Distributivity in lattices of fuzzy subgroups

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ABSTRACT

The main goal of this paper is to study the finite groups whose lattices of fuzzy subgroups are distributive. We obtain a characterization of these groups which is similar to a well-known result of group theory.

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1. Introduction

One of the most important facets of fuzzy logic is constituted by fuzzy set theory (see [29]). This topic has enjoyed a rapid evolution in the last years (for example, see [7]). It is an ingredient in the development of information technologies: information processing, information systems, Artificial Intelligence, and so on. Applications have also been found in industrial engineering [8], in decision and organization sciences (preference modeling and multicriteria evaluation), medical sciences (fuzzy expert systems) or in many other disciplines pertaining to human-oriented studies such as cognitive psychology and some aspects of social sciences. Several new research directions appeared in connection with mathematical branches: category theory, topology, algebra, analysis or probability theory (especially (GTU) – see [28]).

In algebra, a fundamental domain is the lattice theory. The formation of lattices is an important feature of many structures such as subgroups of a group, ideals of a ring, submodules of a module, or ideals of a lattice. Consequently, the study of some basic properties of these structures can be carried out within the purview of lattice theory. One of the most famous examples that illustrate this fact is concerning to finite groups and due to Ore [19]: a finite group is cyclic if and only if its lattice of subgroups is distributive.

The concept of fuzzy subgroup of a group has been introduced by Rosenfeld [20]. After 1971, an important step in fuzzy subgroup theory was made by Liu [12], which formulated the notion of fuzzy normal subgroup. This is closely connected to that of fuzzy congruence, studied in [2,4,9,11], or [13]. The set FL(G) of all fuzzy subgroups of a given group G forms a complete lattice under the usual ordering of fuzzy set inclusion (see [1]). Several remarkable sublattices of FL(G) have been investigated in [1–3]. From these, we mention the lattice FN(G) of all fuzzy normal subgroups which is modular, as show [3], or [6]. Also, recall here the technique initiated in [6] (see [27], too) to derive fuzzy theorems from their crisp versions.

Another significant direction in the above study is to classify the fuzzy subgroups of a finite group. This can be made by introducing some natural equivalence relations on its lattice of fuzzy subgroups. Many papers have treated the particular
case of finite cyclic groups. Thus, in [15] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [16–18] and [26] deal with this number for cyclic groups of order \( p^aq^m \) (\( p, q \) primes). Remind also the paper [25] where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups of an arbitrary finite cyclic group. It plays an essential role in establishing of an explicit formula for the well-known central Delannoy numbers (see [24]).

The distributivity constitutes a very powerful property of a lattice. In this paper it is studied for the lattice \( FL(G) \) of fuzzy subgroups of a finite group \( G \). We shall use a new equivalence relation on \( FL(G) \) that permits us to link this lattice to the lattice of subgroups of \( G \). In this way, a result similar to Ore’s theorem will be obtained.

The paper is organized as follows: in Section 2 we present some preliminary definitions and results on the subgroups and fuzzy subgroups of (finite) groups. Section 3 deals with a connection between the subgroup lattice \( L(G) \) and the fuzzy subgroup lattice \( FL(G) \) associated to a group \( G \). Section 4 is dedicated to proving our main theorem, that characterizes the finite groups \( G \) for which \( FL(G) \) is a distributive lattice. In the final section some conclusions and further research directions are indicated.

Most of our notation is standard and will usually not be repeated here. Basic notions and results on groups (respectively on fuzzy groups) can be found in [22] (respectively in [10]). For subgroup lattice concepts we refer the reader to [21,23].

2. Preliminaries

Let \((G, \cdot, e)\) be a group, where \( e \) denotes the identity of \( G \). Then the set \( L(G) \) of all subgroups of \( G \) is a complete bounded lattice with respect to set inclusion, called the subgroup lattice of \( G \). Note that \( L(G) \) has initial element the trivial subgroup \( \{e\} \) and final element \( G \), and its binary operations \( \wedge, \vee \) are defined by

\[
H \wedge K = H \cap K, \quad H \vee K = (H \cup K) \quad \text{for all } H, K \in L(G).
\]

A remarkable modular sublattice of \( L(G) \) is the normal subgroup lattice \( N(G) \), which consists of all normal subgroups in \( G \). Many results concerning to the relation between the structure of a group and the structure of its (normal) subgroup lattice are presented in [21,23]. We recall here only the following beautiful theorem, constituting the starting point for our discussion.

