Complementation in subgroup lattices

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Abstract. In this survey we bring together a number of results concerning to some classes of groups determined by different types of complementation of their subgroup lattices.

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1 Preliminaries

Let $(L, \wedge, \vee)$ be a bounded lattice with the initial element 0 and the final element 1.

Let $a \in L$. An element $a' \in L$ is called a *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$. We say that L is a *complemented lattice* if all its elements possess complements.

Suppose now that a belongs to the interval $[c/b] = \{x \in L \mid b \leq x \leq c\}$. Then $a' \in L$ is a *relative complement* of a in $[c/b]$ if $a \wedge a' = b$ and $a \vee a' = c$. Remark that a complement of a is a relative complement of a in the interval $[1/0] = L$. If every element of L has a relative complement in any interval containing it, we call L a *relatively complemented lattice*. Clearly, a relatively complemented lattice is also a complemented lattice.

A *boolean algebra* is a complemented distributive lattice. Note that a boolean algebra L is a uniquely complemented lattice (i.e. each element of L possesses a unique complement) and a relatively complemented lattice, too.

In a bounded distributive lattice L, if a' is a complement of a, then a' is the largest element x of L with $a \wedge x = 0$. More generally, let $(L, \wedge, \vee)$ be a lattice with 0 and $a \in L$. An element $a^* \in L$ is called a *pseudocomplement* of a if the following two conditions are verified:

a) $a \wedge a^* = 0$;

b) $a \wedge x = 0$ $(x \in L)$ implies that $x \leq a^*$.

We say that L is a *pseudocomplemented lattice* if every element of L has a pseudocomplement.

An *ortholattice* is a bounded lattice L for which there exists a map $\perp$ from L to L satisfying the next three properties:

a) $\perp$ is an autoduality of L (i.e. $\perp$ is one-to-one and onto map and, for all $a, b \in L$, we have $a \leq b$ iff $b^\perp \leq a^\perp$) ;

b) $a^\perp$ is a complement of a, for every $a \in L$;

c) $a^{\perp\perp} = a$, for all $a \in L$.

An ortholattice L is called *orthomodular* if each its orthogonal pair $(a, a^\perp)$ is a modular pair, that is it verifies the following modular identities: for $x \in L$, $a \leq x$ implies that $x \wedge (a \vee a^\perp) = a \vee (x \wedge a^\perp)$ and $a^\perp \leq x$ implies that $x \wedge (a \vee a^\perp) = a^\perp \vee (x \wedge a)$.

2 Main results

Let $(G, \cdot, e)$ be a group. Then the set $L(G)$ consisting of all subgroups of G is partially ordered with respect to set inclusion. Moreover, any subset of $L(G)$ has a greatest lower bound in $L(G)$ (the intersection of all its elements) and a least upper bound in $L(G)$ (the join of all its elements). Therefore $L(G)$ is a complete lattice with initial element the trivial subgroup $\{e\}$ and final element G, called the *subgroup lattice* of G. Its binary operations will be denoted by $\wedge$ and $\vee$. Note that we have $X \wedge Y = X \cap Y$ (= the intersection of X and Y) and $X \vee Y = < X, Y >$ (= the subgroup generated by X and Y), for any two subgroups X, Y of G.

Definition 1. We say that G is a *K–group* if $L(G)$ is a complemented lattice.

The structure of finite K–groups is in general not known. A finite K–group is not necessarily solvable and subgroups of a finite K–group are not always K–groups (for example, see the alternating group A_5, respectively the symmetric group S_4). Much more can be said about finite K–groups satisfying additional conditions. We begin by indicating two general properties of these groups.

Lemma 2. *The direct product of a family of finite groups is a K–group if and only if each factor is a K–group.*

Lemma 3. *The Frattini subgroup of a finite K–group is trivial.*

From Lemma 3 it obtains the following corollary.

Corollary 4. *A finite nilpotent group is a K–group if and only if it is elementary abelian.*

A characterization of finite K–groups is given by the next result.

Theorem 5 (G. Zacher [30]). *A finite group G is a K–group if and only if the maximal nilpotent normal subgroup N of G is a direct product of minimal normal abelian subgroups of G and N has a complement which is a K–group.*

Proof. Suppose that G is a K–group. Then, for any maximal subgroup M of G, we have that $N \wedge M$ is a normal subgroup of G and $N/N \wedge M$ is isomorphic with one of the factor groups of a principal series. Since $\Phi(G) = \{e\}$, N can be written as a direct product of minimal normal abelian subgroups of G. Let N' be a complement of N ($N \vee N' = G$, $N \wedge N' = \{e\}$). We shall prove that N' is a K–group. Since $N' \cong G/N$, it is sufficient to show that G/N is a K–group. Let H be a subgroup of G such that $N \subseteq H$ and H' be a complement of H in $L(G)$. Using the modular law, it obtains:

$$H \wedge (H' \vee N) = N \vee (H \wedge H') = N \vee \{e\} = N.$$

Hence $H' \vee N/N$ is a complement of H/N in $L(G/N)$.

