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Abstract

The aim of this chapter is to provide new insights of the return map of a given ellipse E. The main tool of this approach is the eccentricity ε of E and hence various expressions and transformations of ε are used in order to obtain new aspects concerning both the return map and the associated delta map. Also, we study the return map in the setting of plane hyperbolic geometry.

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The Return Map of an Ellipse: New Theoretical and Practical Aspects

Mircea Crasmareanu

Abstract The aim of this chapter is to provide new insights of the return map of a given ellipse E. The main tool of this approach is the eccentricity ε of E and hence various expressions and transformations of ε are used in order to obtain new aspects concerning both the return map and the associated delta map. Also, we study the return map in the setting of plane hyperbolic geometry.

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Introduction

Recall that the ellipse is a fundamental geometrical object, more precisely, a remarkable plane curve, that is a commonly encountered model of various practical aspects, for instance in the Newtonian motion, like the orbits of planets, moons orbiting planets, etc. More recently, the ellipse is a very useful form in (both civil and sacral) architecture and industrial design. In the present study we focus on one pure mathematical aspects from this very rich theory.

Fix an ellipse E of eccentricity $\varepsilon \in (0, 1)$ and a positive integer $n \in \mathbb{N}^*$; for the general theory of conics we recommend the very recent book [13]. We recall briefly the construction of the return map of E: a ray emanating from one focus of E is reflected by the ellipse in such a way as to pass through its other focus. After $2n$ such reflections an initial ray from the focus F_1 is related to a final ray from F_1 and the point z where the initial ray meets the unit circle centred at F_1 is related to the point w_n where the final ray meets this circle.

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The aim of this chapter is to study these expressions of the functions $\delta = \delta(\varepsilon)$, $w_n = w_n(z)$ in various settings and to discuss some aspects of them; a strong motivation for our study is the lack of examples in [17]. It is amazing that several numerical constants and angles which we find in this work have also some geometrical and (or) physical significance.

The content of this work is as follows. In the first section we present several aspects of the return map expressed as numbered remarks. A main result is the form of the Iwasawa decomposition of the n-return map as matrix from $SL(2, \mathbb{R})$. Three classes of examples are discussed, two of them starting from hyperbolas, namely the equilateral and a Barning-type one.

The second section studies the return map in the framework of the plane hyperbolic geometry by using as model the Poincaré upper half-plane model. In this setting we find a geometric interpretation for the involved δ map as being half of the hyperbolic distance between the unit complex i and its first return $w_1(i)$. We pointed also that the hyperbolic distance between i and $w_n(i)$ tends to infinity as n tends to ∞ in the scale of 2δ. We discuss also the complex expression of the given ellipse as well as the Hopf invariant of the matrix associated to the n-return map. An example studied in both sections is provided by *self-complementary ellipses* characterized by $\varepsilon = \frac{1}{\sqrt{2}}$.

1 Several Remarks and Some Concrete Examples

In the paper [17] the n-return map of E, namely $z \in S^1 \to w_n \in S^1$, is obtained as the restriction to the unit circle S^1 of the Möbius transformation:

$$w_n = \frac{(\cosh n\delta)z - \sinh n\delta}{(-\sinh n\delta)z + \cosh n\delta}, \quad \cosh \delta = \frac{1+\varepsilon^2}{1-\varepsilon^2}. \tag{1.1}$$

Allowing the map w_n to be extended on the real axis gives the existence of two points $o^n_{1,2} \in \mathbb{R}$ which are mapped to their opposite (or antipodal) $o^n_{\pm}$:

$$o^n_{\pm} = \frac{\cosh(n\delta) \pm 1}{\sinh(n\delta)}, \quad o^1_{-} = \varepsilon = \frac{1}{o^1_{+}}, \quad o^n_{-} = \tanh\left(\frac{n}{2}\delta\right) = \frac{1}{o^n_{+}}.$$

The fixed points of w_n being ± 1 are also universal i.e. do not depend on n or ε.

First of all we clarify the geometrical setting of the return map. Since for an arbitrary point $P \in \mathbb{R}^k$ its tangent space $T_P\mathbb{R}^k$ is naturally identified with $\mathbb{R}^k$ we have the tangent plane $T_{F_1}\mathbb{R}^2 = \mathbb{R}^2$. Given now a point P in a general Riemannian manifold (M^k, g) its indicatrix is the hypersurface of g-unit tangent vectors i.e. $I_P(M, g) = \{X \in T_P M; g(X, X) = 1\}$ of $T_P M$. Hence, returning to the given ellipse E the considered unit circle centred at F_1 is exactly the indicatrix $I_{F_1}(\mathbb{R}^2, g_{Euclidean})$ corresponding to the Euclidean metric of plane. So, the n-return map is $z \in I_{F_1}(\mathbb{R}^2, g_{Euclidean}) \to w_n \in I_{F_1}(\mathbb{R}^2, g_{Euclidean})$ or in the projective manner:

$$[z, 1] \in P^1(\mathbb{C}) := \mathbb{C} \cup \{\infty\} \to [(\cosh n\delta)z - \sinh n\delta, (-\sinh n\delta)z + \cosh n\delta] \in P^1(\mathbb{C}).$$

54

The n-return map can be considered as the 2×2 matrix belonging to $Sym(2) \cap SL(2, \mathbb{R})$, as usually with $exp : sl(2, \mathbb{R}) \to SL(2, \mathbb{R})$:

$$\Gamma_n(\varepsilon) = \begin{pmatrix} \cosh(n\delta) & -\sinh(n\delta) \\ -\sinh(n\delta) & \cosh(n\delta) \end{pmatrix} = exp\left(\begin{pmatrix} 0 & -n\delta \\ -n\delta & 0 \end{pmatrix}\right),$$

$$\Gamma_1(\varepsilon) = \frac{1}{1-\varepsilon^2}\begin{pmatrix} 1+\varepsilon^2 & -2\varepsilon \\ -2\varepsilon & 1+\varepsilon^2 \end{pmatrix},$$

55

56

$$\Gamma_2(\varepsilon) = \frac{1}{(1-\varepsilon^2)^2}\begin{pmatrix} 1+6\varepsilon^2+\varepsilon^4 & -4\varepsilon(1+\varepsilon^2) \\ -4\varepsilon(1+\varepsilon^2) & 1+6\varepsilon^2+\varepsilon^4 \end{pmatrix}. \quad (1.2)$$

The map $\varepsilon \in \mathbb{R} \to \Gamma_1(\varepsilon)$ is a smooth curve on Lie group $SL(2, \mathbb{R}) \simeq Sp(2, \mathbb{R}) \simeq SU(1, 1) \simeq O(2, 1)$ (the first three Lie groups are covering spaces for the last one) through its neutral element $I_2 = \Gamma_1(0)$ and hence the derivative in $\varepsilon = 0$ belongs to its Lie algebra:

$$\frac{d}{dt}\Gamma_1(\varepsilon)|_{\varepsilon=0} = -2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in sl(2, \mathbb{R}) = \{A \in M_2(\mathbb{R}); TrA = 0\},$$

which is (modulo its coefficient -2 and considered as a linear endomorphism of the plane) the matrix of the symmetry with respect to the first bisectrix $B_1 : y = x$. Recall also the standard basis of the Lie algebra $sl(2, \mathbb{R})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, [h, e^\pm] = \pm 2e^\pm, [e^+, e^-] = h.$$

