \varepsilon) + \frac{K_s^* - \beta^*(y_s - \varepsilon)}{\varepsilon}[r - (y_s - \varepsilon)], & r > y_s, \end{cases} \tag{16}$$