A NOTE ON THE HYPERBOLIC CURVATURE OF EUCLIDEAN PLANE CURVES

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Dedicated to the memory of Academician Radu Miron

ABSTRACT. We introduce and study a new curvature function for plane curves inspired by the weighted mean curvature of M. Gromov. We call it *hyperbolic* being the difference between the usual curvature and the inner product of the normal vector field and the hyperbolic vector field. But, since the problem of vanishing of this curvature involves complicated expressions, we computed it for several examples.

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The last forty years known an intensive research in the area of geometric flows. The most simple of them is the *curve shortening flow* and already the excellent survey [1] is almost twenty years old. Recall that the main geometric tool in this last flow is the well-known curvature of plane curves. Hence, to give a re-start to this problem seams to search for variants of the curvature or in terms of [4], deformations of the usual curvature. The goal of this short note is to propose such a deformation.

Fix $I \subseteq \mathbb{R}$ an open interval and $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)), \quad \|r'(t)\| > 0, \quad t \in I. \tag{1}$$

The ambient setting, namely $\mathbb{R}^2$, is an Euclidean vector space with respect to the canonical inner product:

$$\langle u, v\rangle = u^1v^1 + u^2v^2, \quad u = (u^1, u^2), \quad v = (v^1, v^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u\rangle. \tag{2}$$

The infinitesimal generator of the rotations in $\mathbb{R}^2$ is the linear vector field, called *angular*:

$$\xi(u) := -u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \quad \xi(u) = i \cdot u = i \cdot (u^1 + iu^2). \tag{3}$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O, R)$:

$$\begin{cases} \gamma^{\xi}_{u_0}(t) = (u^1_0 \cos t - u^2_0 \sin t, u^1_0 \sin t + u^2_0 \cos t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix} = SO(2) \cdot u_0, \\ R = \|u_0\| = \|(u^1_0, u^2_0)\|, \quad t \in \mathbb{R}, \end{cases} \tag{4}$$

and since the rotations are isometries of the Riemannian metric $g_{can} = dx^2 + dy^2$ it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2(x, y), g_{can})$. The first integrals of ξ are the Gaussian functions i.e. multiples of the square norm: $f_C(x,y) = C(x^2+y^2)$, $C \in \mathbb{R}$. For an arbitrary vector field $X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X,\xi] = (yA_x - xA_y - B)\frac{\partial}{\partial x} + (A + yB_x - xB_y)\frac{\partial}{\partial y}$$

where the subscript denotes the variable corresponding to the partial derivative.

In the following we fix the complete vector field:

$$\Gamma_h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

which we call *hyperbolic* since its integral curves are the equilateral hyperbolas $(x(t), y(t)) = (e^t x_0, e^{-t} y_0)$. Its Lie bracket with ξ is:

$$[\Gamma_h, \xi] = 2y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}.$$

The Frenet apparatus of the curve C is provided by:

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|}, \quad N(t) = i \cdot T(t) = \frac{1}{\|r'(t)\|}(-y'(t), x'(t)) \\ k(t) = \frac{1}{\|r'(t)\|}\langle T'(t), N(t)\rangle = \frac{1}{\|r'(t)\|^3}\langle r''(t), ir'(t)\rangle = \frac{1}{\|r'(t)\|^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases} \tag{5}$$

Hence, if C is naturally parametrized (or parametrized by arc-length) i.e. $\|r'(t)\| = 1$ for all $t \in I$ then $r''(t) = k(t)ir'(t)$. In a complex approach based on $z(t) = x(t) + iy(t) \in \mathbb{C} = \mathbb{R}^2$ we have $2\lambda = Im(\bar{z}dz)$ and:

$$\begin{cases} k(t) = \frac{1}{|z'(t)|^3} Im(\bar{z}'(t) \cdot z''(t)) = \frac{1}{|z'(t)|} Im\left(\frac{z''(t)}{z'(t)}\right) = \frac{1}{|z'(t)|} Im\left[\frac{d}{dt}(\ln z'(t))\right], \\ Re(\bar{z}'(t) \cdot z''(t)) = \frac{1}{2}\frac{d}{dt}\|r'(t)\|^2, \quad f_C(z) = C|z|^2. \end{cases}$$

This short note defines a new curvature function for C inspired by a notion introduced by M. Gromov in [3, p. 213] and concerning with hypersurfaces M^n in a weighted Riemannian manifold $(\tilde{M}, g, f \in C^\infty_+(\tilde{M}))$. More precisely, the *weighted mean curvature* of M is the difference:

$$H^f := H - \langle \tilde{N}, \tilde{\nabla} f\rangle_g \tag{6}$$

where H is the usual mean curvature of M and $\tilde{N}$ is the unit normal to M. This curvature was studied in several papers and we point out that the curve shortening problem associated to a density is studied in the paper [5].

