

# The adjoint map of Euclidean plane curves and curvature problems

Mircea Crasmareanu

**Abstract.** The adjoint map of a pair of naturally parametrized curves in the Euclidean plane is studied from the point of view of the curvature. A main interest is when the given curve and its adjoint curve share the same natural parameter and the same curvature. For the general linear second order differential equation we introduce a function expressing the deformation of curvatures induced by the adjoint map.

## 1 Introduction

Fix the smooth regular curve  $r : I \subseteq \mathbb{R} \rightarrow \mathbb{E}^2 = (\mathbb{R}^2, \langle \cdot, \cdot \rangle_{can})$  having the Wronskians  $W(r) > 0$  (hence  $r$  is not a line through the origin  $O$  of  $\mathbb{R}^2$ ) and  $W(r') \neq 0$ . Expressing the given curve as  $r(\cdot) = (x(\cdot), y(\cdot))$  its component functions  $x, y$  are solutions of the Wronskian linear differential equation:

$$\begin{cases} W(x, y, u = u(\cdot)) := \begin{vmatrix} x & y & u \\ x' & y' & u' \\ x'' & y'' & u'' \end{vmatrix} = 0 \rightarrow \mathcal{E}^2 : u''(t) + p(t)u'(t) + q(t)u(t) = 0, \\ p := -\frac{[W(r)]'}{W(r)}, \quad q := \frac{W(r')}{W(r)}, \quad \mathcal{E}^2 : \frac{d}{dt} \left( \frac{u'}{W(r)} \right) + \frac{W(r')u}{(W(r))^2} = 0. \end{cases} \quad (1.1)$$

It is well-known that the general solution of (1.1) is provided by two real constants  $C_1, C_2$  through the formula:

$$u(t) = C_1x(t) + C_2y(t), \quad C_1 = \frac{W(u, y)}{W(r)}, \quad C_2 = \frac{W(x, u)}{W(r)}. \quad (1.2)$$

For further use, let  $P = P(t)$  be the anti-derivative of the first coefficient function  $p = p(t)$  and  $k = k(t)$  the usual curvature of  $r$ ; we suppose that  $r$  has no inflexion points, so  $k > 0$

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Corresponding author: Mircea Crasmareanu; Submission: October 24, 2023.

1 or  $k < 0$  on  $I$ . A main hypothesis of this short note is that  $t$  is a natural parameter for  
 2  $r$ ; then  $I = (0, L(r))$  with  $L(r)$  the length of  $r$  and the module  $|W(r)|$  is the Euclidean  
 3 distance from the origin  $O$  to the tangent line of the curve. The two functions above are:

$$4 \quad P = -\ln W(r), \quad k = W(r') = q \exp(-P) \rightarrow q \neq 0. \quad (1.3)$$

5 The second order ordinary differential equation (SODE)  $\mathcal{E}^2$  expresses  $u$  as solution  
 6 of the differential operator:

$$\mathcal{D} := \frac{d^2}{dt^2} + p \frac{d}{dt} + q = \sum_{i=0}^2 \mu_i \frac{d^i}{dt^i} : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}) \quad (1.4)$$

7 and we recall that any  $k$ -differential operator  $\mathcal{D} := \sum_{i=0}^k \mu_i \frac{d^i}{dt^i}$  has an *adjoint operator*  
 8 ([3], [6, p. 218]):

$$\mathcal{D}_a := \sum_{i=0}^k (-1)^i \frac{d^i}{dt^i} (\mu_i \cdot). \quad (1.5)$$

9 Fix now *the adjoint curve*  $r_a(t) = (x_a(t), y_a(t))$  which corresponds to *the adjoint*  
 10 *SODE*  $\mathcal{E}_a^2$  provided by  $\mathcal{D}_a$ . Our aim is to study the corresponding curvature transforma-  
 11 tion  $k \rightarrow k_a$  due to the fundamental theorem of plane curves ([1, p. 52]), which states the  
 12 main role of this differential invariant in the geometry of  $r$ . Expressing the initial curve in  
 13 generalized polar coordinates we characterize the curvature-preserving adjoint map and  
 14 two examples are discussed. Since finding non-selfdual examples of adjoint curves with  
 15 the same curvature is a difficult problem we introduce a curvature-deformation function  
 16 of a general linear SODE; again two examples are considered.