**Theorem.** (Ore [19]) A group \( G \) is locally cyclic if and only if \( L(G) \) is a distributive lattice. In particular, a finite group \( G \) is cyclic if and only if \( L(G) \) is a distributive lattice.

In the following, let us denote by \( \mathcal{F}(G) \) the collection of all fuzzy subsets of \( G \) (which is a complete and completely distributive lattice under the usual intersection, union and containment of fuzzy sets) and take an element \( \mu \) of \( \mathcal{F}(G) \). Then \( \mu \) is said to be a fuzzy subgroup of \( G \) if it satisfies the next two conditions:

(a) \( \mu(xy) \geq \min(\mu(x), \mu(y)) \), for all \( x, y \in G \);

(b) \( \mu(x^{-1}) = \mu(x) \), for any \( x \in G \).

In this situation we have \( \mu(x^{-1}) = \mu(x) \), for any \( x \in G \), and \( \mu(e) = \sup \mu(G) \). If \( \mu \) satisfies the supplementary condition

\[
\mu(xy) = \mu(yx) \quad \text{for all } x, y \in G,
\]

then it is called a fuzzy normal subgroup of \( G \). As in the case of subgroups, the sets \( FL(G) \) and \( FN(G) \) consisting of all fuzzy subgroups and all fuzzy normal subgroups of \( G \) form lattices with respect to fuzzy set inclusion (more precisely, \( FN(G) \) is a sublattice of \( FL(G) \)), called the fuzzy subgroup lattice and the fuzzy normal subgroup lattice of \( G \), respectively. Their binary operations \( \wedge, \vee \) are defined by

\[
\mu \wedge \eta = \mu \cap \eta, \quad \mu \vee \eta = \langle \mu \cup \eta \rangle \quad \text{for all } \mu, \eta \in FL(G),
\]

where \( \langle \mu \cup \eta \rangle \) denotes the fuzzy subgroup of \( G \) generated by \( \mu \cup \eta \) (that is, the intersection of all fuzzy subgroups of \( G \) containing both \( \mu \) and \( \eta \)).

For each \( \alpha \in [0, 1] \), we define the level subset

\[
\mu_{\alpha} = \{ x \in G | \mu(x) \geq \alpha \}.
\]

These subsets allow us to characterize the fuzzy (normal) subgroups of \( G \), in the following manner: \( \mu \) is a fuzzy (normal) subgroup of \( G \) if and only if its level subsets are (normal) subgroups in \( G \). We also mention that the fuzzy subgroup \( \mu^* = \langle \mu \rangle \) generated by \( \mu \in \mathcal{F}(G) \) can be described by using the level subsets \( \mu_{\alpha} \), \( \alpha \in [0, 1] \):

\[
\mu^*(x) = \sup \{ \alpha | x \in \mu_{\alpha} \} \quad \text{for any } x \in G.
\]

The fuzzy subgroups of \( G \) can be classified up to some natural equivalence relations on \( \mathcal{F}(G) \). One of them (used in [25,26], too) is defined by

\[
\mu \sim \eta \iff (\mu(x) > \mu(y) \iff \eta(x) > \eta(y)) \quad \text{for all } x, y \in G.
\]
and two fuzzy subgroups $\mu, \eta$ of $G$ are said to be distinct if $s \mu \eta$. This equivalence relation generalizes that used in Murali’s papers [14–18]. It is also closely connected to the concept of level subgroup. In this way, suppose that the group $G$ is finite and let \( \mu : G \to [0,1] \) be a fuzzy subgroup of $G$. Put \( \mu(G) = \{x_1, x_2, \ldots, x_r\} \) and assume that $x_1 > x_2 > \cdots > x_r$. Then $\mu$ determines the following chain of subgroups of $G$ which ends in $G$:

\[ \mu G_{2_2} \subset \mu G_{2_3} \subset \cdots \subset \mu G_r = G. \]

Moreover, for any $x \in G$ and $i = 1, 2, \ldots, r$ we have

\[ \mu(x) = x_i \iff i = \max\{j \mid x \in \mu G_j\} \iff x \in \mu G_i \setminus \mu G_{i+1}, \]

where, by convention, we set \( \mu G_0 = \emptyset \).

