THE GENERALIZED RADIAL CURVATURE OF PLANE CURVES

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Dedicated to the memory of Academician Radu Miron

ABSTRACT. We introduce and study a new curvature function for plane curves inspired by the weighted mean curvature of M. Gromov. We call it *generalized radial* being the difference between the usual curvature and the inner product of the normal vector field and the gradient of a radial smooth functions. But, since the problem of vanishing of this curvature involves complicated expressions, we computed it for several examples.

The *curve shortening flow* is one of the most studied geometric flows and the survey [2] is an excellent introduction in this subject. The main geometric ingredient of this flow is the curvature of plane curves. Hence, to search for variants of the curve shortening flow is necessary to propose new curvatures or in terms of [7], deformations of the usual curvature. The goal of this short note is to propose such a deformation.

Fix $I \subseteq \mathbb{R}$ an open interval and $C \subset \mathbb{R}^2$ a regular parametrized curve of equation:

$$C : r(t) = (x(t), y(t)), \quad \|r'(t)\| > 0, \quad t \in I. \tag{1}$$

The ambient setting, namely $\mathbb{R}^2$, is an Euclidean vector space with respect to the canonical inner product:

$$\langle u, v \rangle = u^1 v^1 + u^2 v^2, \quad u = (u^1, u^2), \quad v = (v^1, v^2) \in \mathbb{R}^2, \quad 0 \leq \|u\|^2 = \langle u, u \rangle. \tag{2}$$

The infinitesimal generator of the rotations in $\mathbb{R}^2$ is the linear vector field, called *angular*:

$$\xi(u) := -u^2 \frac{\partial}{\partial u^1} + u^1 \frac{\partial}{\partial u^2}, \quad \xi(u) = i \cdot u = i \cdot (u^1 + i u^2). \tag{3}$$

It is a complete vector field with integral curves the circles $\mathcal{C}(O, R)$:

$$\begin{cases} \gamma_{u_0}^\xi(t) = (u_0^1 \cos t - u_0^2 \sin t, u_0^1 \sin t + u_0^2 \cos t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} u_0^1 \\ u_0^2 \end{pmatrix} = SO(2) \cdot u_0, \\ R = \|u_0\| = \|(u_0^1, u_0^2)\|, \quad t \in \mathbb{R}, \end{cases} \tag{4}$$

and since the rotations are isometries of the Riemannian metric $g_{can} = dx^2 + dy^2$ it follows that ξ is a Killing vector field of the Riemannian manifold $(\mathbb{R}^2, g_{can})$. The first integrals of ξ are the Gaussian functions i.e. multiples of the square norm: $f_C(x, y) = C(x^2 + y^2)$, $C \in \mathbb{R}$.

For an arbitrary vector field $X = A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X, \xi] = (yA_x - xA_y - B)\frac{\partial}{\partial x} + (A + yB_x - xB_y)\frac{\partial}{\partial y}$$

where the subscript denotes the variable corresponding to the partial derivative. For example, ξ commutes with *the radial* (or Euler) vector field: $E(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, which is also a complete vector field having as integral curves the homotheties $\gamma_{u_0}^E(t) = e^t u_0$ for all $t \in \mathbb{R}$. The vector field E is the basis of the 1-dimensional annihilator of the Liouville (or tautological) 1-form $\lambda = \frac{1}{2}(-ydx + xdy)$ whose exterior derivative is the area 2-form $dx \wedge dy$. We point out also that the opposite vector field $W = -E$ is exactly the wind in the Zermelo navigation problem corresponding to the Funk metric in the unit disk of $\mathbb{R}^2$, [3]. For an arbitrary Euclidean space $\mathbb{R}^n$ with $n \geq 2$ the radial vector field $E = x^i \frac{\partial}{\partial x^i}$ defines the notion of *horizontal* 1-*form* ρ as satisfying $i_E\rho = 0$ with i_E the interior product.