The rotational field ξ is not a g_{can}-gradient vector field but Γ_h is the gradient of the function $f_h(x,y) = \frac{1}{2}(x^2 - y^2)$. Hence we follow this path and we introduce:

Definition 1 The *hyperbolic curvature* of C is the smooth function $k_h : I \to \mathbb{R}$ given by:

$$k_h(t) := k(t) - \langle N(t), \Gamma_h(r(t))\rangle. \tag{7}$$

Before starting its study we point out that this work is dedicated to the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves and besides its theory of *Myller configuration* ([7]) he generalized also a type of curvature for space curves in [6]. Returning to our subject we note:

Proposition 2 *The expression of the hyperbolic curvature is*:

$$k_h(t) = k(t) + \frac{1}{\|r'(t)\|}\frac{d}{dt}[x(t)y(t)]. \tag{8}$$

Proof We have directly:

$$\langle N(t), \Gamma_h(r(t))\rangle = \frac{1}{\|r'(t)\|}\langle(-y'(t), x'(t)), (x(t), -y(t))\rangle = -\frac{1}{\|r'(t)\|}[x(t)y(t)]' \tag{9}$$

and the conclusion (8) follows. $\square$

Example 3 i) If C is the line $r_0 + tu, t \in \mathbb{R}$ with the vector $u = (u^1, u^2) \neq \bar{0} = (0, 0)$ and the point $r_0 = (x_0, y_0) \in \mathbb{R}^2$ then k_h is the affine map:

$$k_h(t) = \frac{1}{\|u\|}[u^1(y_0 + tu^2) + u^2(x_0 + tu^1)], \quad t \in \mathbb{R}. \tag{10}$$

In particular, if $O \in C$ then $k_h(t) = \frac{2u^1u^2t}{\|u\|}$.
ii) If C is the circle $\mathcal{C}(O, R) : r(t) = Re^{it}$ the k_h is again a non-constant function:

$$k_h(t) = \frac{1}{R}\left(1 + R^2 \cos 2t\right) \in \left[\frac{1 - R^2}{R}, \frac{1 + R^2}{R} \geq 2\right]. \tag{11}$$

Hence, the unit circle S^1 has the hyperbolic curvature $k_h(t) = 2\cos^2 t \in [0, 2]$.
iii) The equilateral hyperbola $H_e(R) : xy = R^2$ has the hyperbolic curvature equal to the usual curvature:

$$h_k(t) = k(t) = \frac{2R^2}{t\sqrt{t^4 + R^4}} \geq 0, \quad t \in (0, +\infty). \tag{12}$$

We note that in the paper [2] is studied a product on the set of equilateral hyperbolas. If the equilateral hyperbola is expressed as $H^e(R) : x^2 - y^2 = R^2$ then its curvatures are:

$$k(t) = -\frac{1}{R(\cosh 2t)^{\frac{3}{2}}} < 0, \quad k_h(t) = \frac{R^2(\cosh 2t)^2 - 1}{R(\cosh 2t)^{\frac{3}{2}}}.$$

In particular for $H^e(1)$ its hyperbolic curvature is positive:

$$k_h(t) = \frac{(\sinh 2t)^2}{R(\cosh 2t)^{\frac{3}{2}}} \geq 0.$$

iv) Suppose that C is positively oriented in the terms of Definition 1.14 from [8, p. 17]. Suppose also that C is convex; then applying the Theorem 1.18 of page 19 from the same book it results for the usual curvature the inequality $k \geq 0$; it results that $k_k(t) \geq \frac{1}{\|r'(t)\|} \frac{d}{dt}[x(t)y(t)]$, for all $t \in I$. $\square$