A well-known representation of the basis of $sl(2, \mathbb{R})$ is through vector fields of degree two real polynomial coefficients:

$$h = -2t\frac{d}{dt}, \quad e^- = -\frac{d}{dt}, \quad e^- = t^2\frac{d}{dt}.$$

The associated Hermitian bilinear form on $\mathbb{C}^2$ to the matrix above is:

$$\begin{cases} \Phi(z, w) = (z_1, z_2) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \overline{w_1} \\ \overline{w_2} \end{pmatrix} = z_2\overline{w_1} + z_1\overline{w_2}, \\ \Phi(z, z) = 2Re(z_1\overline{z_2}) = 2 < z_1, z_2 >_{\mathbb{R}^2}, \end{cases}$$

57 for $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in $\mathbb{C}^2$.

Remark 1.1 We start by recalling the Iwasawa decomposition of $\Gamma \in SL(2, \mathbb{R})$ following [2]:

608367_1_En_1_Chapter-online ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:4/12/2024 Pages: 18 Layout: T1-Standard

$$\Gamma = Rotation(t) \cdot \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, Rotation(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, r > 0.$$

A straightforward computation yields:

Proposition 1.2 *The Iwasawa parameters $t_n = t_n(\varepsilon)$, $r_n = r_n(\varepsilon)$ and $\alpha_n = \alpha_n(\varepsilon)$ of Γ_n are given by:*

$$\begin{cases} \cos t_n = \frac{\cosh(n\delta)}{\sqrt{\cosh(2n\delta)}}, & \sin t_n = -\frac{\sinh(n\delta)}{\sqrt{\cosh(2n\delta)}}, \\ r_n = \sqrt{\cosh(2n\delta)}, & \alpha_n = -\tanh(2n\delta) < 0. \end{cases}$$

By denoting the elements of the first column of Γ_n with γ^n_{11} respectively γ^n_{21} it results:

$$r_n = \sqrt{\gamma^{2n}_{11}}, \quad \cos t_n = \frac{\gamma^n_{11}}{r_n}, \quad \sin t_n = \frac{\gamma^n_{21}}{r_n}, \quad \alpha_n = \frac{\gamma^{2n}_{21}}{\gamma^{2n}_{11}}.$$

In particular:

$$\begin{cases} \cos t_1(\varepsilon) = \frac{1+\varepsilon^2}{\sqrt{1+6\varepsilon^2+\varepsilon^4}}, & \sin t_1(\varepsilon) = \frac{-2\varepsilon}{\sqrt{1+6\varepsilon^2+\varepsilon^4}}, \\ r_1(\varepsilon) = \frac{\sqrt{1+6\varepsilon^2+\varepsilon^4}}{1-\varepsilon^2}, & \alpha_1(\varepsilon) = -\frac{4\varepsilon(1+\varepsilon^2)}{1+6\varepsilon^2+\varepsilon^4}. \end{cases}$$

Remark 1.3 The symmetric matrix Γ_n has the eigenvalues $\lambda^1_n = e^{-n\delta}$, $\lambda^2_n = e^{n\delta}$ which are the foci of the ellipse which is the numerical range of Γ_n. The general theory of Möbius transformations gives the identities:

$$\frac{w_n(z) - \lambda^1_n}{w_n(z) - \lambda^2_n} = \rho \frac{z - \lambda^1_n}{z - \lambda^2_n}, \quad \rho = w'_n(\lambda^1_n) = \frac{1}{w'_n(\lambda^2_n)}.$$

With the universal (i.e. for all integers n) matrix:

$$S = Rotation\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in SO(2)$$

we have $S^t \cdot \Gamma_n \cdot S = diag(e^{-n\delta}, e^{n\delta})$. With respect to the usual classification of matrices Γ from $SL(2, \mathbb{R})$ in elliptic, parabolic and hyperbolic if $|Tr\Gamma| < 2$, $|Tr\Gamma| = 2$ respectively $|Tr\Gamma| > 2$ we get that Γ_n is a hyperbolic one and a 2-valued map on the set of hyperbolic matrices is considered in [9].

The product of two matrices Γ_1 is again a matrix in $Sym(2) \cap SL(2, \mathbb{R})$:

$$\Gamma_1(\varepsilon)\Gamma_1(\tilde{\varepsilon}) = \Gamma_1(\tilde{\varepsilon})\Gamma_1(\varepsilon) =$$

$$= \frac{1}{(1-\varepsilon^2)(1-\tilde{\varepsilon}^2)} \begin{pmatrix} (1+\varepsilon^2)(1+\tilde{\varepsilon}^2) + 4(\varepsilon\tilde{\varepsilon})^2 & -2[\tilde{\varepsilon}(1+\varepsilon^2) + \varepsilon(1+\tilde{\varepsilon}^2)] \\ -2[\tilde{\varepsilon}(1+\varepsilon^2) + \varepsilon(1+\tilde{\varepsilon}^2)] & (1+\varepsilon^2)(1+\tilde{\varepsilon}^2) + 4(\varepsilon\tilde{\varepsilon})^2 \end{pmatrix},$$

608367_1_En_1_Chapter-online ✓ TYPESET DISK LE ✓ CP Disp.:4/12/2024 Pages: 18 Layout: T1-Standard

but (in general) is not a Γ_1 matrix. For a fixed eccentricity we have the commutativity $\Gamma_n(\varepsilon) \cdot \Gamma_m(\varepsilon) = \Gamma_m(\varepsilon) \cdot \Gamma_n(\varepsilon)$. □

Remark 1.4 A Wick-type transformation, see [11], $\varepsilon \to i\varepsilon$ yields:

$$\cosh \delta(i\varepsilon) + \sinh \delta(i\varepsilon) = \frac{1-\varepsilon^2}{1+\varepsilon^2} + i\frac{2\varepsilon}{1+\varepsilon^2} \tag{1.3}$$

and in the right-hand side of the last relation we have the Pythagorean parametrization [6] $\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}, \frac{2\varepsilon}{1+\varepsilon^2}\right) := z(\varepsilon) = \frac{1+i\varepsilon}{1-i\varepsilon}$ of the unit circle S^1. The basis vector fields of $sl(2,\mathbb{R})$ applied to the map $\varepsilon \to z(\varepsilon)$ give:

$$\begin{cases} h(z(\varepsilon)) = \frac{-4i\varepsilon}{(1-i\varepsilon)^2}, \quad e^+(z(\varepsilon)) = \frac{-2i}{(1-i\varepsilon)^2}, \quad e^-(z(\varepsilon)) = \frac{2i\varepsilon^2}{(1-i\varepsilon)^2}, \\ h(z(\varepsilon)) = e^+(z(\varepsilon)) \to \varepsilon = \frac{1}{2}, \quad z\left(\frac{1}{2}\right) = \frac{3}{5} + i\frac{4}{5}, \quad \cosh\delta = \frac{5}{3}, \quad \sinh\delta = \frac{4}{3}. \end{cases}$$