A necessary and sufficient condition for two fuzzy subgroups $\mu, \eta$ of $G$ to be equivalent with respect to $\sim$ has been identified in [26]: $\mu \sim \eta$ if and only if $\mu$ and $\eta$ have the same set of level subgroups, that is they determine the same chain of subgroups of type (*). This result shows that there exists a bijection between the equivalence classes (modulo $\sim$) of fuzzy subgroups of $G$ and the set of chains of subgroups of $G$ that terminate in $G$. So, many problems on $FL(G)$ (for example, the counting of all distinct fuzzy subgroups of $G$, studied in [25]) can be translated as problems in $L(G)$. This technique will be used in our paper, too.

Finally, recall that a lattice $L$ is called distributive if the identity

\[ a \land (b \lor c) = (a \land b) \lor (a \land c), \]

holds for all $a, b, c \in L$.

### 3. An equivalence relation on $FL(G)$

Let $(G, \cdot, e)$ be a finite group. As we have seen in the previous section, any fuzzy subgroup of $G$ determines a chain of subgroups in $G$ of type

\[ H_1 \subset H_2 \subset \cdots \subset H_r = G \]

and two fuzzy subgroups $\mu, \eta$ of $G$ are equivalent with respect to $\sim$ if they induce the same chain of this type. Starting from this remark, a new binary relation (denoted by $\approx$) can be defined on $FL(G)$ by deleting all terms $H_i$, $i = \prod_{i=1}^r \prod_{i=1}^r$, of the above chain. So, for $\mu, \eta \in FL(G)$, we put $\mu \approx \eta$ if the first terms in the corresponding chains of $\mu$ and $\eta$ are the same. Clearly, $\approx$ is an equivalence relation on $FL(G)$, which is weaker than $\sim$ ($\mu \sim \eta$ implies that $\mu \approx \eta$). By using the level subgroups, we can describe $\approx$ in a more simple way:

\[ \mu \approx \eta \iff \mu G(e) = \eta G(e). \]

A set of representatives for the equivalence classes of $FL(G)$ modulo $\approx$ is

\[ L = \{\mu_H \mid H \in L(G)\}, \]

where

\[ \mu_H : G \to [0,1], \ \mu_H(x) = \begin{cases} 1, & x \in H \\ 0, & x \notin H \end{cases} \quad \text{for all } H \in L(G). \]

Moreover, it is easy to see that for any two subgroups $H$ and $K$ of $G$, we have

\[ \mu_H \lor \mu_K = \mu_{H \lor K}. \quad (1) \]

Since $\mu_H \lor \mu_K = (\mu_H \lor \mu_K)$ and $\mu_{H \lor K}$ is a fuzzy subgroup of $G$ which contains $\mu_H \lor \mu_K$, one obtains that

\[ \mu_H \lor \mu_K \subseteq \mu_{H \lor K}. \quad (2) \]

Next, let $x \in G$ such that $\mu_{H \lor K}(x) = 1$. Then $x$ belongs to $H \lor K$, the subgroup of $G$ generated by $H \cup K$. Because

\[ \mu_{H \lor K} G_1 = \{a \in G \mid (\mu_H \lor \mu_K)(a) \geq 1\} = \{a \in G \mid \max\{\mu_H(a), \mu_K(a)\} \geq 1\} = \{a \in G \mid (\mu_H(a) \geq 1 \lor \mu_K(a) \geq 1\} = \{a \in G \mid (\mu_H(a) = 1 \lor \mu_K(a) = 1\} = \{a \in G \mid a \in H \lor a \in K\} = H \cup K, \]

it results $x \in (\mu_{H \lor K} G_1)$ and so $\sup\{x \mid x \in (\mu_{H \lor K} G_1)\} = 1$. This equality can be rewritten as $(\mu_H \lor \mu_K)(x) = 1$, therefore the following implication holds

\[ \mu_{H \lor K}(x) = 1 \Rightarrow (\mu_H \lor \mu_K)(x) = 1, \]

showing that

\[ \mu_{H \lor K} \subseteq \mu_H \lor \mu_K. \quad (3) \]

Now, from (2) and (3) we obtain

\[ \mu_H \lor \mu_K = \mu_{H \lor K}. \quad (4) \]

which together with (1) lead us to the following result.
Proposition 3.1. The set \( L = \{ \mu_H \mid H \in L(G) \} \) is a sublattice of \( FL(G) \).