The Frenet apparatus of the curve C is provided by:

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|}, \quad N(t) = i \cdot T(t) = \frac{1}{\|r'(t)\|}(-y'(t), x'(t)) \\ k(t) = \frac{1}{\|r'(t)\|}\langle T'(t), N(t)\rangle = \frac{1}{\|r'(t)\|^3}\langle r''(t), ir'(t)\rangle = \frac{1}{\|r'(t)\|^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases}$$
(5)

Hence, if C is naturally parametrized (or parametrized by arc-length) i.e. $\|r'(t)\| = 1$ for all $t \in I$ then $r''(t) = k(t)ir'(t)$. In a complex approach based on $z(t) = x(t) + iy(t) \in \mathbb{C} = \mathbb{R}^2$ we have $2\lambda = Im(\bar{z}dz)$ and:

$$\begin{cases} k(t) = \frac{1}{|z'(t)|^3}Im(\bar{z}'(t) \cdot z''(t)) = \frac{1}{|z'(t)|}Im\left(\frac{z''(t)}{z'(t)}\right) = \frac{1}{|z'(t)|}Im\left[\frac{d}{dt}(\ln z'(t))\right], \\ Re(\bar{z}'(t) \cdot z''(t)) = \frac{1}{2}\frac{d}{dt}\|r'(t)\|^2, \quad f_C(z) = C|z|^2. \end{cases}$$

This short note defines a new curvature function for C inspired by a notion introduced by M. Gromov in [5, p. 213] and concerning with hypersurfaces M^n in a weighted Riemannian manifold $(\tilde{M}, g, f \in C_+^\infty(\tilde{M}))$. More precisely, the *weighted mean curvature* of M is the difference:

$$H^f := H - \langle \tilde{N}, \tilde{\nabla}f\rangle_g$$
(6)

where H is the usual mean curvature of M and $\tilde{N}$ is the unit normal to M. This curvature was studied in several papers; the curve shortening problem associated to a density is studied in the paper [8].

The rotational field ξ is not a g_{can}-gradient vector field but E is the gradient of the Gaussian function $f_{\frac{1}{2}}$. In fact, we will consider a general radial function:

$$f(x, y) := F(r^2) = F(x^2 + y^2), \quad F : [0, +\infty) \to \mathbb{R}, F \in C^\infty$$

with the derivative $F' > 0$; hence F is a strictly increasing real function. We point out that he subject of plane curves with curvature depending on position is a very interesting one and a recent study is the paper [1]. Then we introduce:

Definition 1 The *F-radial curvature* of C is the smooth function $k_{Fr} : I \to \mathbb{R}$ given by:

$$k_{Fr}(t) := k(t) - \langle N(t), \nabla f(r(t))\rangle.$$
(7)

Before starting its study we point out that this work is dedicated to the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves

and besides its theory of *Myller configuration* ([10]) he generalized also a type of curvature for space curves in [9]. Returning to our subject we note:

Proposition 2 *The expression of the F-radial curvature is:*

$$k_{Fr}(t) = k(t) - 2F'(\|r(t)\|^2)\frac{\langle ir'(t), r(t)\rangle}{\|r'(t)\|}. \tag{8}$$

As consequences:
i) *if $t_0 \in I$ satisfies $r(t_0) = O(0,0) \in C$ then $k_{Fr}(t_0) = k(t_0)$,*
ii) *the curve C and its trigonometrically rotation iC share the same F-radial curvatures,*
iii) *suppose that the curvature function k has a constant sign, say $k > 0$. Then the Frenet decomposition of the position vector $r(t)$ is:*

$$r(t) = \frac{k'_{Fr}(t) - k'(t)}{F'(\|r(t)\|^2)k(t) - [\ln F'(\|r(t)\|^2)]'(k(t) - k_{Fr}(t))}T(t) + \frac{k(t) - k_{Fr}(t)}{2F'(\|r(t)\|^2)}N(t). \tag{9}$$