Also the 1-return map applied exactly to $z(\varepsilon) \in S^1$ gives:

$$z(\varepsilon) = \frac{1+i\varepsilon}{1-i\varepsilon} \in S^1 \to w_1 = z\left(\varepsilon\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2\right) \in S^1 \tag{1.4}$$

and hence we obtain a new rational map which is strictly increasing:

$$R_1 : \varepsilon \in (0,1) \to \varepsilon\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 \in (0,+\infty), \quad R_1\left(\frac{1}{2}\right) = \frac{9}{2}. \tag{1.5}$$

The integral $\int_0^1 R_1(\varepsilon)d\varepsilon$ does not converge and the equation $R_1(\varepsilon) = 1$ is a cubic one:

$$\varepsilon^3 + \varepsilon^2 + 3\varepsilon - 1 = \left(\varepsilon + \frac{1}{3}\right)^3 + \frac{8}{3}\left(\varepsilon + \frac{1}{3}\right) - \frac{52}{27} = 0$$

with an unique real solution $\varepsilon = 0.2956...$ while an associated elliptic curve $EllC$: $y^2 = x^3 + x^2 + 3x - 1$ has two lattice points $(1, \pm 2)$. With the derivative of the cubic polynomial from the right-hand-side of $EllC$ we have an associated hyperbola $H : y^2 = 3x^2 + 2x + 3$ and its lattice point $P(-1,-2)$ provides a rational parametrization:

$$H \setminus \{P\} : (x, \pm y)(t) = \left(-\frac{t^2+4t+1}{t^2-3}, \pm\frac{2t^2+4t+6}{t^2-3}\right), \quad t \in \mathbb{R} \setminus \{\pm\sqrt{3}\}.$$

In fact, H has four families of infinite countable lattice points available with WolframAlpha. The center of H is $C\left(-\frac{1}{3}, 0\right)$ and the eccentricity is $\frac{2}{\sqrt{3}} > 1$ which appears also in our Example 1.14.

608367_1_En_1_Chapter-online ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:4/12/2024 Pages: 18 Layout: T1-Standard

The equality $R_1(\varepsilon) = \cosh\delta$ means the quartic equation:

$$\varepsilon^4 + 4\varepsilon^3 + 2\varepsilon^2 + 2\varepsilon - 1 = 0$$

with the only suitable solution $\varepsilon = 0.3229....$ □

Remark 1.5 The function $\delta = \delta(\varepsilon)$ can be expressed as:

$$\delta = 2artanh(\varepsilon) = 2(\tanh)^{-1}(\varepsilon), \quad o^n_- = \tanh[n\,(\tanh)^{-1}(\varepsilon)], \tag{1.6}$$

and then it is a bijective map with the inverse:

$$\varepsilon = \tanh\left(\frac{\delta}{2}\right) = \frac{\cosh\delta - 1}{\sinh\delta} = o^1_-,\ R_1\left(\tanh\left(\frac{\delta}{2}\right)\right) = \frac{e^{2\delta}(\cosh\delta - 1)}{\sinh\delta}. \tag{1.7}$$

In the paper [16] a *canonical function* on the set of ellipses with the same eccentricity ε is introduced through the formula:

$$C_{ellipse}(\varepsilon) := \frac{1}{8(1-\varepsilon^2)^2}\left[\pi\sqrt{1-\varepsilon^2} - 2\varepsilon(1-\varepsilon^2) - 2\sqrt{1-\varepsilon^2}\arcsin\varepsilon\right],$$

$$C_{ellipse}\left(\frac{1}{2}\right) = \frac{1}{9}\left(\frac{2\pi}{\sqrt{3}} - \frac{3}{2}\right). \tag{1.8}$$

A long but straightforward computation gives an expression of this canonical function as depending of δ:

$$C_{ellipse}(\delta) = \frac{(\sinh\delta)^3}{16\sqrt{2}(\cosh\delta - 1)^{\frac{3}{2}}}\left[\pi - \frac{[2(\cosh\delta - 1)]^{\frac{3}{2}}}{(\sinh\delta)^2} - 2\arcsin\left(\frac{\cosh\delta - 1}{\sinh\delta}\right)\right]. \tag{1.9}$$

□

Remark 1.6 The last section of the paper [5] deals with a remarkable class of ellipses, called *self-complementary*, and characterized by $\varepsilon = \frac{1}{\sqrt{2}}$. The corresponding δ is a transcendental number and $R_1\left(\frac{1}{\sqrt{2}}\right)$ is more greater than 1:

$$\delta = arcosh(3) = \ln(3 + 2\sqrt{2}) = 1.7672... = \int_1^3 \frac{dt}{\sqrt{t^2 - 1}},\ R_1\left(\frac{1}{\sqrt{2}}\right) = 12 + \frac{17}{\sqrt{2}}. \tag{1.10}$$

The matrices of the first two return maps are:

$$\Gamma_1\left(\frac{1}{\sqrt{2}}\right) = \begin{pmatrix} 3 & -2\sqrt{2} \\ -2\sqrt{2} & 3 \end{pmatrix},\ \Gamma_2\left(\frac{1}{\sqrt{2}}\right) = \begin{pmatrix} 17 & -12\sqrt{2} \\ -12\sqrt{2} & 17 \end{pmatrix} \tag{1.11}$$

while the point $z\left(\frac{1}{\sqrt{2}}\right)$ is $\left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right) \in S^1$. The angle $\arccos\left(\frac{1}{3}\right)$ appears in several studies, and recently in [15]. The general matrix Γ_n has the eigenvalues $\lambda_n^1 = (3+2\sqrt{2})^n$ and $\lambda_n^2 = (3-2\sqrt{2})^n$ while the first Iwasawa parameters are given by: $\cos t_1 = \frac{3}{\sqrt{17}}$, $\sin t_1 = -\frac{2\sqrt{2}}{\sqrt{17}}$, $r_1 = \sqrt{17}$, $\alpha_1 = -\frac{12\sqrt{2}}{17}$.