## 17 2 The adjoint map and curvature problems

18 We fix now the expression of the initial curve:

$$r(t) := \rho(t) \exp(i\omega(t)), \rho > 0, \quad r'(t) = \rho'(t) \exp(i\omega(t)) + \rho(t) \omega'(t) \exp\left(i\omega(t) + \frac{\pi}{2}\right) \quad (2.1)$$

19 and hence since  $t$  is a natural parameter we have:

$$(\rho')^2 + (\rho\omega')^2 = 1. \quad (2.2)$$

20 The characterization of the curvature-preserving transformations  $r \rightarrow r_a$  is provided  
 21 by:

1 **Theorem 2.1.** *Suppose that  $t$  is also a natural parameter for the adjoint curve  $r_a$ . Then*  
 2 *the adjoint map  $r \rightarrow r_a$  is curvature-preserving if and only if:*

$$\frac{[\ln(\rho\sqrt{1-(\rho')^2})]''}{\rho^2(1-(\rho')^2)-1} = \frac{[\ln(\rho\sqrt{1-(\rho')^2})]'}{\rho\rho'}. \quad (2.3)$$

3 *Proof.* Since  $t$  is a natural parameter for  $r$  we can express  $q$  as function of  $\rho$ . Indeed the  
 4 system (1.1) means:

$$\begin{cases} x'' + px' + qx = 0 \\ y'' + py' + qy = 0. \end{cases} \quad (2.4)$$

5 Multiplying the first equation with  $x'$ , the second equation with  $y'$  and adding the re-  
 6 sulting relations we obtain:

$$q = -\frac{2p}{(\rho^2)'} = -\frac{p}{\rho\rho'}. \quad (2.5)$$

7 The adjoint SODE of  $\mathcal{E}^2$  is:

$$\mathcal{E}_a^2 : u_a''(t) - p(t)u_a'(t) + (q(t) - p'(t))u_a(t) = 0 \quad (2.6)$$

8 and the hypothesis gives the adjoint curvature:

$$k_a = (q - p') \exp(P). \quad (2.7)$$

9 The equality  $k = k_a$  (which implies  $q \neq p'$ ) combined with (2.5) reads as:

$$q = \frac{P''}{1 - \exp(-2P)} = -\frac{P'}{\rho\rho'}. \quad (2.8)$$

10 From (2.1) we obtain the Wronskian of the initial curve:

$$W(r) = \rho^2\omega' \quad (2.9)$$

11 which, via (2.2), means:

$$W(r) = \rho\sqrt{1-(\rho')^2} \quad (2.10)$$

12 and then:

$$P = -\ln(\rho\sqrt{1-(\rho')^2}). \quad (2.11)$$

13 Replacing this last formula into (2.8) yields the claimed relation (2.3).  $\square$

14 **Example 2.2** The degenerate case of the characterization (2.3) is provided by the  
 15 constancy of  $P$ , equivalently, from (1.2), the constancy of  $W(r)$ . From (2.9) we have:

$$\rho^2\omega' = \text{constant} \quad (2.12)$$

1 which is exactly the Kepler second law ([2, p. 235]). This means that  $r$  is a central  
 2 conic and it is well-known that the operator (1.3) is self-adjoint (i.e.  $p = 0$ ) if and only  
 3 if  $r$  is projectively equivalent to a conic ([6, p. 220]). In particular, if  $\rho = \text{constant} =$   
 4  $R > 0$  then the condition (2.2) gives  $\omega(t) = \frac{t}{R}$  i.e. we have the well-known natural  
 5 parametrization of the circle  $\mathcal{C}(O, R)$ .  $\square$

6 **Example 2.3** We can start now directly with a naturally parametrized curve. It is  
 7 the Cornu spiral ([1, p. 54])  $r : (0, +\infty) \rightarrow \mathbb{E}^2$ :

$$r(t) = \left( \int_0^t \cos(u + \frac{u^2}{2}) du, \int_0^t \sin(u + \frac{u^2}{2}) du \right), \quad k(t) = t + 1 > 1 \quad (2.13)$$