Remark 3.1. In general, \( \sim \) is not a congruence relation on \( FL(G) \), even if the factor set \( FL(G)/\sim \) possesses a lattice structure (isomorphic to \( L \)), by Proposition 3.1. Next, let us define the map \( f : L(G) \to FL(G) \) by
\[
f(H) = \mu_H \quad \text{for all } H \in L(G).
\]
Obviously, \( f \) is one-to-one and \( f(1) = L \). On the other hand, the equalities (1) and (4) show that \( f \) is a lattice homomorphism. Hence the following proposition holds.

Proposition 3.2. The map \( f : L(G) \to FL(G) \) defined by \( f(H) = \mu_H \), for all \( H \in L(G) \), is a lattice embedding. In particular, the lattices \( L(G) \) and \( L \) are isomorphic.

Since \( L(G) \) can be seen as a sublattice of \( FL(G) \), any global property of \( FL(G) \) can be transported on \( L(G) \). This is the technique which will be used for distributivity in the next section.

Remark 3.2. Let \( H \in N(G) \) and \( x, y \) be two arbitrary elements of \( G \) such that \( xy \in H \). Then we have \( (xH)(yH) = xyH = H \). So, \( (yH) = (xH)^{-1} \) in the factor group \( G \), which implies that \( xyH = (yH)(xH) = H \), that is \( xy \in H \). Therefore we have proved that
\[
xy \in H \iff xy \in H
\]
or equivalently
\[
\mu_H(xy) = \mu_H(yx).
\]
This shows that \( \mu_H \) is contained in \( FN(G) \), and hence \( f \) maps the normal subgroups of \( G \) into fuzzy normal subgroups. Because the lattice \( FN(G) \) is modular (see [3]), we can infer the following elementary result of group theory: the normal subgroup lattice of a finite group \( G \) is modular.

4. The distributivity of \( FL(G) \)

In this section we shall establish a necessary and sufficient condition for a finite group \((G, \cdot, e)\) to have a distributive fuzzy subgroup lattice. Suppose first that \( FL(G) \) is a distributive lattice. Then, by Proposition 3.2, the subgroup lattice \( L(G) \) of \( G \) is also distributive and therefore \( G \) is cyclic, according to Ore’s theorem.

Conversely, we shall show that the fuzzy subgroup lattice of a finite cyclic group is distributive. First of all, we need to prove two auxiliary results.

Lemma 4.1. If \( G \) is a finite cyclic \( p \)-group, then the lattice \( FL(G) \) is distributive.

Proof. It is well-known that under our hypothesis \( L(G) \) is a chain. Let \( \mu, \eta \) be two fuzzy subgroups of \( G \) and \( \alpha \in \text{Im} \mu \cup \eta \). Then \( \alpha \) is contained in the image of at least one of these fuzzy subgroups, say \( \alpha \in \text{Im} \eta \). So, \( \eta G_\alpha \) is a subgroup of \( G \). Put \( \text{Im} \mu = \{ \alpha_1, \alpha_2, \ldots, \alpha_k \} \), where \( 1 \geq \alpha_1 > \alpha_2 > \cdots > \alpha_k \geq 0 \). Then we have
\[
\mu G_\alpha = \begin{cases} 
\emptyset & \text{if } 1 \geq \alpha > \alpha_1 \\
\mu G_{\alpha_1} \cup \eta G_\alpha & \text{if } \alpha \geq \alpha > \alpha_{s+1} \quad \text{for some } s = 1, \ldots, k-1 \\
G & \text{if } \alpha \geq \alpha \geq 0.
\end{cases}
\]
which implies that
\[
\mu_\alpha \eta G_\alpha = \mu G_\alpha \cup \eta G_\alpha = \begin{cases} 
\eta G_{\alpha} & \text{if } 1 \geq \alpha > \alpha_1 \\
\mu G_{\alpha} \cup \eta G_\alpha & \text{if } \alpha \geq \alpha \geq \alpha_{s+1} \quad \text{for some } s = 1, \ldots, k-1 \\
G & \text{if } \alpha \geq \alpha \geq 0.
\end{cases}
\]
Since \( L(G) \) is fully ordered, the union of any two subgroups of \( G \) is also a subgroup in \( G \), and therefore \( \mu_\alpha \eta G_\alpha \in L(G) \). Thus, all level subsets of \( \mu \cup \eta \) are subgroups of \( G \), which show that \( \mu \cup \eta \in FL(G) \). Therefore \( \mu \vee \eta = \mu \cup \eta \), that is the binary operations in the lattices \( FL(G) \) and \( \mathcal{P}(G) \) are the same. Hence \( FL(G) \) is distributive, in view of the distributivity of \( \mathcal{P}(G) \). □