The explicit value of δ allows us to verify two theoretical results of [17]. The first one is the corollary 2:

$$\int_{\pi(1-e^{-n\delta/2})}^{\pi(1+e^{-n\delta/2})} \frac{d\varphi}{\cosh 2n\delta + (\sinh 2n\delta)\cos\varphi} > 2\pi(1-e^{-n\delta}).$$

For $n = 1$ the WolframAlpha gives the estimate:

$$\int_{\pi\frac{\sqrt{2}}{1+\sqrt{2}}}^{\pi\sqrt{2}} \frac{d\varphi}{17+12\sqrt{2}\cos\varphi} - \frac{4+4\sqrt{2}}{3+2\sqrt{2}}\pi = 0.923....$$

The second one is the theorem 4 which states that if $a > b > 0$ and $\lambda > 1$ then:

$$\int_{\pi(1-\frac{1}{\sqrt{\lambda}})}^{\pi(1+\frac{1}{\sqrt{\lambda}})} \frac{d\varphi}{a+b\cos\varphi} < \frac{2\pi\lambda}{a+b}.$$

A self-complementary ellipse has $a = \sqrt{2}b$ and then the inequality above becomes:

$$\int_{\pi(1-\frac{1}{\sqrt{\lambda}})}^{\pi(1+\frac{1}{\sqrt{\lambda}})} \frac{d\varphi}{\sqrt{2}+\cos\varphi} = 2\arctan[(\sqrt{2}-1)\tan\frac{x}{2}]|_{\pi(1-\frac{1}{\sqrt{\lambda}})}^{\pi(1+\frac{1}{\sqrt{\lambda}})} < \frac{2\pi\lambda}{\sqrt{2}+1}.$$

For example if $\lambda = 4$ the integral is $\frac{3\pi}{2}$, obviously lower than $\frac{8\pi}{\sqrt{2}+1}$. □

Remark 1.7 In [5] is recalled a method to obtain ellipses by using the Joukowski map $J : \mathbb{C}^* \to \mathbb{C}$, $J(z) := z + \frac{1}{z}$. Namely, the circle of radius $r > 1$ is transformed into the ellipse $E(r)$ with $a = r + \frac{1}{r}$ and $b = r - \frac{1}{r}$. Since its eccentricity is $\varepsilon(r) = \frac{2r}{r^2+1}$ we get:

$$\cosh\delta(r) = \frac{r^4+6r^2+1}{(r^2-1)^2}, R_1(r) = \frac{2r}{r^2+1}\left(\frac{r+1}{r-1}\right)^4 \to R_1(2) = \frac{4\cdot 3^4}{5}. \tag{1.12}$$

Inspired by the usual formulae of the Pythagorean triples we can exemplify with $r = \frac{m}{n}$ for

$1 \leq n < m \in \mathbb{N}^*$ and then:

$$\cosh \delta(m,n) = \frac{m^4 + 6m^2n^2 + n^4}{(m^2 - n^2)^2}, \; R_1(m,n) = \frac{2mn}{m^2 + n^2}\left(\frac{m+n}{m-n}\right)^4. \quad (1.13)$$

For example $\cosh \delta(2,1) = \frac{41}{9}$.

In the theory of quasiconformal mappings w there exists a number $0 < \|\mu\|_\infty < 1$ and an associated *dilatation* [2]:

$$K(w) := \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$$

considered as a supremum of the major to minor axes of infinitesimal ellipses into which w takes infinitesimal circles. If there exists an ellipse E with K above then its eccentricity is:

$$\varepsilon^2 = 1 - K^{-2} = \frac{4\|\mu\|_\infty}{(1 + \|\mu\|_\infty)^2}$$

and then an ansatz $\|\mu\|_\infty = r^2$ yields exactly our $\varepsilon = \frac{2r}{r^2+1}$ by allowing also a value $\|\mu\|_\infty > 1$. □

Remark 1.8 Another possible expression of the eccentricity studied in [5] is $\varepsilon = \sin \varphi$ with the angle $\varphi \in \left(0, \frac{\pi}{2}\right)$. Then:

$$\sinh \delta(\varphi) = \frac{2 \sin \varphi}{\cos^2 \varphi}, \quad R_1(\varphi) = \sin \varphi \left(\frac{1 + \tan \frac{\varphi}{2}}{1 - \tan \frac{\varphi}{2}}\right)^4. \quad (1.14)$$

The unit-complex number $z(\varepsilon) = \frac{1+i\varepsilon}{1-i\varepsilon} \in S^1$ is in fact $z(\varepsilon) = e^{i\delta(\varepsilon)}$ i.e.:

$$\cos \delta = \frac{1}{\cosh \delta} = \frac{1 - \varepsilon^2}{1 + \varepsilon}, \quad \sin \delta = \tanh \delta = \frac{2\varepsilon}{1 + \varepsilon^2}. \quad (1.15)$$

The function $\varepsilon \to \sin \delta(\varepsilon)$ has as unique fixed point $\varepsilon = 0$ which means circles while the same problem for the function $\varepsilon \to \cos \delta(\varepsilon)$ yields the cubic equation:

$$\varepsilon^3 + \varepsilon^2 + \varepsilon - 1 = 0 \to \left(\varepsilon + \frac{1}{3}\right)^3 + \frac{2}{3}\left(\varepsilon + \frac{1}{3}\right) - \frac{34}{27} = 0 \quad (1.16)$$

with the unique real solution:

$$\varepsilon = \frac{1}{3}\left(\sqrt[3]{17 + 3\sqrt{22}} - 1 - \frac{2}{\sqrt[3]{17 + 3\sqrt{22}}}\right) = 0.5436... \quad (1.17)$$

so, it provides an unique ellipse with eccentricity $\varepsilon = \cos\delta(\varepsilon)$. The Pisot number $\frac{1}{\varepsilon} = 1.8392...$ is used in [18]. In a similar manner to the Remark 1.4 we associate to the cubic equation above a hyperbola $\tilde{H} : y^2 = 3x^2 + 2x + 1$ and its lattice point $\tilde{P}(0, -1)$ gives the rational parametrization:

$$\tilde{H} \setminus \{\tilde{P}\} : (x, \pm y)(t) = \left(\frac{2t+2}{t^2-3}, \pm\frac{t^2+2t+3}{t^2-3}\right), \quad t \in \mathbb{R} \setminus \{\pm\sqrt{3}\}.$$

The eccentricity of $\tilde{H}$ is again $\frac{2}{\sqrt{3}}$. □

Remark 1.9 The Weierstrass-type invariants of the ellipse are expressed in [5] directly in terms of ε as:

$$g_2(\varepsilon) = \frac{\sqrt[3]{4}}{3}(\varepsilon^4 - 3\varepsilon^2 + 1), \quad g_3(\varepsilon) = \frac{1}{27}(2\varepsilon^6 - 9\varepsilon^4 + 3\varepsilon^2). \tag{1.18}$$

Then we express these invariants in terms of δ as:

$$\begin{cases} g_2(\delta) = \frac{\sqrt[3]{4}}{3} \cdot \frac{(\cosh\delta-1)^4+\sinh^4\delta-3\sinh^2\delta(\cosh\delta-1)^2}{\sinh^4\delta}, \\ g_3(\delta) = \frac{2(\cosh\delta-1)^6-9\sinh^2\delta(\cosh\delta-1)^4+3\sinh^4\delta(\cosh\delta-1)^2}{27\sinh^6\delta}. \end{cases} \tag{1.19}$$

The function g_2 has the unique zero $\varepsilon = \frac{\sqrt{5}-1}{2}$ and its δ and R_1 are provided by:

$$\cosh\delta = \sqrt{5} \to g_3\left(\frac{\sqrt{5}-1}{2}\right) = \frac{-27+8\sqrt{5}}{2} < 0,\ R_1\left(\frac{\sqrt{5}-1}{2}\right) = \frac{17+9\sqrt{5}}{29}. \tag{1.20}$$