8 for which we obtain the coefficient functions:

$$\begin{cases} P(t) = -\ln \left[ \sin(t + \frac{t^2}{2}) \int_0^t \cos(u + \frac{u^2}{2}) du - \cos(t + \frac{t^2}{2}) \int_0^t \sin(u + \frac{u^2}{2}) du \right], \\ q(t) = (t + 1) \left[ \sin(t + \frac{t^2}{2}) \int_0^t \cos(u + \frac{u^2}{2}) du - \cos(t + \frac{t^2}{2}) \int_0^t \sin(u + \frac{u^2}{2}) du \right]^{-1}. \end{cases} \quad (2.14)$$

9 Due to the very complicated computations we use the Wolfram Alpha to find the SODE  
 10 satisfies by the components of  $r$ . Unfortunately, this free software provides not a SODE  
 11 but a third order differential equation:

$$(1 + t)U''' - U'' + (1 + t)^3U' = 0. \quad (2.15)$$

12 In fact, for a naturally parametrized plane curve it is well-known to satisfy the third  
 13 order differential equation:

$$\mathcal{E}^3 : kU''' - k'U'' + k^3U' = 0 \quad (2.16)$$

14 so, the relation (2.15) is exactly a recognition of this fact. The adjoint equation to (2.16)  
 15 is:

$$\mathcal{E}_a^3 : U_a''' + \frac{k'}{k}U_a'' + \left[ 2 \left( \frac{k'}{k} \right)' + k^2 \right] U_a' + \left[ \left( \frac{k'}{k} \right)'' + 2kk' \right] U_a = 0. \quad (2.17)$$

16  $\square$

17 **Example 2.4** Trying to connect the SODE (1.1) with the third order differential  
 18 equation (2.16) we derive  $\mathcal{E}^2$  in order to obtain  $d\mathcal{E}^2 = \mathcal{E}^3$ ; then the equality of the  
 19 coefficients with (2.16) means:

$$\begin{cases} p = -\frac{k'}{k}, \\ p' + q = k^2, \\ q' = 0. \end{cases} \quad (2.18)$$

20 Therefore, both coefficients  $p, q$  are given as functions of  $k$ , which is supposed to be  
 21 strictly positive:

$$q = k^2 + \left( \frac{k'}{k} \right)' = \text{constant} = C, \quad P = \ln \frac{1}{k}. \quad (2.19)$$

1 The choice  $C = 1$  is inspired by the hypothesis of natural parameter for  $t$ ; then the above  
 2 second order non-linear differential equation in the unknown  $k$  has an implicit solution,  
 3 provided by Wolfram Alpha:

$$C + t = \int_1^{k(t)} \frac{du}{\sqrt{Cu^2 - u^4 + 2u^2 \ln u}}, \quad C \in \mathbb{R}. \quad (2.20)$$

4 We study now the same problem for the adjoint equations i.e. when  $d\mathcal{E}_a^2 = \mathcal{E}_a^3$ . The  
 5 derivative of  $\mathcal{E}_a^2$  is:

$$u_a''' - pu_a'' + (q - 2p')u_a' + (q' - p'')u_a = 0 \quad (2.21)$$

6 and this equation coincides with (2.17) if and only if:

$$p = -\frac{k'}{k}, \quad q = k^2. \quad (2.22)$$

7 It follows then that (1.1) is similar to (2.16) but as second order differential equation  
 8 and not as a third order one. Comparing (2.22) with the initial expression (1.1) of the  
 9 coefficient functions  $p, q$  it results:

$$W(r) = Ck, \quad W(r') = Ck^3, \quad C \in \mathbb{R}^*. \quad (2.23)$$

10 The supplementary hypothesis of natural parametrization for  $r$  reduces the equations  
 11 above to  $k^2 = \frac{1}{C}$  and then  $r$  is a circle and (1.1) is a self-adjoint SODE.  $\square$

12 Let us recall now some known facts concerning the transformation of parameter in  
 13 a linear SODE  $\mathcal{E} : u'' + p' + qu = 0$ . Let  $\tilde{t} = \tilde{t}(t)$  such a change of parameter; then the  
 14 new linear SODE is:

$$u_{\tilde{t}\tilde{t}} + \tilde{p}u_{\tilde{t}} + \tilde{q}u = 0, \quad p = \tilde{t}'\tilde{p} - \frac{\tilde{t}''}{\tilde{t}'}, \quad q = (\tilde{t}')^2\tilde{q}. \quad (2.24)$$