Conversely, suppose that G satisfies the above conditions and let N' be a complement of N in $L(G)$ such that N' is a K–group. Since $G/N \cong N'$, we have that G/N is also a K–group. Let $U \in L(G)$. Then there exists a subgroup W of G such that $W \vee (U \vee N) = G$ and $W \wedge (U \vee N) = N$. By the modular law, it results $W \wedge (U \vee N) = N \vee (U \wedge W)$ and so we have $U \wedge W \subseteq N$. Take a normal subgroup Q of G such that $Q \subseteq N$, $Q \wedge U = \{e\}$ and assume that Q is maximal under these conditions. We shall verify that $U' = Q \vee (W \wedge N')$ is a complement of U in $L(G)$. We have:

$$\begin{aligned} U \wedge U' &= U \wedge [Q \vee (W \wedge N')] = U \wedge [W \wedge (Q \vee N')] = \\ &= U \wedge N \wedge (Q \vee N') = U \wedge [Q \vee (N \wedge N')] = U \wedge Q = \{e\}. \end{aligned}$$

If $N \not\subseteq U \vee U'$, then we can choose a maximal subgroup M containing $U \vee U'$. Since $U \vee U' \vee N = G$, we have not $N \subseteq M$. Thus there exists a minimal normal subgroup T of G such that $T \vee M = G$, $T \wedge M = \{e\}$ and $T \subseteq N$. Then $T \vee Q$ is a normal subgroup of G and $T \vee Q \subseteq N$, $(T \vee Q) \wedge U = \{e\}$. This contradicts the maximality of Q. Therefore $N \subseteq U \vee U'$ and so:

$$U \vee U' = U \vee U' \vee N = U \vee N \vee (W \wedge N') = U \vee N \vee W = G.$$

Hence U' is a complement of U. ∎

Concerning to the solvable K–groups, it is well–known (see [22], pp. 119–120) that every normal subgroup of a finite solvable K–group is also a K–group and that an arbitrary solvable group G is a K–group iff every normal subgroup of G has a complement in G. The following theorem establishes a necessary and sufficient condition for a finite solvable group in order to be a K–group.

Theorem 6 (G. Zacher [30]). *A finite solvable group G is a K–group if and only if G has a series of normal subgroups*

$$\{e\} = N_0 \subset N_1 \subset \cdots \subset N_r = G$$

such that N_i/N_{i-1} is a maximal nilpotent normal subgroup of G/N_{i-1} and the Frattini subgroup of G/N_{i-1} is trivial, for all $i = \overline{1,r}$.

Proof. Assume that G is a K–group. Then, by Lemma 3, we have $\Phi(G) = \{e\}$. The factor group G/N_1 is also a K–group and so our conditions are necessary.

Conversely, assume that G has a series of normal subgroups satisfying the above conditions. Using induction on r, it is sufficient to prove only that N_1 has a complement in $L(G)$. N_1 is a direct product of minimal abelian subgroups, because $\Phi(G) = \{e\}$. Let M be a maximal subgroup of G such that $M \subseteq N_1$ and M has a complement M' in $L(G)$. We shall prove that $M = N_1$. Suppose that $M \neq N_1$ and let H be a minimal normal subgroup of $N_1 \wedge M'$. Then H possesses a complement H' and we have:

$$\begin{aligned}(H \vee M) \wedge (H' \wedge M') &= [(H \vee M) \wedge M'] \wedge H' = \\ &= [H \vee (M \wedge M')] \wedge H' = H \wedge H' = \{e\}, \\ (H \vee M) \vee (H' \wedge M') &= M \vee [H \vee (H' \wedge M)] = \\ &= M \vee [M' \wedge (H \vee H')] = M \vee M' = G.\end{aligned}$$

Therefore $H' \wedge M'$ is a complement of $H \vee M$. This contradicts the maximality of M. Hence $M = N_1$ and N_1 has a complement M'. ■

The structure of finite supersolvable K–groups is indicated in the next result.