The corresponding unit-complex number is $z\left(\frac{\sqrt{5}-1}{2}\right) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and the angle $\arccos\left(\frac{1}{\sqrt{5}}\right)$ appears, for example, in [14]. The image w_1 of z is $w_1 = \frac{\sqrt{5}}{25}(-11, 2) \in S^1$. With $\Phi = \frac{\sqrt{5}+1}{2}$ the famous Golden mean [10] it results $\varepsilon = \Phi - 1$ and $\cosh\delta = 2\Phi - 1$. The Iwasawa parameters corresponding to $\varepsilon = \frac{\sqrt{5}-1}{2}$ are: $\cos t_1 = \frac{\sqrt{5}}{3}$, $\sin t_1 = \frac{2}{3}$, $r_1 = 3$, $\alpha_1 = -\frac{4\sqrt{5}}{9}$. □

Remark 1.10 In the paper [5] the number $L = L(rri; \varepsilon) := \frac{\sqrt{1+\varepsilon^2}}{\varepsilon} > \sqrt{2}$ is introduced as upper bound for the regular refraction interval of the given ellipse E. Since $\varepsilon^2 = \frac{1}{L^2-1}$ it results:

$$\cosh\delta = \frac{L^2}{L^2-2},\ R_1(\varepsilon(L)) = R_1(L) = \frac{L^2+2\sqrt{L^2-1}}{\sqrt{L^2-1}(L^2-2\sqrt{L^2-1})},\ L\left(\varepsilon = \frac{1}{2}\right) = \sqrt{5}. \tag{1.21}$$

608367_1_En_1_Chapter-online ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:4/12/2024 Pages: 18 Layout: T1-Standard

We can express the canonical function in terms of L as:

$$C_{elipse}(L) = \frac{(L^2-1)^{\frac{3}{2}}}{8(L^2-2)^{\frac{3}{2}}}\left[\pi - \frac{2\sqrt{L^2-2}}{L^2-1} - 2\arcsin\frac{1}{\sqrt{L^2-1}}\right]. \tag{1.22}$$

The equation $\cosh\delta = L$ has only one suitable solution $L = 2$ corresponding to $\varepsilon = \frac{1}{\sqrt{3}}$ and:

$$C_{ellipse}(L=2) = \frac{3\sqrt{3}}{8\sqrt{2}}\left[\frac{\pi}{2} - \frac{\sqrt{2}}{3} - \arcsin\frac{1}{\sqrt{3}}\right].$$

In the paper [1] another constant is associated to the family of ellipses with the same eccentricity:

$$C = C(\varepsilon) := 2 + \frac{2}{\sqrt{4-e^2}}.$$

Hence:

$$\begin{cases} \varepsilon = 2\frac{\sqrt{(C-3)(C-1)}}{C-2}, \\ \cosh\delta = \frac{5C^2-20C+16}{-3C^2+12C-8}, \quad \sinh\delta = 4\frac{(C-2)\sqrt{(C-3)(C-1)}}{-3C^2+12C-8}. \end{cases}$$

Also $L = \frac{\sqrt{5C^2-20C+16}}{2\sqrt{(C-3)(C-1)}}$ and the self-complementary ellipse have $C\left(\frac{1}{\sqrt{2}}\right) = 2 + \frac{2\sqrt{2}}{\sqrt{7}} \simeq 3.069...$ For the eccentricity $\varepsilon = \varepsilon(r)$ of the Remark 1.7 we have:

$$C = C(r) = 2 + \frac{r^2+1}{\sqrt{r^4+r^2+1}}$$

and the equality $C(\varepsilon) = \cosh\delta$ has an unique solution: $\varepsilon = \frac{1}{3}\sqrt{19-4\sqrt{13}} \simeq 0.7131....$ □

Remark 1.11 In [9] a quaternionic (but non-internal) product is introduced on (0, 1]:

$$u_1 \odot_c u_2 := \sqrt{\frac{1-(u_1u_2)^2}{u_1^2+u_2^2}}.$$

The non-internal character means that it is possible for $u_1 \odot_c u_2$ to belongs to $(1, +\infty)$. So, we search for a possible eccentricity ε such that its $\odot_c$-square to be:

$$\varepsilon^2_{\odot_c} = \cosh\delta = \frac{1+\varepsilon^2}{1-\varepsilon^2}.$$

This means:

$$\frac{1-\varepsilon^2}{2\varepsilon^2} = \frac{1+\varepsilon^2}{(1+\varepsilon^2)^2}$$

608367_1_En_1_Chapter-online ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:4/12/2024 Pages: 18 Layout: T1-Standard

or, denoting $x = \varepsilon^2 \in (0, 1)$, the cubic equation:

$$f(x) := x^3 - x^2 + 5x - 1 = \left(x - \frac{1}{3}\right)^3 + \frac{14}{3}\left(x - \frac{1}{3}\right) + \frac{16}{27} = 0.$$

Its unique real solution belongs indeed to (0, 1) since is $x = 0.20678...$ and hence the unique eccentricity satisfying $\varepsilon^2_{\odot_c} = \cosh\delta$ is:

$$\varepsilon = \sqrt{0.20678...} \simeq 0.45473.$$

The associated hyperbola $H : y^2 = f'(x) = 3x^2 - 2x + 5$ has the eccentricity $\frac{2}{\sqrt{3}}$.

□

In the following we consider some (classes of) concrete examples.

Example 1.12 In [5] the value $\cosh\delta = \sqrt{2}$ is associated to the equilateral hyperbola. This value corresponds to $\varepsilon = \sqrt{2} - 1$, $\delta = \ln(\sqrt{2} + 1)$ and:

$$R_1(\sqrt{2} - 1) = \sqrt{2} + 1,\, z(\sqrt{2} - 1) = e^{i\frac{\pi}{4}},\, L(rrl; \sqrt{2} - 1) = \frac{\sqrt{4 - 2\sqrt{2}}}{\sqrt{2} - 1}. \quad (1.23)$$

The value $\varepsilon = \sqrt{2} - 1$ is the unique positive solution of the equation $\varepsilon \cdot R_1(\varepsilon) = 1$ the others being $\pm i$ and $-1 - \sqrt{2} < -2$. The corollary 1 of [17] expresses the cosine of $arg(w_n)$ as function of cosine of $\arg(z)$ as:

$$\cos\arg w_n = \frac{\cosh(2n\delta)\cos\arg z - \sinh(2n\delta)}{-\sinh(2n\delta)\cos\arg z + \cosh(2n\delta)} \quad (1.24)$$

and hence for our $z(\sqrt{2} - 1) = e^{i\frac{\pi}{4}}$ it results:

$$\cos\arg w_n = \frac{\cosh(2\sqrt{2}n) - \sqrt{2}\sinh(2\sqrt{2}n)}{\sqrt{2}\cosh(2\sqrt{2}n) - \sinh(2\sqrt{2}n)}. \quad (1.25)$$

The Iwasawa parameters are: $\cos t_1 = \sqrt{\frac{2}{3}}$, $\sin t_1 = \frac{1}{\sqrt{3}}$, $r_1 = \sqrt{3}$, $\alpha_1 = -\frac{1}{\sqrt{2}}$.