An important problem is the class of curves with prescribed hyperbolic curvature. Using the formalism of [9, p. 2] if $r : S^1 \simeq [0, 2\pi) \to \mathbb{R}^2$ is naturally parametrized then there exists the smooth function $\theta : S^1 \to \mathbb{R}$, called *normal angle*, such that:

$$N(t) = e^{i\theta(t)} = (\cos\theta(t), \sin\theta(t)), \quad T(t) = -iN(t) = -ie^{i\theta(t)} = e^{i(\theta(t) - \frac{\pi}{2})} \tag{13}$$

and then the Frenet equations yields:

$$\frac{d\theta}{dt}(t) = k(t). \tag{14}$$

It follows that the hyperbolic curvature is a derivative:

$$k_h(t) = \frac{d}{dt}[\theta(t) + x(t)y(t)]. \tag{15}$$

Proposition 4 *Suppose that t is a natural parameter on the curve C. Then C is hyperbolic-flat i.e. $k_h \equiv 0$ if and only if $\theta + x \cdot y$ is a constant.*

In the following we present other two examples in order to remark the computational aspects of our approach.

Example 5 Recall that for $R > 0$ the cycloid of radius R has the equation:

$$C : r(t) = R(t - \sin t, 1 - \cos t) = R[(t, 1) - e^{i(\frac{\pi}{2} - t)}], \quad t \in \mathbb{R}. \tag{16}$$

We have immediately:

$$r'(t) = R(1 - \cos t, \sin t) = R[(1, 0) - e^{it}], \quad k(t) = -\frac{1}{4R\sin\frac{t}{2}}, \quad \|r'(t)\| = 2R|\sin\frac{t}{2}| \tag{17}$$

and then we restrict our definition domain to $(0, \pi)$. It follows:

$$k_h(t) = -\frac{1}{4R\sin\frac{t}{2}} + \frac{R[2(\cos^2 t - \cos t) + t\sin t]}{2\sin\frac{t}{2}}. \tag{18}$$

A natural parameter s for C is provided by: $t = 2\arccos\left(1 - \frac{s}{4R}\right)$. $\square$

Example 6 Fix a graph $C : y = f(x)$, $x \in I$ with the second derivative f'' strictly positive. With the usual parametrization $C : r(t) = (t, f(t))$ we have:

$$r'(t) = (1, f'(t)), \quad k(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} > 0, \quad \|r(t)\|^2 = t^2 + f^2(t) \tag{19}$$

which gives that C is convex and:

$$k_k(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} + \frac{f(t) + tf'(t)}{[1 + (f'(t))^2]^{\frac{1}{2}}}. \tag{20}$$

It follows that the function f making k_h constant zero is a solution of the non-autonomous differential equation:

$$f''(t) = -[f(t) + tf'(t)][1 + (f'(t))^2]. \tag{21}$$

$\square$

References

[1] K.-S Chou and X.-P. Zhu, *The curve shortening problem*, Boca Raton, FL: Chapman & Hall/CRC, 2001. Zbl 1061.53045

[2] M. Crasmareanu and M. Popescu, *Quaternionic product of equilateral hyperbolas and some extensions*, Mathematics, Special Issue "Geometric Methods and their Applications" **8(10)** (2020), no. 10, paper 1686.

[3] M. Gromov, *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal. **13(1)** (2003), 178-215; erratum ibid. **18(5)** (2008), 1786-1786. Zbl 1044.46057

[4] B. Mazur, *Perturbations, deformations, and variations (and "near-misses") in geometry, physics, and number theory*, Bull. Am. Math. Soc., New Ser. **41(3)** (2004), 307-336. Zbl 1057.11033

[5] V. Miquel and F. Vinado-Lereu, *The curve shortening problem associated to a density*, Calc. Var. Partial Differ. Equ. **55(3)** (2016), Paper No. 61, 30 p. Zbl 1346.53061

[6] R. Miron, *Une généralisation de la notion de courbure de parallélisme*, Gaz. Mat. Fiz., Bucureşti, Ser. A **10(63)** (1958), 705-708. Zbl 0087.36101

[7] R. Miron, *The geometry of Myller configurations. Applications to theory of surfaces and nonholonomic manifolds*, Bucharest: Editura Academiei Române, 2010. Zbl 1206.53003

[8] L. Younes, *Shapes and diffeomorphisms. 2nd updated edition*, Applied Mathematical Sciences 171. Berlin: Springer, 2019. Zbl 1423.53002

[9] X.-P. Zhu, *Lectures on mean curvature flows*, AMS/IP Studies in Advanced Mathematics vol. 32, Providence, RI: American Mathematical Society, 2002. Zbl 1197.53087

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