In the following, assume that \( G \) is an arbitrary finite cyclic group. Then it has a direct decomposition of type
\[
G = \prod_{i=1}^k \mathbb{Z}/p_i, 
\]
where \( G_i \) is a finite cyclic \( p_i \)-group, for all \( i = 1, \ldots, k \). Since \( G_i, i = 1, \ldots, k \), are of coprime orders, one obtains that
\[
L(G) \cong \prod_{i=1}^k L(G_i).
\]
a lattice isomorphism which is frequently used in order to reduce many problems on \( L(G) \) to the subgroup lattices of finite cyclic \( p \)-groups. Our next aim is to study whether a similar connection holds between the fuzzy subgroup lattices of \( G \) and \( G_i \), \( i = \mathbb{T}, k \).

**Lemma 4.2.** Let \( G \) be a finite cyclic group and \( G = \bigoplus_{i=1}^{k} G_i \) be the direct decomposition of \( G \) as product of finite cyclic \( p \)-groups. Then the map \( g : FL(G) \to \bigoplus_{i=1}^{k} FL(G_i) \) defined by

\[
g(\mu) = (\mu|G_1, \mu|G_2, \ldots, \mu|G_k), \quad \text{for all } \mu \in FL(G),
\]

is an one-to-one homomorphism of lattices.

**Proof.** Clearly, \( g \) is a lattice homomorphism and thus we have to prove only that \( g \) is one-to-one. Remark also that it is sufficient to verify this for \( k = 2 \). So, let \( G \) be a finite cyclic group of order \( n \) which possesses a direct decomposition of type

\[
G = G_1 \times G_2,
\]

where \( G_i \cong \mathbb{Z}_{p^m_i}, \ i = 1, 2 \ (p_1, p_2 \text{ distinct primes}). \) Taking a generator \( x \) of \( G \), we have

\[
G_1 = \langle x^{m_1} \rangle = \{x^{m_1u} \mid u = 0, 1, \ldots, p_1 - 1 \}
\]

and

\[
G_2 = \langle x^{m_2} \rangle = \{x^{m_2v} \mid v = 0, 1, \ldots, p_2 - 1 \}.
\]

We shall prove that any fuzzy subgroup \( \mu \) of \( G \) is uniquely determined by its restrictions on \( G_1 \) and \( G_2 \). Let \( s \in \{0, 1, \ldots, n - 1 \} \) and \( d = (s, n) \). Then \( ds \) and therefore there exists \( t \in \mathbb{N} \) such that \( s = dt \). By using the first condition in the definition of a fuzzy subgroup, one obtains that

\[
\mu(x^s) = \mu(x^{dt}) \geq \mu(x^t).
\]

On the other hand, \( d \) can be written as \( d = as + bn \) with \( a, b \in \mathbb{Z} \). Then \( x^d = x^{as+bn} = x^ax^bn = x^s \), which implies that

\[
\mu(x^d) = \mu(x^{as}) \geq \mu(x^s).
\]

Obviously, the inequalities (5) and (6) give us

\[
\mu(x^s) = \mu(x^d).
\]