The same corollary 2 verified in the Remark 1.6 has the new shape:

$$\int_{\pi\left(1-\frac{1}{\sqrt{1+\sqrt{2}}}\right)}^{\pi\left(1+\frac{1}{\sqrt{1+\sqrt{2}}}\right)} \frac{d\varphi}{3 + 2\sqrt{2}\cos\varphi} - \frac{2\sqrt{2}\pi}{\sqrt{2} + 1} = 5.854... - 3.680.. = 2.174...$$

□

Example 1.13 Fix the point $(b, u, v) \in S^2$ hence $b^2 + u^2 + v^2 = 1$. In [7] a family of conics (called $SU(2)$-*conics*) depending on (b, u, v) is constructed using the

608367_1_En_1_Chapter-online ☑ TYPESET ☐ DISK ☐ LE ☑ CP Disp.:4/12/2024 Pages: 18 Layout: T1-Standard

adjoint representation $Ad : SU(2) \to GL(su(2)) \simeq GL(3, \mathbb{R})$. The eccentricity of an $SU(2)$-conic Γ depends only on v as:

$$\varepsilon_\Gamma = \sqrt{2(1 - v^2)} \tag{1.26}$$

and then it is an ellipse only for $v \in [-1, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, 1]$. The corresponding δ is provided by:

$$\cosh \delta(v) = \frac{3 - 2v^2}{2v^2 - 1}. \tag{1.27}$$

The linear fractional transformation:

$$f(t) = \frac{-2t + 3}{2t - 1} \tag{1.28}$$

has the fixed points $t_1 = 1$ and $t_2 = -\frac{3}{2}$. Hence $v = 1$, which corresponds the unit circle as $SU(2)$-conic, is the only fixed point of the map $v^2 \to \cosh \delta(v)$. □

Example 1.14 In [4] there are studied conics associated to symmetric Pythagorean triple preserving matrices and a hyperbola (called Barning) with eccentricity $\frac{2}{\sqrt{3}}$ (as in the Remark 1.11) is a main example. So, we consider the inverse $\frac{\sqrt{3}}{2} = \varepsilon = \sin(\varphi = \frac{\pi}{3})$ and then:

$$\begin{cases} \cosh \delta = 7, \quad \sinh \delta = 4\sqrt{3}, \\ z\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{7}(1, 4\sqrt{3}), \quad R_1\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}(97 + 56\sqrt{3}), \quad L = \sqrt{\frac{7}{3}}. \end{cases} \tag{1.29}$$

The example 2.5 of the book [12] discusses the ellipse:

$$\tilde{E} : 4\left(x - \frac{1}{4}\right)^2 + \frac{16}{3}y^2 = 1$$

from the Poncelet's Theorem point of view. Its eccentricity is $\tilde{e} = \frac{1}{2}$ and since $e^2 + \tilde{e}^2 = 1$ it provides a pair of complementary ellipses [5] with our initial E. □

2 The Return Map and the Plane Hyperbolic Geometry

The expression of the return map in terms of hyperbolic functions of δ raises the natural question of a possible relationship between the return map and the hyperbolic geometry. The answer is affirmative and we use the upper half-plane model of hyperbolic geometry $\mathbb{C}_+ = \{z \in \mathbb{C}; \Im z > 0\}$ (here $\Im z$ means the imaginary part of the complex number z and similar for $\Re$) endowed with the Riemannian metric given by Poincaré:

$$g(x, y) = \frac{dx^2 + dy^2}{y^2} = \left(\frac{|dz|}{y}\right)^2. \quad (2.1)$$

The Lie group $SL(2, \mathbb{R})$ acts in the Möbius way on $\mathbb{C}_+$ and a matrix is identified with its image of the unit complex i. Then $\mathbb{C}_+$ is $SL(2, \mathbb{R})/SO(2)$ and the indicatrix or unit tangent bundle of $(\mathbb{C}_+, g)$ is the quotient $PSL(2, \mathbb{R})/(\pm I_2)$. The hyperbolic measure $dV = \frac{1}{y^2} dx \wedge dy$ is invariant with respect to the action of $SL(2, \mathbb{R})$. For every $T \in SL(2, \mathbb{R})$ it is known that:

$$|T'(z)| = \frac{\Im T(z)}{\Im z}. \quad (2.2)$$

The geodesic polar coordinates $(\rho, \varphi) \in (0, +\infty) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ on the Poincaré upper half-plane $(\mathbb{C}_+, g)$ are given by $\rho := dist_g(i, z), \varphi := \arctan \frac{x^2+y^2-1}{2x}$ and then:

$$x = \frac{\sinh \rho \cos \varphi}{\cosh \rho - \sinh \rho \sin \varphi}, \quad y = \frac{1}{\cosh \rho - \sinh \rho \sin \varphi}. \quad (2.3)$$

An important tool of the hyperbolic geometry is the Lobachevsky's *angle of parallelism function* Π defined by: $\sin \Pi(x) = \frac{1}{\cosh x}$. It follows the Lobachevsky's angle of our ellipse:

$$\Pi(E) = \arcsin \frac{1 - \varepsilon^2}{1 + \varepsilon^2} = \frac{\pi}{2} - \arg z(\varepsilon) \quad (2.4)$$

with $z(\varepsilon)$ the complex number of the Remark 1.4. With $\varepsilon = \varepsilon(L)$ for L of the Remark 1.10 we have $\Pi(E(L)) = \arcsin \left(1 - \frac{2}{L^2}\right)$.

A straightforward computation gives:

$$\Im w_n(z) = \frac{\Im z}{|(\cosh n\delta - \Re z \sinh n\delta) + (\Im z \sinh n\delta)i|^2} \quad (2.5)$$

which means that w_n preserves the upper-half plane and we can study the restriction $w_n : S^1_+ := S^1 \cap \mathbb{C}_+ \to S^1_+$. For example:

$$w_n(i) = \frac{-\sinh(2n\delta) + i}{\cosh(2n\delta)} \quad (2.6)$$

and since the hyperbolic distance of $(\mathbb{C}_+, g)$ is provided by the formula:

$$\cosh dist_g(z, w) := 1 + \frac{|z - w|^2}{2\Im z \Im w}, \quad (2.7)$$

we get the distance between i and its image:

$$dist_g(i, w_n(i)) = 2n\delta, \quad \lim_{n \to +\infty} dist_g(i, w_n(i)) = +\infty \quad (2.8)$$

and the formulae (2.3) work in the case of $w_n(i)$ for $\varphi = \pi$. The first part of (2.8) provides a geometric interpretation of δ: is half of the hyperbolic distance between i and its first return $w_1(i)$.

