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ALMOST COMPLEX INTEGRAL SUBMANIFOLDS AND TOTALLY UMBILICAL INTEGRAL SUBMANIFOLDS

FATMA ÖZDEMIR, ALI DOĞANAKSOY and MIRCEA CRÂŞMĂREANU

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Dans cet article on considère une variété Riemanniene avec une métrique Hermitienne qui admet un champ de planes doué avec une structure presque complexe. On donne un critère tel que l'operateur de Weingarten associé a une distribution integrable soit linearement complexe. On preuve aussi certaines conditions telles que les sous-variété integrales soient totalement ombiliques. Quelques examples soint aussi presentes.

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1. INTRODUCTION

In [4] and [3] the integrability conditions of the distribution P are given. Here for such a distribution P and the shape operator L_Z of P, we obtain a criteria for the total umbilicity of the integral submanifolds.

By a manifold, we mean a smooth manifold, that is, a Hausdorff space with a fixed complete atlas compatible with the pseudogroup of transformations of class C^∞ of R^n. Similarly, plane fields and vector fields that take a part in the work are all supposed to be smooth. On a manifold M, we denote the algebra of vector fields by $\chi(M)$. For vector fields X and Y on M, by $[X,Y]$ we denote their Lie bracket, and for a tensor field K of type $(1,1)$ on M, $[K,K]$ denotes Nijenhuis tensor (or torsion) of K i.e., $[K,K]$ is a tensor field of type $(1,2)$, which, regarded as a mapping $\chi(M)\times\chi(M)\to\chi(M)$, is given by

$$[\mathrm{K},\mathrm{K}](X,Y)=2([KX,KY]+K^2[X,Y]-K[KX,Y]-K[X,KY])$$

for vector fields X and Y on M, [5].

Let M be an n-dimensional Riemannian manifold and P be a k-dimensional distribution. Assume that P admits a tensor field J of type $(1,1)$ such that, when considered as a linear endomorphism of $P(x)$, $J_x^2=-1_{P(x)}$ for all $x\in M$, where $J_x^2=J_x\circ J_x$ and $1_{P(x)}$ is the identity map of $P(x)$. As J_x defines a complex structure on $P(x)$ at each $x\in M$, P should be an even dimensional distribution.

Such a pair (P,J) is called an *almost complex k-substructure* (abbreviated a.c.k-s) on M. Now we will give certain definitions related to J.

In the following we will always denote the orthogonal complement of P by Q. Let p and q be the projections $TM \to P$ and $TM \to Q$ respectively, Q being the $(n-k)$-dimensional plane field with $Q(x) = P(x)^{\perp}$ at each $x \in M$. We can also consider p and q as $(1,1)$ tensors on M this should be clear from the context. Then the tensor field J_p defined on M, of type $(1,1)$ maps TM onto P, such that its restriction to $P(x)$ is the tensor J and it vanishes on $Q(x)$ at each $x \in M$.

We define the extension of J to be the $(1,1)$ tensor $\tilde{J}$ by setting

$$\tilde{J} = q + J_p \tag{1.1}$$

Then it can be seen that

$$\tilde{J}^2 = q - p \tag{1.2}$$

$$\tilde{J}^3 = q - J_p = 1 - \tilde{J} + \tilde{J}^2 \tag{1.3}$$

$$\tilde{J}^4 = 1 \tag{1.4}$$

From (1.4) it results that the tensor field $\tilde{J}$ defines a structure of *electromagnetic type* on M([6]).

A Hermitian metric on a manifold with an a.c.k-s (P,J) is a Riemannian metric g which is invariant by the almost complex structure J, i.e., $g(JX, JY) = g(X,Y)$ for any vector fields X and Y that belong to P.

2. TOTALLY UMBILIC INTEGRAL SUBMANIFOLDS

Let now J be a Hermitian complex structure on P. We define the shape operator L_Z of P by $L_Z(X) = -p(\nabla_X Z)$ for $Z \in Q$ and $X \in P$, and $\overline{L}_Z = J\, L_Z$ be the conjugate shape operator. Here ∇ is the covariant derivative operator of the Riemannian connection determined by the Hermitian metric.

Proposition 2.1. *Let P be an integrable distribution with a Hermitian almost complex structure J. It can be seen that L_Z is a self-adjoint operator. Then $\overline{L}_Z$ is skew-adjoint if and only if $JL_Z = L_Z J$, i.e. L_Z is complex linear.*

Proof. Assume $\overline{L}z$ is skew-adjoint. Since L_Z is self-adjoint and J is skew-adjoint then

$$(\overline{L}z)^* = (J\,L_Z)^* = L_Z^{\,*}\,J^* = -L_Z^{\,*}J = -L_Z J \tag{2.1}$$

If $(\overline{L}z)$ is skew-adjoint then $(\overline{L}z)^* = -\overline{L}z = -J\,L_Z$. So we have $L_Z J = JL_Z$.

