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The flow-curvature of spacelike parametrized curves in the Lorentz plane

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Abstract. We introduce and study a new frame and a new curvature function for a fixed parametrization of a spacelike curve in the Lorentz plane. This new frame is called *flow* since it involves the time-dependent rotation of the usual Frenet flow.

The theory of geometric flows is a new, fascinating field of research in geometric analysis. The most simple of them is the *curve shortening flow* and already the excellent survey [?] is twenty years old. Recall that the main geometric tool in this last flow is the well-known curvature of plane curves. Hence, to give a re-start to this problem seams to search for variants of the curvature, or in terms of [6], *deformations* of the usual curvature. The goal of this short note is to propose such a deformation, not in the plane Euclidean geometry, but in its Lorentzian counterpart.

The setting of this paper is the Lorentz plane $\mathbb{R}^2_1 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$:

$$\begin{cases} \langle u, v\rangle_L = -u^1v^1 + u^2v^2, \quad u = (u^1, u^2) \in \mathbb{R}^2, \quad v = (v^1, v^2) \in \mathbb{R}^2, \\ 0 \leq \|u\|^2_L = |\langle u, u\rangle_L|. \end{cases} \tag{1}$$

Fix an open interval $I \subseteq \mathbb{R}$ and consider $C \subset \mathbb{R}^2$ a spacelike parametrized curve of equation:

$$\begin{cases} C : r(t) = (x(t), y(t)) = x(t)\bar{i} + y(t)\bar{j}, \quad \bar{i} = (1, 0), \quad \bar{j} = (0, 1), \\ \langle r'(t), r'(t)\rangle_L > 0, \quad t \in I. \end{cases} \tag{2}$$

The infinitesimal generator of the Lorentz rotations in $\mathbb{R}^2_1$ is the linear vector field:

$$\xi_L(u) := u^2 \frac{\partial}{\partial u^1} + u^1 \frac{\partial}{\partial u^2}, \quad \xi_L(u) = j \cdot u = j \cdot (u^1 + iu^2) \tag{3}$$

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where $(\mathbb{R}^2, j)$, $j^2 = 1$, is the two-dimensional paracomplex algebra, [?]. The first integrals of ξ_L are the functions: $f_\alpha(x,y) = \alpha(x^2 - y^2)$, $\alpha \in \mathbb{R}$. For an arbitrary vector field $X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ its Lie bracket with ξ is:

$$[X, \xi_L] = (A - xA_x - yA_y)\frac{\partial}{\partial x} + (B - xB_x - yB_y)\frac{\partial}{\partial y}, \tag{4}$$

where the subscript denotes the variable corresponding to the partial derivative. For example, ξ_L commutes with *the radial* (or Euler) vector field $E(x,y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, which is also a complete vector field having as integral curves the Euclidean homotheties $\gamma^E_{u_0}(t) = e^t u_0$ for all $t \in \mathbb{R}$ and $u_0 \in \mathbb{R}^2$.

The Frenet apparatus of the curve C is provided by:

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|_L}, N(t) = j \cdot T(t) = \frac{1}{\|r'(t)\|_L}(y'(t), x'(t)), \\ \langle T(t), T(t)\rangle_L = 1 = -\langle N(t), N(t)\rangle_L \\ k_L(t) = \frac{1}{\|r'(t)\|_L}\langle T'(t), N(t)\rangle_L \\ k_L(t) = \frac{1}{\|r'(t)\|^3_L}\langle r''(t), jr'(t)\rangle_L = \frac{1}{\|r'(t)\|^3_L}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases} \tag{5}$$

Hence T is an unit spacelike vector field along C while N is an unit timelike vector field along C; we denote with $\mathcal{X}_C$ the set of vector fields along the curve C. The curvature function can be expressed using a 2×2 determinant:

$$k_L(t) = \frac{1}{\|r'(t)\|^3_L} \det \begin{pmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{pmatrix} \tag{6}$$

and the difference to the Euclidean curvature consists in the ratio in front of this determinant; in the Euclidean case is the Euclidean norm $\|r'(t)\|^{-3}$. The Frenet equations can be unified by means of the column matrix $\mathcal{F}(t) = \begin{pmatrix} T \\ N \end{pmatrix}(t)$ as:

$$\frac{d}{dt}\mathcal{F}(t) = -\|r'(t)\|_L k_L(t) R'_L(0)\, \mathcal{F}(t), \quad R'_L(0) \in so(1,1) \tag{7}$$

with the Lorentz rotation $R_L(t) \in SO(1,1)$ given by the symmetric matrices:

$$R_L(t) := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, t \in \mathbb{R}, \quad R_L^{-1}(t) := \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}. \tag{8}$$

The Lorentz rotated curve $jC : r_j(t) := j \cdot r(t) = (y(t), x(t))$ is a timelike curve since $< r'_j(t), r'_j(t) >_L = - < r'(t), r'(t) >_L$.