Proof We have directly:

$$\langle N(t), E(r(t))\rangle = \langle iT(t), r(t)\rangle \tag{10}$$

and the conclusion (8) follows. The first consequence derives from the inequality:

$$|k_{Fr}(t) - k(t)| \leq 2F'(\|r(t)\|^2)\|r(t)\|. \tag{11}$$

Concerning the second consequence it is obvious that the C and $iC : t \to (-y(t), x(t))$ share the same curvature k and the same second term from (8). The last consequence follows from deriving (8) and the use of the Frenet equations. $\square$

Example 3 i) If C is the line $r_0 + tu, t \in \mathbb{R}$ with the vector $u \neq \bar{0} = (0,0)$ then k_{Fr} is the non-constant function:

$$k_{Fr}(t) = -2F'(\|r_0 + tu\|^2)\frac{\langle r_0, iu\rangle}{\|u\|}. \tag{12}$$

In particular, if $O \in C$ then C is a F-radial-flat curve i.e. $k_{Fr} \equiv 0$.
ii) If C is the circle $\mathcal{C}(O, R > 0) : r(t) = Re^{it}$ the k_{Fr} is a constant:

$$k_{Fr} \equiv \frac{1}{R} + 2RF'(R^2) \geq 2\sqrt{2F'(R^2)} > 0. \tag{13}$$

If we want a F-radial-flat curve then we allow the variant of the negative F' and it results the function $F(r) = -\frac{1}{2}\ln r$ for $r \in (0, +\infty)$; remark that the function $e^{-\frac{\|r(t)\|^2}{2}}$ is considered as example in the paper [6, p. 170] in the setting of Gromov type deformation of curvature for plane curve.

For the case of logarithmic spiral expressed in polar coordinates as $\rho_{R,\alpha}(t) = Re^{\alpha t}$ with $R, \alpha > 0$ and $t \in \mathbb{R}$ we have the increasing function:

$$k_{Fr}(t) = \frac{1}{\sqrt{\alpha^2 + 1}}\left(\frac{1}{Re^{\alpha t}} + 2F'(R^2)Re^{\alpha t}\right) \geq \frac{2\sqrt{2F'(R^2)}}{\sqrt{\alpha^2 + 1}} > 0 \tag{14}$$

and for $\alpha \to 0$ we re-obtain the radial curvature of the circle $\mathcal{C}(O, R)$.
iii) Suppose that C is positively oriented in the terms of Definition 1.14 from [11, p. 17]. Suppose also that C is convex; then applying the Theorem 1.18 of page 19 from the same book it results for the usual curvature the inequality $k \geq 0$. From the inequality (11) it results that $k_{Fr}(t) \geq -2F'(\|r(t)\|^2)\|r(t)\|$, for all $t \in I$. $\square$

Let $J \subseteq \mathbb{R}$ be another open interval and fix the diffeomorphism $\varphi : s \in J \to t \in I$ with the smooth inverse $\varphi^{-1} : t \in I \to s \in J$. Since $r'(s) = \varphi'(s)r'(t(s))$ we restrict our study to the class $Diff_+(J, I)$ of orientation-preserving diffeomorphisms: $\varphi'(s) > 0$, for all $s \in J$. The transformation of the rotational curvature under the action of φ is:

$$k_{Fr}(s) = k(t) - 2F'(\|r(s)\|^2)\langle r(\varphi^{-1}(t)), N(t)\rangle, \quad t = t(s) \tag{15}$$

and then:

$$k_{Fr}(s) - k_{Fr}(t) = 2\langle F'(\|r(t)\|^2)r(t) - F'(\|r(\varphi^{-1}(t))\|^2)r(\varphi^{-1}(t)), N(t)\rangle. \tag{16}$$

Proposition 4 *The orientation-preserving diffeomorphism φ preserves also the F-radial curvature of C if and only if the vector $T(t)$ is parallel to the vector $F'(\|r(t)\|^2)r(t) - F'(\|r(\varphi^{-1}(t))\|^2)r(\varphi^{-1}(t))$ for all $t \in I$.*