15 It follows two *relative invariants*:

$$\sqrt{q}dt = \sqrt{\tilde{q}}d\tilde{t}, \quad \left(2p + \frac{q'}{q}\right)dt = \left(2\tilde{p} + \frac{\tilde{q}'}{\tilde{q}}\right)d\tilde{t} \quad (2.25)$$

16 and therefore we have *the absolute invariant*:

$$I(\mathcal{E}) := \frac{2p + \frac{q'}{q}}{\sqrt{q}} = \frac{2pq + q'}{q^{\frac{3}{2}}}. \quad (2.26)$$

17 Hence two linear SODEs,  $\mathcal{E}$  and  $\tilde{\mathcal{E}}$ , are equivalent through a transformation  $\tilde{t} = \tilde{t}(t)$  if  
 18 and only if  $I(\mathcal{E}) = I(\tilde{\mathcal{E}})$ . For the two parts of the example 2.4 above we have:

$$I\left(\mathcal{E}\left(p = -\frac{k'}{k}, q = 1\right)\right) = 2p = -2\frac{k'}{k}, \quad I\left(\mathcal{E}\left(p = -\frac{k'}{k}, q = k^2\right)\right) = 0. \quad (2.27)$$

1 It is worth to remark that in both cases studied in the example 2.4 the first coefficient  
 2 function is  $p = -\frac{k'}{k}$ . In this case we can perform a change of parameter which reduces  
 3 the SODE  $\mathcal{E}^2$  to a simpler form. Namely, let  $\tau = \tau(t)$  be the structural angle function of  
 4  $r$  i.e.  $\tau(t) = \int_{t_0}^t k(u)du$ ; then  $r'(t) = \exp(i\tau(t))$ . For this new parameter we have:

$$r' = k \frac{dr}{d\tau}, \quad r'' = k^2 \frac{d^2r}{d\tau^2} + k' \frac{dr}{d\tau} \quad (2.28)$$

5 and then the SODE  $\mathcal{E}^2 : r'' - \frac{k'}{k}r' + q(t)r(t) = 0$  reduces to:

$$k^2(t(\tau)) \frac{d^2r}{d\tau^2}(\tau) + q(t(\tau))r(t(\tau)) = 0. \quad (2.29)$$

6 We finish this section with an example showing again the complexity of finding  $\mathcal{E}^2$   
 7 even for a simple curve.

8 **Example 2.5** The well-known catenary curve is ([4]):

$$r(t) = (\ln(t + \sqrt{1+t^2}), \sqrt{1+t^2}) \quad (2.30)$$

9 and then:

$$\begin{cases} k(t) = \frac{1}{1+t^2} \in (0, 1), \tau(t) = \arctan t \rightarrow r'(\tau) = \exp(i\tau), \\ W(r)(t) = \frac{t \ln(t + \sqrt{1+t^2})}{\sqrt{1+t^2}} - 1, \\ W(r)(\tau) = \sin \tau \ln \frac{1 + \sin \tau}{\cos \tau} - 1 = \sin \tau \ln \frac{\cos \frac{\tau}{2} + \sin \frac{\tau}{2}}{\cos \frac{\tau}{2} - \sin \frac{\tau}{2}} - 1. \end{cases} \quad (2.31)$$

10 It results:

$$\begin{cases} p(t) = -\frac{1}{t} \left[ \frac{1}{1+t^2} + \frac{\sqrt{1+t^2}}{t \ln(t + \sqrt{1+t^2}) - \sqrt{1+t^2}} \right], \\ q(t) = \frac{1}{\sqrt{1+t^2} [t \ln(t + \sqrt{1+t^2}) - \sqrt{1+t^2}]}. \end{cases} \quad (2.32)$$

11 Obviously, considered separately the functions  $x, y$  satisfy more simple SODE. For  $y$  we  
 12 have  $y' = \frac{t}{1+t^2}y$  which by derivation means:

$$y'' - \frac{t}{1+t^2}y' + \frac{t^2-1}{(1+t^2)^2}y = 0. \quad (2.33)$$

13 The adjoint SODE of this last SODE is  $y''_a + \frac{t}{1+t^2}y'_a = 0$  with the solution  $y_a(t) = x(t) =$   
 14  $\ln(t + \sqrt{1+t^2})$ .  $\square$

### 15 3 The curvature-deformation function of a linear SODE

16 Motivated by the complexity to find curvature-preserving adjoint maps, in this section  
 17 we introduce a measure of the difference of curvatures. It is worth to point out that this

1 notion works directly for a SODE (1.1), irrespective if it represents or not a given plane  
 2 curve.