This short note defines a new frame and correspondingly a new curvature function for C:

Definition 1 The *flow-frame* of C consists in the pair of vector fields $(E^f_1, E^f_2) \in \mathcal{X}_C$ given by:

$$\mathcal{E}(t) := \begin{pmatrix} E^f_1 \\ E^f_2 \end{pmatrix}(t) = R_L(t)\mathcal{F}(t) = \begin{pmatrix} \cosh tT(t) + \sinh tN(t) \\ \sinh tT(t) + \cosh tN(t) \end{pmatrix} \quad (9)$$

the letter f being the initial of the word "flow". The *flow-curvature* of C is the smooth function $k_{Lf} : I \to \mathbb{R}$ given by *the flow-equations* as the flow-variant of (7):

$$\frac{d}{dt}\mathcal{E}(t) = -\|r'(t)\|_L k_{Lf}(t) R'_L(0)\mathcal{E}(t). \quad (10)$$

Before starting its study we point out that this work is dedicated to the memory of Academician Radu Miron (1927-2022). He was always interested in the geometry of curves and besides his theory of *Myller configurations* ([8]) he generalizes also a type of curvature for space curves in [7]. Returning to our subject we note as first main result:

Proposition 2 *E^f_1 is an unit spacelike vector field and E^f_2 is an unit timelike vector field. The expression of the flow-curvature is:*

$$k_{Lf}(t) = k_L(t) - \frac{1}{\|r'(t)\|_L} < k_L(t). \quad (11)$$

If the decomposition of the position vector $r(t)$ with respect to the Frenet frame is $r(t) = A(t)T(t)+B(t)N(t)$ with the smooth functions $A, B : I \to \mathbb{R}$ then its decomposition with respect to the frame flow is:

$$r(t) = [A(t)\cosh t - B(t)\sinh t]E^f_1(t) + [B(t)\cosh t - A(t)\sinh t]E^f_2(t). \quad (12)$$

Proof We have directly in the flow-frame:

$$-\|r'(t)\|_L k_{Lf}(t) R'_L(0) = R'_L(t)R_L^{-1}(t) - \|r'(t)\|_L k_L(t) R_L(t) R'_L(0) R_L^{-1}(t)$$

and the conclusion follows. Also the decomposition (12). $\square$

Examples 3 i) If C is the line $r_0 + tu, t \in \mathbb{R}$ with the spacelike vector $u \neq \bar{0} = (0, 0)$ then both k_L and k_{Lf} are constant:

$$k_{Lf} = -\frac{1}{\|u\|} = constant < 0 = k_L. \quad (13)$$

In particular, if u is an unit spacelike vector then $k_{Lf} = -1$.
ii) The corresponding of the circles $\mathcal{C}(O, R > 0)$ of Euclidean plane geometry is provided by the equilateral hyperbola as integral curve of ξ_L:

$$\begin{cases} H_e(R) : x^2 - y^2 = R^2, \quad k_L = constant = -\frac{1}{R} < 0, \\ T(t) = (\sinh t, \cosh t), \quad N(t) = \frac{1}{R}r(t) = (\cosh t, \sinh t), \\ (R_L(u) \circ r)(t) = r(t + u) = (R_L(t) \circ r)(u). \end{cases} \quad (14)$$

The parametrization by arc-length of $H_e(R)$ is:

$$r_e(s) = R\left(\cosh\frac{s}{R}, \sinh\frac{s}{R}\right). \tag{15}$$

The new frame and curvature are:

$$\begin{cases} E_1^f(t) = T(2t), \quad E_2^f(t) = N(2t), \quad k_{Lf} = -\frac{2}{R}, \\ r(t) = (-R\sinh t)E_1^f(t) + (R\cosh t)E_2^f(t). \end{cases} \tag{16}$$