An important problem is the class of curves with prescribed F-radial curvature. For example, if we ask the vanishing of the F-radial curvature for a naturally parametrized curve then it follows the characterizing equation:

$$2F'(\|r(t)\|^2)\langle r(t), ir'(t)\rangle = \|r''(t)\| = \langle r''(t), ir'(t)\rangle \tag{15}$$

which says that the vectors $r''(t) - 2F'(\|r(t)\|^2)r(t)$, $r'(t)$ are collinear: there exists a smooth function $\lambda : I \to \mathbb{R}$ such that:

$$r''(t) - 2F'(\|r(t)\|^2)r(t) = \lambda(t)r'(t). \tag{16}$$

Using the formalism of [12, p. 2] if $r : S^1 \simeq [0, 2\pi] \to \mathbb{R}^2$ is naturally parametrized then there exists the smooth function $\theta : S^1 \to \mathbb{R}$, called *normal angle*, such that:

$$N(t) = e^{i\theta(t)} = (\cos\theta(t), \sin\theta(t)), \quad T(t) = -iN(t) = -ie^{i\theta(t)} = e^{i(\theta(t) - \frac{\pi}{2})} \tag{17}$$

and then the Frenet equations yields:

$$\frac{d\theta}{dt}(t) = k(t). \tag{18}$$

Suppose again that C is with $k > 0$. Then the tangential component of the identity (9) means:

$$\langle r(t), r'(t)\rangle = \frac{k'(t)F'(\|r(t)\|^2)}{2k(t)[F''(\|r(t)\|^2) - F'(\|r(t)\|^2)^2]} \tag{19}$$

respectively the value:

$$\lambda(t) = \frac{k'(t)[F'(\|r(t)\|^2)]^2}{k(t)[F'(\|r(t)\|^2)^2 - F''(\|r(t)\|^2)]}. \tag{20}$$

We point out that the denominator of the last two ratios is zero only for $F(r) = -\ln r$ but then the derivative $F'(r)$ is strictly negative.

Proposition 5 *Suppose that t is a natural parameter on the curve C and θ is a strictly increasing function. Then C is F-radial-flat if and only if:*

$$r(t) = \frac{1}{2F'(\|r(t)\|^2)}[\theta'(t)e^{i\theta(t)} - \lambda(t)e^{i(\theta(t) - \frac{\pi}{2})}]. \tag{21}$$

subject to the unitary condition:

$$[\theta'(t)]^2 + \lambda^2(t) = 4[F'(\|r(t)\|^2)]^2. \tag{22}$$

Now, we work the same vanishing problem but in the polar coordinates $(\rho, t) \in \mathbb{R}_{>0} \times [0, 2\pi)$ of $\mathbb{R}^2$; hence $E = \rho \frac{\partial}{\partial \rho} = z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}$ and the equation of C is:

$$C : \rho = \rho(t), \quad t \in [0, 2\pi).$$

Recall that the curvature of C is then:

$$k(t) = \frac{2(\rho'(t))^2 + \rho(t)^2 - \rho(t)\rho''(t)}{[\rho(t)^2 + (\rho'(t))^2]^{\frac{3}{2}}}. \tag{23}$$

Then the vanishing of k_{rad} means the equality:

$$\frac{2(\rho'(t))^2 + \rho(t)^2 - \rho(t)\rho''(t)}{[\rho(t)^2 + (\rho'(t))^2]^{\frac{3}{2}}} = \frac{-2F'(\rho^2(t))\rho^2}{[\rho(t)^2 + (\rho'(t))^2]^{\frac{1}{2}}} \tag{24}$$

which implies:

Proposition 6 *Let C be curve not containing O. Its F-radial curvature is constant zero if and only if ρ is the solution of the following non-linear second-order ODE*:

$$\rho'' = \frac{2}{\rho}(\rho')^2 + \rho[1 + 2F'(\rho^2)(\rho^2 + (\rho')^2)]. \tag{25}$$

In the following we present a couple of examples in order to remark the computational aspects of our approach.