3 **Definition 3.1** The *curvature-deformation function* of the linear SODE  $\mathcal{E}$  expressed  
 4 in (1.1) is the smooth function  $CD(\mathcal{E}) : I \rightarrow \mathbb{R}$ :

$$CD(\mathcal{E}) := q \exp(-P) - (q - p') \exp(P), \quad P = \int p. \quad (3.1)$$

5 It follows directly:

$$CD(\mathcal{E}) = P'' \exp(P) - 2q \sinh(P) \quad (3.2)$$

6 and then given a pair of smooth functions  $(p, \mathcal{F})$  there exists a unique  $q$  such that  
 7  $CD(\mathcal{E}) = \mathcal{F}$ , namely:

$$q = \frac{P'' \exp(P) - \mathcal{F}}{2 \sinh(P)}. \quad (3.3)$$

8 **Example 3.2** Fix the constants  $\alpha, \beta, \gamma$  and the *Gauss hypergeometric equation* with  
 9  $I = (1, +\infty)$ :

$$\mathcal{E}(\alpha, \beta, \gamma) : t(t-1)u'' + [(\alpha + \beta + 1)t - \gamma]u' + \alpha\beta u = 0. \quad (3.4)$$

10 For  $\alpha + \beta = \gamma = 1$  we obtain  $P(t) = \ln(t^2 - t)$  and finally:

$$CD(\mathcal{E})(t) = \alpha\beta \left[ \frac{1}{(t^2 - t)^2} - 1 \right] - 2 - \frac{1}{t^2 - t}. \quad (3.5)$$

11 The adjoint SODE to the particular hypergeometric SODE ( $\alpha + \beta = \gamma = 1$ ) is:

$$\mathcal{E}_a(\alpha + \beta = 1 = \gamma) : t(t-1)u_a'' - (2t-1)u_a' + \left( \alpha\beta + 2 + \frac{1}{t^2 - t} \right) u_a = 0. \quad (3.6)$$

12 The absolute invariant of our particular hypergeometric SODE is (supposing  $\alpha\beta > 0$ ):

$$I(\alpha + \beta = 1 = \gamma) = \frac{(2t-1)(t^2-t)}{\sqrt{\alpha\beta}} > 0. \quad (3.7)$$

13  $\square$

14 **Example 3.3** Let  $\mathcal{E} : u'' = \Lambda(u', u, t)$  be a general SODE. In the paper [5] is  
 15 considered a Wunschmann-type condition for it:

$$\Lambda_{ut} + \Lambda_{uu}u' + \Lambda_{uu'}\Lambda = 2\Lambda_u\Lambda_{u'}. \quad (3.8)$$

1 This condition means that on the two-dimensional manifold of solutions of  $\mathcal{E}$  (conform  
2 (1.2)) there exists a diagonal Riemannian (or semi-Riemannian) metric satisfying the  
3 Hamilton-Jacobi equation. If  $\mathcal{E}$  is a linear one:

$$\Lambda(u', u, t) := -p(t)u' - q(t)u \quad (3.9)$$

4 then the relation (3.8) means:

$$-q' = 2pq \rightarrow I(\mathcal{E}) = 0 \quad (3.10)$$

5 and then supposing  $q > 0$  it results  $P = \ln \frac{1}{\sqrt{q}}$ . We compute the curvature-deformation  
6 function in terms of  $q$ :

$$CD(\mathcal{E}) = q^{\frac{3}{2}} - q^{\frac{1}{2}} - \frac{1}{2q} \left( \frac{q'}{q} \right)'. \quad (3.11)$$

7 Then the case  $q = 1$  discussed in the previous section gives a vanishing  $CD(\mathcal{E})$ .  $\square$

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22 **Mircea Crasmareanu** Faculty of Mathematics, University "Al. I. Cuza", Iași,  
23 700506, Romania

24 E-mail: [mcrasm@uaic.ro](mailto:mcrasm@uaic.ro)

25 <http://www.math.uaic.ro/~mcrasm>

26 ORCID: 0000-0002-5230-2751