We note that $H_e(R)$ is called *pseudo-circle* in [10, p. 110] and is denoted $H^1(-R)$.
iii) The quarter-circle $C_{1/4} : r(t) = R(\cos t, \sin t)$, $t \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ has:

$$\begin{cases} T(t) = \frac{1}{\sqrt{\cos 2t}}(-\sin t, \cos t), \quad N(t) = \frac{1}{\sqrt{\cos 2t}}(\cos t, -\sin t), \\ k_L(t) = \frac{1}{R(\cos 2t)^{\frac{3}{2}}} > 0. \end{cases} \tag{17}$$

Its flow-curvature and total flow-curvature are:

$$k_{Lf}(t) = \frac{2\sin^2 t}{R(\cos 2t)^{\frac{3}{2}}} \geq 0, \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} k_{Lf}(t)dt = \frac{2}{R}\left(2.66435... \times 10^8\right) \tag{18}$$

and the minimum value of k_{Lf}, namely 0, is attained at $t = 0$.
iv) The previous example of the equilateral hyperbola suggests to consider the curve $t \in I \to E_2^f(t)$. Since:

$$< (E_2^f)'(t), (E_2^f)'(t) >_L = [1 - \|r'(t)\|_L k_L(t)]^2 \tag{19}$$

it follows that this new curve is spacelike if and only if the initial curve C is not Lorentzian flow-flat i.e. $k_{LF} \equiv 0$.
v) Fix the smooth function $f : I \to \mathbb{R}$ and its graph $\Gamma_f : r(t) = (t, f(t))$. This curve is spacelike if and only if the range of the derivative of f is included in $(-\infty, -1) \cup (1, +\infty)$ and then its curvatures are:

$$k_L(t) = \frac{f''(t)}{[(f'(t))^2 - 1]^{\frac{3}{2}}}, \quad k_{Lf}(t) = \frac{f''(t)}{[(f'(t))^2 - 1]^{\frac{3}{2}}} - \frac{1}{[(f'(t))^2 - 1]^{\frac{1}{2}}}. \tag{20}$$

Looking for a flow-flat graph we obtain the Lorentz version of the Grim-Reaper:

$$f(t) = -\ln(\sinh t), \quad t \in (0, +\infty), \quad f'(t) = -\frac{\cosh t}{\sinh t} < -1. \tag{21}$$

Without the minus sign this curve appears in [1, p. 671] (in Corollary 5.1) and in the formula (5) of [5, p. 4]; we point out that another curve is also called *Lorentzian grim-reaper* in Corollary 8.1 of [2, p. 769]. The usual curvature of the graph of (21) is:

$$k_L(t) = \sinh t = \frac{1}{\|r'(t)\|}. \tag{22}$$

Since $\sinh(\ln(1+\sqrt{2}))=1$ we compute two integrals:

$$\int_0^{\ln(1+\sqrt{2})} f(t)dt = 0.955202...., \quad \int_0^{\ln(1+\sqrt{2})} k_L(t)dt = \sqrt{2}-1 = 0.41421...$$

The frames of this flow-flat graph are:

$$\begin{cases} T(t) = (\sinh t, -\cosh t), \quad N(t) = (-\cosh t, \sinh t), \\ E_1^f = (0,-1), \quad E_2^f = (-1,0). \end{cases} \tag{23}$$

□

Remarks 4 i) An important tool in dynamics is the Fermi-Walker derivative. Then the Lorentz Fermi-Walker derivative is the hyperbolic variant of derivative from ([**?**]), namely $\nabla_L^{FW} : \mathcal{X}_C \to \mathcal{X}_C$:

$$\nabla_L^{FW}(X) := \frac{d}{dt}X - \|r'(\cdot)\|_L k_L(\cdot)[\langle X, N\rangle_L \cdot T - \langle X, T\rangle_L \cdot N]. \tag{24}$$

It is natural to make here a remark concerning *rotation-minimizing fields* $X \in \mathcal{X}_C$ i.e. fields satisfying:

$$\frac{d}{dt}X(t) = \lambda(t)T(t), \quad \langle X(t), T(t)\rangle_L = 0 \tag{25}$$

for a smooth function $\lambda = \lambda(t)$. Then the Fermi-Walker derivative of such X is parallel with the tangent vector field T:

$$(\nabla_L^{FW}X)(t) = [\lambda(t) - \|r'(t)\|_L k_L(t)\langle X(t), N(t)\rangle_L]T(t). \tag{26}$$

Computing the Fermi-Walker derivative on our frames we get:

$$\nabla_L^{FW}(T) = \nabla_L^{FW}(N) = 0, \quad \nabla_L^{FW}(E_1^f) = E_2^f, \quad \nabla_L^{FW}(E_2^f) = E_1^f. \tag{27}$$