A second natural question is if the action of w_n on S^1_+ is a transitive one and we search for a partial answer considering as source the remarkable point $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} \in F$=the fundamental domain of the modular group $PSL(2,\mathbb{Z})$=$\{z \in \mathbb{C}_+; |z| \geq 1, |\Re z| \leq \frac{1}{2}\}$ and the target being i. We obtain again a positive answer:

$$w_n(\omega) = i \to \cosh(n\delta) = \frac{1}{\sqrt{4\sqrt{3}-6}} = 1.03795..., \sinh(n\delta) = \frac{2-\sqrt{3}}{\sqrt{4\sqrt{3}-6}}. \tag{2.9}$$

Also, $w_1(\omega) = \omega - 1 \in F$ for $\varepsilon = 2-\sqrt{3} = 0.2679...$, $\cosh\delta = \frac{2}{\sqrt{3}}$, $\sinh\delta = \frac{1}{\sqrt{3}}$:

$$L(rri; \varepsilon = 2-\sqrt{3}) = \frac{2\sqrt{2}}{\sqrt{3}-1}, C(\varepsilon = 2-\sqrt{3}) = 2 + \frac{2}{\sqrt{4\sqrt{3}-3}}, \Pi(E) = \frac{\pi}{3}. \tag{2.10}$$

Recall that ω and $\omega - 1$ are respectively the right and left corner of the fundamental domain F.

Remark 1.6 revisited For the self-complementary ellipses we have:

$$\sin\Pi(E) = \frac{1}{3}, \quad dist_g(i, w_n(i)) = \ln(3+2\sqrt{2})^{2n}. \tag{2.11}$$

The geodesic polar coordinates of $z\left(\frac{1}{2}\right) = \frac{1}{3} + i\frac{2\sqrt{2}}{3} \in S^1_+ \cap F$ are $\varphi = 0$, $\cosh\rho = \frac{3}{2\sqrt{2}}$, $\sinh\rho = \frac{1}{2\sqrt{2}}$. □

Remark 2.1 Returning to the general complex algebra $\mathbb{C} \ni z := x + iy$ we note that the canonical equation of E:

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

becomes in the complex variable z:

$$E : \alpha(z^2 + \bar{z}^2) + 2\beta|z|^2 - 4 = 0, \quad \alpha = \frac{1}{a^2} - \frac{1}{b^2} < 0, \quad \beta = \frac{1}{a^2} + \frac{1}{b^2} > 0. \tag{2.12}$$

Conversely an equation $E : \alpha(z^2 + \bar{z}^2) + 2\beta|z|^2 - 4 = 0$ with $\alpha < 0 < \beta$ means the canonical equation with:

$$\frac{1}{a^2} = \frac{\beta+\alpha}{2}, \quad \frac{1}{b^2} = \frac{\beta-\alpha}{2}, \quad \varepsilon^2 = \frac{-2\alpha}{\beta-\alpha} \tag{2.13}$$

and then for our return map:

$$\cosh\delta = \frac{\beta - 3\alpha}{\beta+\alpha}, \quad \sinh\delta = \frac{2\sqrt{-2\alpha(\beta-\alpha)}}{\beta+\alpha}. \tag{2.14}$$

For the example of self-complementary ellipse $b = \frac{a}{\sqrt{2}}$ implies $\alpha = -\frac{1}{a^2}$ and $\beta = \frac{3}{a^2}$. □

Remark 2.2 Until now we do not use the symmetrical character of the matrix $\Gamma_n(\varepsilon)$. Recall after [8] that to any symmetric 2×2 matrix:

$$\Gamma = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in Sym(2)$$

we can associate a complex number $A(\Gamma) := \frac{a-c}{2} - bi$ called *the Hopf invariant of* Γ, which together with the real number $B(\Gamma) := \frac{a+c}{2} = \frac{1}{2}Tr(\Gamma)$ extends the real matrix Γ to a Hermitian one:

$$\Gamma^c := \begin{pmatrix} B & \bar{A} \\ A & B \end{pmatrix} \in H(2), \quad Tr(\Gamma) = Tr(\Gamma^c), \quad \det(\Gamma) = \det(\Gamma^c). \tag{2.15}$$

For our n-return map it results:

$$A_n(\varepsilon) = i\sinh(n\delta) = (-i)r_n \sin t_n, \quad B_n(\varepsilon) = \cosh(n\delta) = r_n \cos t_n \tag{2.16}$$

and then $A_n(\varepsilon) \in F$ if and only if $|\sinh(n\delta)| \geq 1$; for example this holds for the self-complementary ellipses. The complex number $A_n + B_n$ is $r_n e^{i(-t_n)}$.

The group $SL(2, \mathbb{C})$ acts on the 4-dimensional real linear space $H(2)$ of 2×2 Hermitian matrices:

$$\Psi : SL(2,\mathbb{C}) \times H(2) \to H(2), \quad \Psi(T, \Gamma^c) = \Psi_T(\Gamma^c) := T \cdot \Gamma^c \cdot T^*, \quad T^* := \bar{T}^t. \tag{2.17}$$

For our n-return matrix it follows:

$$\Psi_{\Gamma_n}(\Gamma_n^c) = \begin{pmatrix} \cosh(n\delta)\cosh(2n\delta) & \sinh(n\delta)(-2\cosh^2(n\delta) - i) \\ \sinh(n\delta)(-2\cosh^2(n\delta) + i) & \cosh(n\delta)\cosh(2n\delta) \end{pmatrix}. \tag{2.18}$$

This last Hermitian matrix belongs also to the range of the map $\Gamma \in Sym(2) \to \Gamma^c \in H(2)$; more precisely is the image of the matrix:

$$\begin{cases} \tilde{\Gamma}_n = \begin{pmatrix} e^{-n\delta}\cosh(n\delta) & -\sinh(n\delta) \\ -\sinh(n\delta) & e^{n\delta}\cosh(n\delta) \end{pmatrix} \in Sym(2) \cap SL(2,\mathbb{R}), \\ A(\tilde{\Gamma}_n) = \sinh(n\delta)[-\cosh(n\delta) + i], \quad B(\tilde{\Gamma}_n) = \cosh^2(n\delta), \end{cases} \tag{2.19}$$

and we pointed out the occurrence of the eigenvalues of $\Gamma_n(\varepsilon)$ in the main diagonal of $\tilde{\Gamma}_n$.