Theorem 7 (P. Hall [11]). *For a finite group G, the following conditions are equivalent:*

a) *G is a supersolvable K–group.*

b) *G is isomorphic to a subgroup of a direct product of groups with square–free order.*

c) *Every subgroup H of G has a complement H' such that $HH' = H'H$.*

Proof. a)$\Longrightarrow$b) Suppose that G is a supersolvable K–group. If G contains two normal subgroups N_1 and N_2 such that $N_1 \wedge N_2 = \{e\}$, then G can be embedded into the direct product $G/N_1 \times G/N_2$. Thus it is sufficient to prove that $|G|$ is square–free if G has only one minimal normal subgroup. Let p be the largest prime divisor of $|G|$. Since G is supersolvable, it has a unique Sylow p–subgroup S and this is normal. By Theorem 5, the maximal nilpotent normal subgroup of G is a direct product of minimal normal abelian subgroups. In our situation, G has only one minimal subgroup, which is of order p. Therefore S must be a minimal normal subgroup and so $|S| = p$. Let $S' \in L(G)$ such that $SS' = G$ and $S \wedge S' = \{e\}$. Then the centralizer C of S is a normal subgroup of G and it is the direct product of S and $S' \wedge C$. But $S' \wedge C = \{e\}$, as $S' \wedge C$ is a normal subgroup of G contained in S'. This implies that S' is cyclic, as it is isomorphic with a subgroup of the group of all automorphisms of the cyclic group S. From Theorem 5 we have that S' is a K–group and so all its Sylow subgroups are elementary abelian. Hence $|G|$ is square–free.

b)$\Longrightarrow$c) We remark that all groups of square–free order satisfy the condition c). We shall prove our implication in two steps.

First we show that if G is a direct product of two groups G_1 and G_2 both of which satisfy the condition c), then G itself satisfies c). Let $H \in L(G)$. Every element $h \in H$ can be written as $h = h_1 h_2$, where $h_1 \in G_1$ and $h_2 \in G_2$. Let N_1 be the subgroup of G_1 consisting of all elements in G_1 which appear as G_1–factors of elements in H and $N_2 = H \wedge G_2$. By our assumption, N_1 and N_2 have complements N_1' and N_2' such that :

$$N_i N_i' = G_i,\ N_i \wedge N_i' = \{e\},\ i = 1, 2.$$

We verify that $N_1' \times N_2'$ is a complement of H in $L(G)$. Let $h \in H \wedge (N_1' \times N_2')$. Then $h = h_1 h_2$, where $h_1 \in G_1$ and $h_2 \in G_2$. Since $h \in H$, it results $h_1 \in N_1$ and so $h_1 \in N_1 \wedge N_1'$. Thus $h_1 = e$ and $h = h_2$ is contained in N_2. But $h_2 \in N_2'$, so that $h_2 \in N_2 \wedge N_2'$. Hence $h = h_2 = e$. This implies that:

$$H \wedge (N_1' \times N_2') = \{e\}.$$

Now, let $g = g_1 g_2 \in G$ ($g_1 \in G_1$, $g_2 \in G_2$). Then there exist $a_1 \in N_1$ and $a_1' \in N_1'$ such that $g_1 = a_1 a_1'$. By the definition of N_1, we can take an element $a \in H$ with $a = a_1 a_2$, where $a_1 \in G_2$. Suppose $a_2^{-1} g_2 = b_2 b_2'$ with $b_2 \in N_2$ and $b_2' \in N_2'$. It obtains:

$$g = g_1 g_2 = a_1 a_1' g_2 = a_1 a_2 a_2^{-1} a_1' g_2 = (ab_2) a_1' b_2' \in H(N_1' \times N_2').$$

Hence $H(N'_1 \times N'_2) = G$.

Next we show that if G satisfies the condition c), then any subgroup of G satisfies c), too. Let $U \in L(G)$ and $V \in L(U)$. Then there exists a complement V' of V such that $VV' = G$ and $V \wedge V' = \{e\}$. It obtains $V(U \wedge V') = G$ and $V \wedge (U \wedge V') = \{e\}$, therefore U satisfies c).

c)$\Longrightarrow$a) Suppose that G satisfies the condition c). Then it is solvable. Let $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_r = G$ be a principal series of G. Each factor group G_i/G_{i-1} is abelian. Let $i \in \{1, 2, ..., r\}$ and H be a subgroup of G such that $G_{i-1} \subseteq H \subseteq G_i$. By our hypothesis, there exists a complement H' of H such that $HH' = G$ and $H \wedge H' = \{e\}$. Let $N = G_i \wedge (G_{i-1} \vee H')$. N is a normal subgroup of G and $G_{i-1} \subseteq N \subseteq G_i$. Then $N = G_{i-1}$ or $N = G_i$. In the first case we obtain:

$$\begin{aligned} H &= H \vee N = H \vee [G_i \wedge (G_{i-1} \vee H')] = H \vee G_{i-1} \vee (G_i \wedge H') = \\ &= H \vee (G_i \wedge H') = G_i \wedge (H \vee H') = G_i. \end{aligned}$$

In the second case we have:

$$\begin{aligned} H &= H \wedge N = H \wedge G_i \wedge (G_{i-1} \vee H') = H \wedge (G_{i-1} \vee H') = \\ &= G_{i-1} \vee (H \wedge H') = G_{i-1}. \end{aligned}$$

Therefore G_i/G_{i-1} contains no proper subgroup and so it is cyclic of prime order. Hence G is supersolvable. ∎

A long–standing open question of finite K–group theory has been answered in a very recent paper of M. Constantini and G. Zacher. They proved the following remarkable theorem.