We need to show $(L_Z)^* = L_Z$ (L_Z is self-adjoint). This is equivalent to

$g(L_Z X, Y) - g(X, L_Z Y) = 0$; i.e., $< p(\nabla_X Z), Y > - < X, p(\nabla_Y Z) > = 0$:

Since $Y \in P$, $Z \in Q$

$$< Y, Z > = 0 \tag{2.2}$$

$$X < Y, Z > = 0 \tag{2.3}$$

$$X < Y, Z > = < \nabla_X Y, Z > + < Y, \nabla_X Z > = 0 \tag{2.4}$$

or

$$< \nabla_X Y, Z > = - < Y, \nabla_X Z > \tag{2.5}$$

if $Y \in P$ then

$$< Y, \nabla_X Z > = < Y, p(\nabla_X Z) > \tag{2.6}$$

Then

$$< \nabla_X Y, Z > = - < Y, p(\nabla_X Z) > \tag{2.7}$$

$$< \nabla_Y X, Z > = - < X, p(\nabla_Y Z) > \tag{2.8}$$

From (2.4), (2.7) and (2.8) we get

$$< p(\nabla_X Z), Y > - < X, p(\nabla_Y Z) > \tag{2.9}$$

$$= - < \nabla_X Y, Z > + < \nabla_Y X, Z > = < Z, \nabla_Y X - \nabla_X Y > .$$

But $[X,Y] = \nabla_X Y - \nabla_Y X$ and since P is integrable $[X,Y] \in P$. Then $< Z, [X,Y] > = 0$ which shows $g(L_Z X, Y) = g(X, L_Z Y)$.

Conversely, if $JL_Z = L_Z J$ then we have

$$(\overline{L}z)^* = (J\,L_Z)^* = L_Z^{\,*}\,J^* = -L_Z J = -JL_Z = -\overline{L}z \tag{2.10}$$

that is, $\overline{L}z$ is skew-adjoint. □

Proposition 2.2. *Let P be an integrable distribution with a Hermitian almost complex structure J. If the integral submanifolds are totally umbilic, then $\overline{L}z$ is skew-adjoint.*

Proof. If P is integrable then $JL_Z = L_Z\,J$ if and only if $(\overline{L}z)^* = -\overline{L}z$.

Recall that, if P is integrable then $L_Z^* = L_Z$. Since $L_Z^* = L_Z$,

$(\bar{L}_Z)^* = (J\,L_Z)^* = L_Z^*\,J^* = -L_Z^*J = -L_Z J.$

If $L_Z = \lambda\, I$ then $J(\lambda\, I) = (\lambda\, I)\, J$, and

$$(\bar{L}_Z)^* = (J\,L_Z)^* = (J(\lambda\, I))^* = (\lambda\, I)J^* = -(\lambda\, I)J = -J(\lambda\, I) = -J\,L_Z \qquad (2.11)$$

which implies that L_Z is skew-adjoint. □

Proposition 2.3. *Let P be integrable. Integral submanifolds are totally umbilic if and only if L_Z is self-adjoint, L_Z is complex linear on P, and every complex subbundle of P is L_Z invariant for every $Z \in Q$.*

Proof. Assume that integral manifolds are totally umbilic, i.e. $L_Z = \lambda I$. $< L_Z\, X, Y > = < \lambda\, X, Y > = < X, \lambda Y > = < X, L_Z\, Y >$ which implies that $L_Z = L_Z^*$. L_Z is complex linear if and only if $L_Z\, J = J\, L_Z$. If $L_Z = \lambda\, I$ then $(\lambda\, I)\, J = J(\lambda\, I)$. It follows that L_Z is complex linear. Let R be a complex subbundle of P $(R \subset P)$. For any $X \in R$, and if L_Z is totally umbilic, we get $L_Z\,(X) = \lambda\, X$ and $L_Z\,(X)$ is in R.

Now we assume that, $L_Z = L_Z^*$, $L_Z\, J = J\, L_Z$ and for any complex subbundle R of P, $X \in R$ implies that $L_Z X \in R$.

Assume $L_Z \neq \lambda\, I$ (i.e. integral manifolds are not totally umbilic). Since L_Z is self-adjoint, it has eigenvalues. Let λ and μ be two distinct eigenvalues and X and Y be the corresponding eigenvectors, i.e. $L_Z\, X = \lambda\, X$, $L_Z\, Y = \mu\, Y$. Let R be the complex subbundle of P spanned by $X + Y$. If $\lambda \neq \mu$, $L_Z\,(X + Y) = (\lambda\, X + \mu\, Y)$, which implies that $L_Z(X + Y) \in R$ if and only if λ is equal to μ. Hence if L_Z is self adjoint, complex linear and R is invariant by L_Z, then L_Z has to be totally umbilic.

REMARKS

The conditions of Proposition 2.3. are in fact quite restrictive. As P is an integrable distribution admitting an almost complex structure J, the integral manifolds of P are almost complex, and M admits a foliation whose leaves are almost complex, connected, immersed submanifolds [1]. On the other hand, it is known that a totally umbilic connected and complete immersed submanifold is either a hyperplane or a hypersphere [5]. Thus if the integral manifolds are compact they should be spheres. Since they should admit complex structures, these compact integral manifolds can be only S^2 and S^6 [2]. The shape operator for S^2 and S^6 is $L_Z = \frac{1}{a_n} I_n$, where I_n is the identity matrix and a_n is the radius for the spheres S^2 and S^6 respectively.

As examples of manifolds foliated by S^2 and S^6 and any Hermitian manifold N, we can cite the product manifolds $S^2 \times N$, $S^6 \times N$, and total spaces of unit tangent bundles of 3 and 7 dimensional manifolds.

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Faculty of Science and Letters
Istanbul Technical University
80626, Maslak, Istanbul,
Turkey
e-mail: fozdemir@itu.edu.tr

Faculty of Science and Arts
Middle East Technical University 06531,
Ankara, Turkey

Faculty of Mathematics
University "Al. I. Cuza"
Iaşi, 6600
Romania
Email: mcrasm@uaic.ro