Example 7 We study completely a curve with non-constant rotational curve. Namely, the involute of the unit circle S^1 is:

$$C : r(t) = (\cos t + t \sin t, \sin t - t \cos t) = (1 - it)e^{it}, \quad t \in (0, +\infty). \tag{26}$$

A direct computation gives:

$$r'(t) = (t \cos t, t \sin t) = te^{it}, \quad k(t) = \frac{1}{t} > 0, \quad \|r(t)\|^2 = 1 + t^2 \tag{27}$$

and then the F-radial curvature is:

$$k_{Fr}(t) = \frac{1}{t} + 2tF'(1 + t^2) \geq 2\sqrt{2F'(1 + t^2)}. \tag{28}$$

This example can be treated also with respect to a natural parameter $s \in (0, +\infty)$ which is provided by $t := \sqrt{2s}$. For example, the normal angle function is $\theta(s) = \frac{\pi}{2} + \sqrt{2s}$ since $r'(s) = e^{i\sqrt{2s}}$. $\square$

Example 8 Fix a graph $C : y = f(x)$, $x \in I$ with the second derivative f'' strictly positive. With the usual parametrization $C : r(t) = (t, f(t))$ we have:

$$r'(t) = (1, f'(t)), \quad k(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} > 0, \quad \|r(t)\|^2 = t^2 + f^2(t) \tag{29}$$

which gives that C is convex and:

$$k_{Fr}(t) = \frac{f''(t)}{[1 + (f'(t))^2]^{\frac{3}{2}}} - 2F'(t^2 + f^2(t))\frac{f(t) - tf'(t)}{[1 + (f'(t))^2]^{\frac{1}{2}}}. \tag{30}$$

It follows that the function f making k_{Fr} constant zero is a solution of the non-autonomous differential equation:

$$f''(t) = 2F'(t^2 + f^2(t))[f(t) - tf'(t)][1 + (f'(t))^2]. \tag{31}$$

$\square$

Example 9 The derivative curve r' from (28) is an Archimedes' spiral. The general such spiral is given in polar coordinates as:

$$A(spiral) : \rho(t) = Rt, \quad R > 0, \quad t > 0 \tag{32}$$

and hence:

$$k_{Fr}(t) = \frac{2 + t^2}{R(t^2 + 1)^{\frac{3}{2}}} + \frac{2F'(R^2 t^2)Rt}{(t^2 + 1)^{\frac{1}{2}}} > 0. \tag{33}$$

For the case of the g_{can}-harmonic radial function $F(r) = \ln r$ we get:

$$k_{Fr}(t) = \frac{2 + t^2}{R(t^2 + 1)^{\frac{3}{2}}} + \frac{2}{Rt(t^2 + 1)^{\frac{1}{2}}} > 0. \tag{34}$$

□

We end this note with an approach in terms of Fermi-Walker derivative. Let $\mathfrak{X}_C$ be the set of vector fields along the curve C. Then the Fermi-Walker derivative is the map ([4]) $\nabla_C^{FW} : \mathfrak{X}_C \to \mathfrak{X}_C$:

$$\nabla_C^{FW}(X) := \frac{d}{dt}X + k\|r'(\cdot)\|[\langle X, N\rangle T - \langle X, T\rangle N] = \frac{d}{dt}X + k[X^{\flat}(N)T - X^{\flat}(T)N] \tag{35}$$

with $X^{\flat}$ the differential 1-form dual to X with respect to the Euclidean metric. For $X = E \circ r = r$ we have:

$$\nabla_C^{FW}(r) = \|r'(\cdot)\|(T + k[\langle r, N\rangle T - \langle r, T\rangle N]) \tag{36}$$

and then, using (9):

$$(\nabla_C^{FW}r)(t) = \|r'(t)\| \left[1 + \frac{k^2(t) - k(t)k_{Fr}(t)}{2F'(\|r(t)\|^2)}\right] T(t) - \|r'(t)\|k(t)A(t)N(t). \tag{37}$$

Hence, for a F-radial-flat C we have:

$$(\nabla_C^{FW}r)(t) = \|r'(t)\| \left[1 + \frac{k^2(t)}{2F'(\|r(t)\|^2)}\right] T(t) - \|r'(t)\|k(t)A(t)N(t) \tag{38}$$

which can not be the zero vector due to the strictly positive coefficient of $T(t)$.

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