Theorem 8 (M. Constantini and G. Zacher [6]). *Every finite simple group is a K–group.*

Remark. The previous statement does not hold for infinite simple groups. A class of infinite simple groups which are not K–groups was constructed by V.N. Obraztsov in [17].

Our next aim is to investigate the classes of groups whose subgroup lattices are relatively complemented and pseudocomplemented, respectively.

Definition 9. A group G is called an *RK–group* if its subgroup lattice $L(G)$ is relatively complemented.

First of all we indicate a property which is verified by all RK–groups.

Proposition 10. *Let G be an RK–group and $W \subseteq V \subseteq U \subseteq G$ be a chain of subgroups of G such that W is normal in V and V is normal in U. Then W is normal in U.*

Proof. Since G is an RK–group, V has a relative complemented V_1 in the interval $[U/W]$ ($V \vee V_1 = U$, $V \wedge V_1 = W$). Then W is a normal subgroup of V_1 and so W is normal in $V \vee V_1 = U$. ■

A group G is called a *T–group* if in G the normality is a transitive relation and a *$\overline{T}$–group* if all subgroups of G are T–groups. Many problems concerned with these classes of groups have been studied by W. Gaschütz ([9]), S.N. Černikov ([3]), D.J.S. Robinson ([19], [20], [21]) or, more recently, M. de Falco and F. de Giovanni ([8]). Remark that the above proposition says nothing else than all RK–groups are $\overline{T}$–groups.

The following result gives us a characterization of finite RK–groups.

Theorem 11 (G. Zacher [30]). *A finite group G is an RK–group if and only if G is a $\overline{T}$–group with elementary abelian Sylow subgroups.*

Proof. Assume that G is an RK–group. Then, from the above proposition, G is a $\overline{T}$–group. On the other hand, G is a K–group and, using Corollary 4, it obtains that all Sylow subgroups of G are elementary abelian.

Conversely, assume that G is a $\overline{T}$–group with elementary abelian Sylow subgroups. Then G is solvable and every nilpotent subgroup of G is elementary abelian. By Theorem 6, it results that G is a K–group. It is sufficient to show that any interval of type $[G/H]$ is complemented. Let $U \in [G/H]$. We shall prove the existence of a complemented of U in $[G/H]$ by induction on $|G|$. Let p be the largest prime divisor of $|G|$. If $|H|$ is divisible by p, then H contains a subgroup P of order p. Since the Sylow p–subgroup S of G is normal, P is also normal. Hence we may apply inductive hypothesis on G/P in order to prove the existence of a complemented of U. If $(|U|, p) = 1$, then U contains a Sylow p–complement V containing H and let W be a Sylow p–complement such that $V \subseteq W$. As $W \cong G/S$, W is an RK–group by inductive hypothesis. Thus there exists a subgroup T with $V \vee T = W$ and $V \wedge T = H$. Let S_1 be a complement of $S \wedge U$ in S. Since S_1 is a normal subgroup of S, S_1 is also normal in G. We shall show that $U_1 = S_1T$ is a relative complement of U. We have:

$$U_1 \vee U = (S_1 \vee T) \vee [V \vee (U \wedge S)] = [S_1 \vee (U \wedge S)] \vee (T \vee V) = S \vee W = G.$$

Let $a \in U_1 \wedge U$. Then $a = vs$, where $v \in V$ and $s \in S \wedge U$. Since $a \in U_1$, we also have $a = ts_1$ with $t \in T$ and $s_1 \in S_1$. It results $vs = ts_1$, which implies that $v^{-1}t = ss_1^{-1}$. But $v^{-1}t \in W$ and $ss_1^{-1} \in S$, therefore $v^{-1}t = ss_1^{-1} = e$, i.e. $v = t$ and $s = s_1$. Since $(S \wedge U) \wedge S_1 = \{e\}$, it obtains $s = s_1 = e$ and so $a \in V \wedge T = H$. Hence $U_1 \wedge U \subseteq H$. As $H \subseteq U_1 \wedge U$ by definition, it follows that $U_1 \wedge U = H$. Hence U_1 is a relative complement of U. ■

Corollary 12. *A finite group G is an RK–group if and only if G is a solvable K–group which satisfies the following condition: a chain of subgroups*

$$\{e\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$$

is a principal series of G whenever each $|G_i : G_{i-1}| = p_i$ is a prime ($i = \overline{1, r}$) and $p_1 \geq p_2 \geq \cdots \geq p_r$.