The action of Γ_n on an arbitrary $\Gamma^c \in H(2)$ is:

$$2\Psi_{\Gamma_n}(\Gamma^c) = \begin{pmatrix} (a+c)\cosh(2n\delta) - (a-c)\sinh(2n\delta) & (a-c)\cosh(2n\delta) - (a+c)\sinh(2n\delta) + 2bi \\ (a-c)\cosh(2n\delta) - (a+c)\sinh(2n\delta) - 2bi & (a+c)\cosh(2n\delta) - (a-c)\sinh(2n\delta) \end{pmatrix} \in H(2). \tag{2.20}$$

□

Remark 2.3 Recall also the Hopf bundle $H : S^3 \subset \mathbb{C}^2 \to S^2(\frac{1}{2}) \subset \mathbb{R} \times \mathbb{C}$:

$$H(z, w) = \left(\frac{1}{2}(|z|^2 - |w|^2), z\bar{w}\right). \tag{2.21}$$

Since the Clifford torus $\frac{1}{\sqrt{2}}T^2 \subset S^3$ one follows for any $z \in S^1$ the image:

$$H\left(\frac{1}{\sqrt{2}}(z, w_n(z))\right) = \left(0, \frac{1}{2}z\overline{w_n(z)}\right) = \left(0, \frac{1}{2}\frac{\sinh(n\delta)z - \cosh(n\delta)}{\overline{\sinh(n\delta)z - \cosh(n\delta)}}\right), \tag{2.22}$$

and hence the pairs $\frac{1}{\sqrt{2}}(z_1, w_n(z_1))$ and $\frac{1}{\sqrt{2}}(z_2, w_n(z_2))$ belong to same fibre if and only if $\sinh(n\delta)z_1\overline{z_2} - \cosh(n\delta)(z_1 + \overline{z_2}) \in \mathbb{R}$.

Proposition 2.4 *The pairs $\frac{1}{\sqrt{2}}(z, w_n(z))$, $\frac{1}{\sqrt{2}}(\bar{z}, w_n(\bar{z}))$ belong to the same fibre only for $z = \pm 1 \in S^1$.*

Proof The condition above means $\sinh(n\delta)z^2 - 2\cosh(n\delta)z \in \mathbb{R}$. With $z = e^{it} \in S^1$ the imaginary part of the previous complex number is
$2\sin t[\sinh(n\delta)\cos t - \cosh(n\delta)]$ and this real number is zero only for $t \in \{0, \pi\}$. Indeed, we have: $H\left(\frac{1}{\sqrt{2}}(1, 1)\right) = H\left(\frac{1}{\sqrt{2}}(-1, -1)\right) = \left(0, \frac{1}{2}\right)$. □

Remark 2.5 At the end of the paper [3] to a 2-dimensional metric a expressed in diagonal form as $a = diag(a_{11}, a_{22})$ a class of 2×2 matrices depending on two parameters is associated in a natural way:

$$H = H(u, v) = \begin{pmatrix} ua_{11} & \frac{1-u^2}{v}a_{11} \\ va_{22} & -ua_{22} \end{pmatrix}, \quad \det H = -\det a. \tag{2.23}$$

If the metric a is exactly the hyperbolic i.e. $a_{11} = 1 = -a_{22}$ then for $u = \cosh(n\delta)$ and $v = \sinh(n\delta)$ the matrix H is exactly $\Gamma_n(\varepsilon)$. □

Remark 2.6 Let $\Delta^1 := \{z \in \mathbb{C}; |z| < 1\}$ be the unit disk and the well-known correspondence between $\mathbb{C}_+$ and Δ^1:

$$\mathcal{U} : \mathbb{C}_+ \to \Delta^1, \quad z \to \mathcal{U}(z) = i\frac{z-i}{z+i} = \eta, \quad \mathcal{U}^{-1}(\eta) = \frac{i\eta - 1}{-\eta + i}. \tag{2.24}$$

It follows the unit disk-variant of the return map:

$$\begin{cases} \mathcal{U} \circ w_n \circ \mathcal{U}^{-1} : \Delta^1 \to \Delta^1, \\ \mathcal{U} \circ w_n \circ \mathcal{U}^{-1}(\eta) = \frac{[-\sinh(n\delta)+i\cosh(n\delta)](i\eta-1)+[\cosh(n\delta)-i\sinh(n\delta)](i-\eta)}{[\cosh(n\delta)-i\sinh(n\delta)](i\eta-1)+[-\sinh(n\delta)+i\cosh(n\delta)](i-\eta)} = \frac{(\cosh n\delta)\eta-\sinh n\delta}{(-\sinh n\delta)\eta+\cosh n\delta} \end{cases} \tag{2.25}$$

and we remark the total similarity with the initial expression (1.1). This map does not have fixed points since $\pm 1 \notin \Delta^1$.

The equation $\mathcal{U}^{-1}(\varepsilon) = z(\varepsilon)$ has the solutions $\varepsilon_\pm = -1 \pm \sqrt{2}$ and hence the only possible eccentricity is $\varepsilon_+ = \sqrt{2} - 1$ of the Example 1.12. □

Remark 2.7 The isomorphism between $SL(2, \mathbb{R})$ and $O(2, 1)$ as the Lie group with Lie algebra $(\mathbb{R}^{2,1}, < \cdot, \cdot >_{Lorentz} = (+ + -))$ is given by the map:

$$\Gamma = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \to \Psi$$

$$\Psi(a, b, c, d) := \begin{pmatrix} 1 + 2bc & -ab + cd & ab + cd \\ -ac + bd & \frac{1}{2}(a^2 - b^2 - c^2 + d^2) & \frac{1}{2}(-a^2 + b^2 - c^2 + d^2) \\ ac + bd & \frac{1}{2}(-a^2 - b^2 + c^2 + d^2) & \frac{1}{2}(a^2 + b^2 + c^2 + d^2) \end{pmatrix} \tag{2.26}$$

and then the image of Γ_n is:

$$\Psi(a = d = \cosh(n\delta), b = c = -\sinh(n\delta)) := \begin{pmatrix} \cosh(2n\delta) & 0 & -\sinh(2n\delta) \\ 0 & 1 & 0 \\ -\sinh(2n\delta) & 0 & \cosh(2n\delta) \end{pmatrix}. \tag{2.27}$$

Thinking of $\mathbb{R}^{2,1}$ as the set of triples (x, y, z) it follows that the projection of the linear map $\Psi(\Gamma_n)$ on planes $y = constant$ corresponds to the linear map Γ_{2n}. The matrix Ψ is symmetrical if and only if the initial matrix Γ is symmetrical and the induced isometry of the Lorentzian Lie algebras is:

$$\begin{cases} \psi : \gamma = \begin{pmatrix} \alpha & \rho \\ \beta & -\alpha \end{pmatrix} \in sl(2, \mathbb{R}) \to \left(\alpha, \frac{1}{2}(\beta + \rho), \frac{1}{2}(\beta - \rho)\right) \in \mathbb{R}^{2,1}, \\ \psi(Ad(\Gamma)\gamma) = \Psi(\Gamma)\psi(\gamma). \end{cases} \tag{2.28}$$

For example, for a traceless symmetrical matrix and its Hermitian parameters A, B of Remark 2.2:

$$\gamma = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in sl(2, \mathbb{R}) \cap Sym(2) \to (a, b, 0) = (\overline{A(\gamma)}, B(\gamma)) \in \mathbb{R}^{2,1}. \tag{2.29}$$

□

3 Conclusion

As main conclusion of this chapter we point out the richness of informations associated to the return map of an ellipse. This approach is open to other interesting topics from two-dimensional geometries and we hope to develop it in future papers.

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