With the matrix notation we can express these relations as:

$$\nabla_L^{FW}(\mathcal{F}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \nabla_L^{FW}(\mathcal{E}) = R'_L(0)\mathcal{E} \tag{28}$$

and the Fermi-Walker derivative can be expressed in terms of k_{Lf} as:

$$\nabla_L^{FW}(X) = \frac{d}{dt}X - (\|r'\|_L k_{Lf} + 1)[\langle X, N\rangle_L \cdot T - \langle X, T\rangle_L \cdot N]. \tag{29}$$

ii) The nature and the relationship between our frames can be put in the framework of *moving frames* of [9, p. 32]. Recall that the set of all orientation-preserving Lorentzian isometries forms a Lie group, $L(2) := \mathbb{R}^2 \times SO(1,1)$, with the standard projection π_1 on the first factor making $L(2) \to \mathbb{R}^2$ an $SO(1,1)$-principal bundle. A moving frame along C is a map

$F : I \to L(2)$ such that $\pi_1 \circ F = r$. But C defines also a 1-parameter family of bijections of $SO(1,1)$:

$$\begin{cases} L^C : I \to Bijections(SO(1,1)), \\ t \to L^C(t) : SO(1,1) \to SO(1,1), A \to R_L(t)A. \end{cases} \tag{30}$$

Then our frames are $\mathcal{F} : I \to L(2)$ as $\mathcal{F}(t) = (r(t), T(t), N(t))$ and $\mathcal{E} : I \to L(2)$ as $\mathcal{E}(t) = (r(t), (L^C(t) \circ \pi_2 \circ \mathcal{F})(t))$. $\square$

We finish this note with the problem raised in the beginning, namely the possible variants of the curve shortening flow; note that the self-similar solutions in the hyperbolic variant of this problem are studied in [11]. Recall that the (Lorentzian) setting of this question consists in a 1-parameter family of plane curves $C_u : r = r(t,u)$ satisfying:

$$\frac{\partial r(t,u)}{\partial u} = k_L(t,u)N(t,u). \tag{31}$$

It follows immediately an expression in terms of flow-apparatus:

$$\frac{\partial r(t,u)}{\partial u} = \left(k_{Lf}(t,u) + \frac{1}{\|r'_t(t,u)\|_L}\right)[\sinh t E^f_1(t,u) + \cosh t E^f_2(t,u)]. \tag{32}$$

and due to the commutativity of partial derivatives with respect to t and u we have the u-variation of the Frenet frame:

$$\begin{cases} \frac{\partial T(t,u)}{\partial u} = -\|r'_t(t,u)\|_L k^2_L(t,u)T(t,u) + \frac{\partial k_L(t,u)}{\partial t}N(t,u), \\ \frac{\partial N(t,u)}{\partial u} = \frac{\partial k_L(t,u)}{\partial t}T(t,u) - \|r'_t(t,u)\|_L k^2_L(t,u)N(t,u). \end{cases} \tag{33}$$

The first variant which we propose as an open problem is to study the flow-variant of (31):

$$\frac{\partial r(t,u)}{\partial u} = k_{Lf}(t,u)E^f_2(t,u) \tag{34}$$

while the second variant is to restrict only to the normal component of E^f_2:

$$\frac{\partial r(t,u)}{\partial u} = k_{Lf}(t,u)\cosh t N(t,u). \tag{35}$$

We point out that a solution to the Lorentzian curve flow (31) is provided by the 1-parameter family of equilateral hyperbolas with an initial equilateral hyperbola $H_e(R_0 > 0)$:

$$H_e(u) : x^2 - y^2 = R_0^2 - 2u, \quad 0 \le u < \frac{R_0^2}{2}, \quad k_{Lf}(u) = \frac{-2}{\sqrt{R_0^2 - 2u}} \tag{36}$$

for which the expression defined in (34) is:

$$\frac{\partial r(t,u)}{\partial u} - k_{Lf}(t,u)E_2^f(t,u) = \frac{1}{\sqrt{R_0^2 - 2u}}(2\cosh 2t - \cosh t, 2\sinh 2t - \sinh t) \tag{37}$$

and the expression of (35) is:

$$\frac{\partial r(t,u)}{\partial u} - k_{Lf}(t,u)\cosh t N(t,u) = \frac{\cosh 2t}{\sqrt{R_0^2 - 2u}}(\cosh t, \sinh t) = timelike. \tag{38}$$

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