Since a $\overline{T}$–group of finite order is supersolvable, by Theorem 11 we also obtain:

Corollary 13. *All finite RK–groups are supersolvable.*

Definition 14. A group G is called a *PK–group* if its subgroup lattice $L(G)$ is pseudocomplemented.

It is easy to see that any subgroup of a PK–group is a PK–group, but a factor group of a PK–group, as well as a direct product of PK–groups or an extension of a PK–group by another PK–group are not necessarily PK–groups.

A group is called a *D–group* if its subgroup lattice is distributive. Note that the structure of D–groups was determined by O. Ore ([18]): a group is a D–group iff it is locally cyclic. Clearly, all D–groups are PK–groups. The converse does not hold (the simplest example of a PK–group which is not a D–group is the quaternion group Q_8).

Necessary and sufficient conditions for a finite group in order to be a solvable PK–group are indicated in the next theorem.

Theorem 15 (M. Tărnăuceanu [27] and [28]). *Let G be a finite group of order $2^n m$, where $n, m \in \mathbb{N}$ and $m \equiv 1 \pmod 2$.*

a) *For $n \in \{0, 1, 2\}$, G is a PK–group if and only if it is cyclic.*

b) *For $n \geq 3$, G is a solvable PK–group if and only if it is cyclic or isomorphic to the direct product of the generalized quaternion group Q_{2^n} and the cyclic group $\mathbb{Z}_m$.*

Proof. a) Assume that G is a PK–group. Our hypothesis imply that all Sylow subgroups of G are cyclic, therefore G is solvable. Then, in order to prove that G is cyclic, it is sufficient to verify that any Sylow subgroup of G is normal. Suppose that G has a Sylow p–subgroup S which is not normal and let S_G be the core of S in G. Since S_G is a normal subgroup of G, we have $S_G \neq S$.

Case 1. $S_G \neq \{e\}$.

Let S_G^* be a pseudocomplement of S_G. Then we have:

$$(|S_G|, |S_G^*|) = 1. \tag{1}$$

Indeed, if $p \mid |S_G^*|$, then there exists $M \in L(S_G^*)$ with $|M| = p$. Let $x \in G$ such that $M \subseteq S^x$. Thus M is the unique subgroup of order p in S^x. Hence $M \subseteq S_G$ and so $M \subseteq S_G \wedge S_G^* = \{e\}$; contradiction.

Since G is solvable, we can choose a Sylow p–complement S_p satisfying $S_G^* \subseteq S_p$. On the other hand, we have $S_G \wedge S_p = \{e\}$, therefore $S_p \subseteq S_G^*$ and so $S_G^* = S_p$. Let $a \in G$. Because $(|S_G|, |(S_G^*)^a|) = (|S_G|, |S_G^*|) = 1$, it results $S_G \wedge (S_G^*)^a = \{e\}$. This implies that $(S_G^*)^a \subseteq S_G^*$, and thus $(S_G^*)^a = S_G$. Hence S_G^* is a normal subgroup of G.

Let S_G^{**} be a pseudocomplement of S_G^*. Then, from the equality $(|S_G^{**}|, |S_G^*|)$ $= 1$, it follows that S_G^{**} is a p–subgroup of G. Moreover, since $S_G^* \wedge S_G = \{e\}$, we have $S_G \subseteq S_G^{**}$. Now, we consider $x \in G$ such that $S \neq S^x$. By the equalities $S \wedge S_G^* = \{e\}$ and $S^x \wedge S_G^* = \{e\}$, it obtains $S \subseteq S_G^{**}$ and $S^x \subseteq S_G^{**}$. Hence $S = S_G^{**} = S^x$; contradiction.

Case 2. $S_G = \{e\}$.

We have $\{e\} = S_G = \bigwedge_{x \in G} S^x = \bigwedge_{x \in G} (S \wedge S^x)$. This equality implies that there exists $x \in G$ with $S \wedge S^x = \{e\}$. Thus:

$$S^x \subseteq S^*. \tag{2}$$

By $(|S_p|, |S|) = 1$, we obtain $S_p \wedge S = \{e\}$, therefore:

$$S_p \subseteq S^*. \tag{3}$$

Now, the relations (2) and (3) give us $S^* = G$ and so $S = S \wedge G = S \wedge S^* = \{e\}$; contradiction.