Because \( d = p_1^r_1 p_2^r_2 \) and \( d | n \), we have \( d = p_1^r_1 p_2^r_2 \) for some \( r_1 \in \{0, 1, \ldots, m_1 - 1 \} \) and \( r_2 \in \{0, 1, \ldots, m_2 - 1 \} \). Let \( x^{s_d} = x^{s_1} x^{s_2} = x^{s_1} x^{p_2^r_2} \), in view of the direct decomposition of \( G \). Then \( p_1^{r_1} \mu(x^s) \) and \( p_2^{r_2} \mu(x^t) \); therefore \( u \) and \( v \) are of type \( u = p_1^{r_1} u^* \) and \( v = p_2^{r_2} v^* \), respectively. It follows that

\[
\mu(x^{s_d}) = \mu(x^{s_1}) = \mu(x^{p_2^r_2}) \mu(x^{s_2}) \geq \min \{\mu(x^{p_2^r_2}), \mu(x^{s_2})\} = \min \{\mu(x^{p_2^r_2}), \mu(x^{s_1} x^{p_2^r_2})\}
\]

Thus, we have

\[
\mu(x^{s_d}) = \mu(x^{s_1}) = \mu(x^{p_2^r_2}) \mu(x^{s_2}) = \mu(x^{s_1} x^{p_2^r_2}) = \mu(x^{s_1}) \mu(x^{p_2^r_2}) = \mu(x^{s_1} x^{p_2^r_2}) = \mu(x^{s_d}).
\]

which together with (7) show that the values \( \mu(x^s), s = 0, n - 1 \), depend only on \( \mu|G_1 \) and \( \mu|G_2 \). Hence the map \( g \) is one-to-one.

**Remark 4.1.** Suppose that \( k \geq 2 \) in the above lemma. Then, by taking two fuzzy subgroups \( \mu_1 \in FL(G_1) \) and \( \mu_2 \in FL(G_2) \) such that \( \mu_1(e) \neq \mu_2(e) \), there is no fuzzy subgroup \( \mu \) of \( G \) satisfying \( \mu|G_1 = \mu_1 \) and \( \mu|G_2 = \mu_2 \). Therefore the map \( g \) is not onto, and so \( FL(G) \not\cong \bigoplus_{i=1}^{k} FL(G_i) \). Hence the decomposability of the subgroup lattice of a finite cyclic group is a property which cannot be extended to the fuzzy case.

Now we are able to prove the distributivity of the fuzzy subgroup lattice \( FL(G) \) of a finite cyclic group \( G \). From **Lemma 4.2** we know that \( FL(G) \) can be embedded in \( \bigoplus_{i=1}^{k} FL(G_i) \). Since by **Lemma 4.1** all the lattices \( FL(G_i), i = \mathbb{T}, k \), are distributive, so is their direct product \( \bigoplus_{i=1}^{k} FL(G_i) \). Thus \( FL(G) \) is distributive, too.

Hence the following theorem holds.
Theorem 4.3. A finite group $G$ is cyclic if and only if $FL(G)$ is a distributive lattice.

Moreover, by using Ore's theorem, we also infer the following corollary.

Corollary 4.4. Given a finite group $G$, the subgroup lattice $L(G)$ is distributive if and only if the fuzzy subgroup lattice $FL(G)$ is distributive.

5. Conclusions and further research

The main idea of the present paper (as well as of [25]) is to define an equivalence relation on the fuzzy subgroup lattice $FL(G)$ of a (finite) group $G$, which leads us to a connection between $FL(G)$ and the subgroup lattice of $G$. By using this method, the problem of classifying fuzzy subgroups of finite abelian groups is treated in [25] and the distributivity of $FL(G)$ is studied above. It can successfully be applied to study other important properties of $FL(G)$, as modularity, complementation, decomposability, ... and so on. The same properties (excepting modularity, that has been exhaustively investigated in [3]) can be also studied for $FN(G)$. These will surely constitute the subject of some further research.

We finish our paper by indicating another two interesting open problems concerning to fuzzy subgroup lattices, which are derived from similar problems on subgroup lattices.

Problem 1. Let $L$ be a lattice. Study the existence and the uniqueness of a (finite) group $G$ such that $FL(G)$ is isomorphic to $L$.

Problem 2. Given a lattice $L$, a group $G$ is said to be $F$–$L$–free if $FL(G)$ does not contain a sublattice isomorphic to $L$. For some particular lattices $L$ (as the pentagon lattice or the diamond lattice – see [5]), study the class of $F$–$L$–free groups.

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