The converse is obvious.

b) Of course, a cyclic group, as well as a direct product of a generalized quaternion group and a cyclic group are solvable PK–groups. Assume that G is a solvable PK–group. Then, for $p \geq 3$, all Sylow p–subgroups of G are cyclic. Moreover, by induction on n, it is easy to see that a Sylow 2–subgroup of G is cyclic or isomorphic to the generalized quaternion group Q_{2^n}. It obtains that G is nilpotent and so it is cyclic or isomorphic to $Q_{2^n} \times \mathbb{Z}_m$. ∎

Corollary 16. *Let G be a finite group of odd order. Then G is a PK–group if and only if it is a D–group.*

Corollary 17. *Let G be a finite abelian group. Then G is a PK–group if and only if it is a D–group.*

A generalization of Corollary 17 can be seen in Gr.G. Călugăreanu [2]. Here it is proved, using another methods, that for the subgroup lattice of an abelian group (finite or not) the pseudocomplementation and the distributivity are equivalent.

Many papers on complementation in subgroup lattices deal with permutable complements of subgroups $H \in L(G)$, that is subgroups $H' \in L(G)$ satisfying $H \wedge H' = \{e\}$ and $HH' = G$. A finite group G in which every maximal primary cyclic subgroup is permutably complemented is called a *KM–group*. This class of finite groups was first studied by V.A. Kreknin and V.F. Malik (see [12]). The structure of KM–groups of prime power order is indicated in the following.

Theorem 18 (V.A. Kreknin and V.F. Malik [12]). *Let G be a finite p–group with $p > 2$. Then G is a KM–group if and only if there exist cyclic subgroups $A_1, A_2, ..., A_n$ of G such that $G = A_1 A_2 ... A_n$ and, for all $i, j = \overline{1, n}$, we have $A_i A_j = A_j A_i$, $A_i \wedge (A_1 ... A_{i-1} A_{i+1} ... A_n) = \{e\}$ and $|A_i|/|A_j| \in \left\{1, p, \frac{1}{p}\right\}$.*

Mention that a similar result is also obtained in the case $p = 2$ (see [13] and [14]). Finally, in [15] it is shown that a nonprimary KM–group G is either supersolvable or has a normal supersolvable KM–subgroup N such that G/N is a KM–group of exponent 12.

Groups G in which for every subgroup H there exists a subgroup K such that $H \wedge K = \{e\}$ and $X = H \vee (X \wedge K)$, for all subgroups X of G containing H, have been considered by M. Emaldi in [7]. Such a subgroup K is called an *S–complement* of H in G and a group G in which every subgroup has an S–complement is called an *SC–group*. In particular, Emaldi proved that a locally finite group G has the property SC if and only if it is a *C–group* (i.e. any subgroup H of G has a complement K in G such that $HK = KH$). Since the structure of C–groups is well–known (see [4] and [11]), a characterization of locally finite SC–groups can be derived.

M.C. Cirino Groccia and C. Musella ([5]) considered groups G in which for every subgroup H there exists a subgroup K such that $HK = G$ and $X = H \wedge (X \vee K)$, for all subgroups X of G contained in H. Such a subgroup K is called an S^*–*complement* of H in G and a group G in which every subgroup has an S^*–complement is called an S^*C–*group*. Obviously, all S^*C–groups are C–groups and so finite S^*C–groups are metabelian. A characterization of finite S^*C–groups will be presented in Theorem 23, but first we establish (without proofs) four auxiliary results. Our first lemma shows in particular that the class of S^*C–groups is closed under subobjects and homomorphic images.

Lemma 19. *Let G be a group, H a subgroup of G and K an S^*–complement of H in G.*

a) *If M is a subgroup of G containing H, then $K \wedge M$ is an S^*–complement of H in M.*

b) *If N is a normal subgroup of G contained in H, then KN/N is an S^*–complement of H/N in G/N.*

A lattice L is called *weakly* $\vee$–*complemented* if it contains a greatest element 1 and, for every pair $(a, b) \in L^2$ with $a < b$, there exists $c \in L$ such that $a \vee c \neq 1$ and $b \vee c = 1$. The subgroup lattice $L(G)$ of a group G is weakly $\vee$–complemented iff G has the property that all its proper subgroups are intersections of maximal subgroups (such a group is called an IM–*group*). The second lemma shows that all S^*C–groups are IM–groups.

Lemma 20. *Let G be an S^*C–group. Then the subgroup lattice $L(G)$ is weakly $\vee$–complemented.*

Lemma 21. *Let G be a group and N be a normal subgroup of G such that G/N is an S^*C–group. If H is a subgroup of G with $H \wedge N = \{e\}$, then H has an S^*–complement in G.*

A useful remark is given by the our last lemma.

Lemma 22. *Let G be a group and $\langle a \rangle$ be a cyclic normal subgroup of G with prime order p. If b is an element of G of prime order q and $ab \neq ba$, then ab has order q and $a^i b \notin \langle ab \rangle$, for all integers i such that $1 < i < p$.*

Now, we can prove the next theorem.

Theorem 23 (M.C. Cirino Groccia and C. Musella [5]). *Let G be a finite group. Then G is an S^*C–group if and only if it is a semidirect product of two coprime abelian subgroups A and B satisfying:*

a) *All Sylow subgroups of A and B have prime exponent.*

b) *Every subgroup of A is normal in G.*

c) *If x and y are elements of $B\backslash C_B(A_p)$ with the same prime order q, then there exists integers r, s such that $0 < r, s < q$ and $x^r y^s$ belongs to $C_G(A)$.*

d) *For every prime p such that A_p is not contained in the center $Z(G)$ of G, the group $G/C_G(A_p)$ has prime order.*

Proof. Suppose that G is an S^*C–group. Then G is an IM–group and a well–known result of Menegazzo yields that G is a semidirect product of two coprime abelian subgroups A and B which satisfy the properties a) and b) (see [16], Theorem 2.1). Let $x, y \in B\backslash C_B(A_p)$ with the same prime order q and $M = \langle x, y, A\rangle$. By Lemma 19, M is also an S^*C–group, so that $H = A\langle x\rangle$ has an S^*–complement K in M. Clearly, it can be assumed that $\langle x\rangle \neq \langle y\rangle$ and so K has order q. We have $\langle x\rangle = H \wedge \langle K, x\rangle$ and $\langle ax\rangle = H \wedge \langle K, ax\rangle$, for all $a \in A_p\backslash\{e\}$. Since $ax \neq xa$, Lemma 22 yields that ax has order q, therefore both $\langle K, x\rangle$ and $\langle K, ax\rangle$ are q–subgroups of M. Then they are abelian, so that $K \subseteq C_M(A_p) = AC_{\langle x,y\rangle}(A_p)$. It results that K and $C_{\langle x,y\rangle}(A_p)$ are both Sylow q–subgroups of $AC_{\langle x,y\rangle}(A_p)$, thus they are conjugate in M. Since H is normal in M, we obtain that $C_{\langle x,y\rangle}(A_p)$ is also an S^*–complement of H in M. In the same manner, $C_{\langle x,y\rangle}(A_p)$ is an S^*–complement of $H_1 = A\langle y\rangle$ in M. Let $C_{\langle x,y\rangle}(A_p) = \langle z\rangle$, where $z = x^r y^s$, $0 < r, s < q$ and assume that there exists a Sylow subgroup A_t of A such that $z \notin C_G(A_t)$. If $x \notin C_G(A_t)$, then ux has order q and $(ux)^z \notin \langle ux\rangle$, for all $u \in A_t\backslash\{e\}$, a contradiction because $\langle ux\rangle = H \wedge \langle z, ux\rangle$. Then $x \in C_G(A_t)$. Thus $y \notin C_G(A_t)$ and we obtain a similar contradiction since $\langle z\rangle$ is an S^*–complement of H_1 in M. Hence $x^r y^s \in C_G(A)$ and c) is verified. Assume finally that the Sylow p–subgroup A_p of A is not contained in $Z(G)$. By c), it is sufficient to prove that, if x_1 is an element of B of prime order q_1 such that $x_1 \notin C_B(A_p)$, then $x_2 \in C_B(A_p)$, for all x_2 of prime order $q_2 \neq q_1$. Assume that this statement is false and consider the subgroup $M' = \langle x_1, x_2\rangle A$. Since $H' = A\langle x_1\rangle$ is a Hall normal subgroup of L, the subgroup $\langle x_2\rangle$ must be an S^*–complement of H' in M'. Let a be an element of $A_p\backslash\{e\}$ so that $\langle ax_1\rangle = H' \wedge \langle ax_1, x_2\rangle$. By Lemma 22, it obtains that $(ax_1)^{x_2} \notin \langle ax_1\rangle$ and this contradiction shows that d) holds.

Conversely, suppose that G is a semidirect product of two coprime abelian subgroups A and B satisfying a), b), c) and d). By induction on $|G|$, we shall prove that G is an S^*C–group. Let H be a subgroup of G and assume first that A is not contained in H. Then A can be written as $A = (A \wedge H) \times A_1$ with $A_1 \neq \{e\}$. Since the assumptions are inherited by homomorphic images, the factor group G/A_1 is an S^*C–group. Then, using

Lemma 21, H has an S^*–complement in G. Let assume now that H contains A. We have $H = A(H \wedge B)$, therefore a complement K of $H \wedge B$ in B is also a complement of H in G. If $K \subseteq C_G(A)$, then K is an S^*–complement of H in G. Assume that K is not contained in $C_G(A)$ and decompose K as a direct product $K = \prod_{i=1}^{n} \langle c_i \rangle$ of subgroups of prime order such that the number of direct factors not centralizing A is minimal. Let $\langle c_i \rangle$ be one of such factor and $\{A_{p_{i_1}}, A_{p_{i_2}}, ...A_{p_{i_m}}\}$ be the set of Sylow subgroups of A that are not centralized by c_i. Suppose that $H \wedge B$ is not contained in $C = \bigwedge_{j=\overline{1,m}} C_G(A_{p_{i_j}})$. Then there exists an element h of prime order in $H \wedge B$ such that $h \notin C_G(A_{p_{i_t}})$, for some $t \in \{1, 2, ..., m\}$. By d), the elements h and c_i have the same order q, so that c) implies the existence of two integers r, s with $0 < r, s < q$ and $h^r c_i^s \in C_G(A)$. Then the subgroup $K_1 = \langle h^r c_i^s \rangle \times \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \langle c_j \rangle \right)$ is also a complement of H in G, a contradiction by the minimal choice of K. Then $H \wedge B \subseteq C$ and so c_i normalized every subgroup of H. Hence K is an S^*–complement of H in G and G is an S^*C–group. ■

In the following we shall give a direct product characterization of those finite groups G such that $L(G)$ is an ortholattice. Since the lattice we obtain is modular, we are also giving a characterization of those finite groups G such that $L(G)$ is orthomodular.

Theorem 24 (G. Whitson [29]). *Let G be a finite group and $L(G)$ be its subgroup lattice. Then $L(G)$ is an ortholattice if and only if $G \cong \bigtimes_{i \in I} G_i$, where $(G_i)_{i \in I}$ is a family of P–groups with projective dimension ≤ 1 and having relatively prime orders.*

Proof. If $G \cong \bigtimes_{i \in I} G_i$, where $(G_i)_{i \in I}$ is a family of P–groups with projective dimension ≤ 1 and having relatively prime orders, then $L(G) \cong \bigtimes_{i \in I} L(G_i)$, by [24], Theorem 4, page 5. Since each $L(G_i)$ is an ortholattice, $L(G)$ is itself an ortholattice.

Conversely, suppose that $L(G)$ is an ortholattice. Then $\perp$ is an auto–duality of $L(G)$, which implies that $G \cong \bigtimes_{i \in I} G_i$, where $(|G_i|, |G_j|) = 1$ for all $i \neq j$ and each G_i is either solvable or simple (see G. Zacher [31]). If G_i is a simple group with a dual, then it is a simple group of prime order (see

G. Zacher [32]). If G_i is a solvable group with a dual, then it is an M–group and so $G_i \cong \bigtimes_{j\in I_i} G_{ij}$, where $(|G_{ij}|, |G_{ik}|) = 1$ for all $j \neq k$ and each G_{ij} is a P–group or an M–group of prime power order which is not a hamiltonian 2–group. Every $L(G_{ij})$ is an irreducible complemented modular lattice and so G_{ij} is a P–group. Hence G is isomorphic to a direct product of P–groups with projective dimension ≤ 1 and having relatively prime orders. ∎

Corollary 25. *Let G be a finite group and $L(G)$ be its subgroup lattice. Then $L(G)$ is orthomodular if and only if $G \cong \bigtimes_{i\in I} G_i$, where $(G_i)_{i\in I}$ is a family of P–groups with projective dimension ≤ 1 and having relatively prime orders.*

Definition 26. We say that a group G is a *B–group* if its subgroup lattice $L(G)$ is a boolean algebra.

Using Theorem 24, it is very easy to characterize finite B–groups, as shows our last result.

Theorem 27 (G. Whitson [29]). *A finite group G is a B–group if and only if it is cyclic of square–free order.*

Proof. If G is a B–group, then $G \cong \bigtimes_{i\in I} G_i$, where $(G_i)_{i\in I}$ is a family of P–groups satisfying the same properties as in Theorem 24. Since $L(G) \cong \bigtimes_{i\in I} L(G_i)$, every $L(G_i)$ is a distributive lattice. Hence the projective dimension of every G_i is 0 and so G_i is a cyclic group of prime order.

The converse implication is obvious. ∎

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