

A new generalization of Tzitzeica curves in arbitrary dimension

Dedicated to the memory of Professor Dr. Vasile Cruceanu

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Abstract We generalize the notion of a Tzitzeica curve r in the n th dimension by using the constant ratio of the α th power of the Wronskians $W(r')$ of the derivative curve r' and the β th power of the Wronskian $W(r)$ of the original curve. In this context, the powers α and β may be seen as control parameters whose interplay determine special classes of (α, β) -Tzitzeica curves. Our paper is intended to be example-oriented and, therefore, several intriguing, new families of generalized Tzitzeica curves in the n th dimension, such as the power and exponential curves, are introduced and discussed in detail. In particular, for $n = 2$, we show how the (α, β) -Tzitzeica curves are related to the second order homogeneous linear Schrödinger equation.

Keywords centro-affine invariant · Tzitzeica curve · Wronskian

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Introduction

The mathematician Gheorghe Țițeica (1873-1939) who wrote in French under the name of Georges Tzitzeica and founded the Romanian school of differential geometry was a brilliant student of Gaston Darboux. Since he introduced in 1907 and 1911, respectively, two important geometric invariants called today *Tzitzeica surfaces* [12] and *Tzitzeica curves* [13], he has been considered the pioneer of the centro-affine differential geometry. The relationship between these two invariant objects is influenced by the sign of the surface's Gaussian curvature, namely, the asymptotic lines of a Tzitzeica surface with negative Gaussian curvature are Tzitzeica curves. Although both Tzitzeica curves and surfaces are defined in the usual Euclidean space $\mathbb{E}^3 := (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ by using the metric-submanifold tools (the torsion and the Gaussian curvature, respectively), they also represent main centro-affine objects in \mathbb{R}^3 . Since they have been introduced, Tzitzeica's differential invariants have represented topics of continuous research interest, always suitable

for further generalizations (see, for instance, [1]–[5], [9]–[11], and the references within).

In this paper, we are interested in generalizing the Tzitzeica curves to an arbitrary dimension of the ambient space. Namely, let us consider a fixed open real interval $I \subseteq \mathbb{R}$ and a parametric smooth 3D curve $r : I \rightarrow \mathbb{E}^3$, $t \in I \rightarrow r(t) = (x(t), y(t), z(t))$ which is biregular. Hence, the usual Frenet–Serret apparatus provides its curvature $k = k(t)$ and torsion $\tau = \tau(t)$ functions. The curve r is called a *Tzitzeica curve* if the function $t \rightarrow Tz(r)(t) := \frac{\tau(t)}{d^2(t)}$ is constant for all $t \in I$, where $d(t)$ is the Euclidean distance from the origin $O(0, 0, 0)$ to the curve’s osculating plane at an arbitrary point $r(t)$. We will refer to $Tz(r) \in \mathbb{R} \setminus \{0\}$ as the *Tzitzeica constant* of the curve r . Next, we introduce a real function $Tz(r)$ called the *Tzitzeica function* that is defined on the set \mathcal{C}_3 of all smooth 3D curves r by

$$Tz(r) = \frac{(r', r'', r''')}{(r, r', r'')^2}, \quad r \in \mathcal{C}, \quad (0.1)$$

where (\cdot, \cdot, \cdot) denotes the mixed Euclidean product. Notice that the level sets of the Tzitzeica function are given by the Tzitzeica curves which are also centro-affine invariants. For instance, in [5], a remarkable class of cylindrical Tzitzeica curves are determined, and these represent examples of level sets for $Tz(r)$.

The starting point of the present study is the remark that the mixed product from the denominator in (0.1) is exactly *the Wronskian* $W(r)$ of the vectorial function r (see [3, p. 41]) and, therefore, the Tzitzeica function (0.1) of a 3D curve may be rewritten as

$$Tz(r) := \frac{W(r')}{[W(r)]^2}. \quad (0.2)$$

In fact, in [14], Tzitzeica himself generalized the above function to a n th-dimensional curve r in \mathbb{E}^n by

$$Tz(r) := \frac{[W(r')]^{n-1}}{[W(r)]^{n+1}}, \quad (0.3)$$

for $n \geq 2$. This general case was discussed in [9] and [10] while another interesting generalization can be found in the papers [2] and [4].

The paper is structured as follows. A generalization involving the Wronskians’ powers occurring in the ratio (0.3) is proposed in section 1. More precisely, we allow the new powers, denoted by α and β , respectively, to be arbitrary nonzero real numbers. By considering this generalization, the new ratio (1.1) is still invariant under centro-affine transformations. We also focus on the case $n = 2$ that leads to an approach involving the second order homogeneous linear Schrödinger equation. Since this is a first paper on the

proposed generalization (1.1), we give two new examples of generalized Tzitzeica curves, namely, the *power curve* (2.1) and the *exponential curve* (3.1), that are analyzed in detail in sections 2 and 3, respectively. Last section is reserved to conclusions and remarks.

This work is dedicated to the memory of Professor Dr. Vasile Cruceanu (1931-2023), a very appreciated Romanian geometer. In fact, his PhD Thesis (from which the part related to curves is the paper [6]) concerns exactly with the centro-affine geometry.

1 Generalized Tzitzeica curves

Let $\alpha, \beta \in \mathbb{R}^*$ be two fixed real numbers and $r : I \rightarrow \mathbb{E}^n$ a smooth curve. Consider the Wronskians $W(r')$ of the derivative curve r' and $W(r)$ of the original curve, respectively, and assume that $W(r) \neq 0$ and $W(r') \neq 0$ on I . Notice that, in particular, for $n = 2$, the curve r is not a line through the origin of \mathbb{R}^2 .

Definition 1.1 i) The *Tzitzeica function of (α, β) -type* associated with a smooth curve r is defined as the smooth function

$$Tz(r; \alpha, \beta) := \frac{[W(r')]^\alpha}{[W(r)]^\beta}. \quad (1.1)$$

If α or (and) β are rational numbers of type $\frac{1}{2k}$ for $k \in \mathbb{N}^*$ then we consider the absolute value of $W(r')$ or (and) $W(r)$ in the ratio above.

ii) The smooth curve r for which $Tz(r; \alpha, \beta)$ is constant is called a *Tzitzeica curve of (α, β) -type*.

The function (1.1) generalizes the original 3D Tzitzeica curves defined by (0.1). Indeed, for $\alpha = 1$, $\beta = 2$, and $n = 3$, we find the Tzitzeica constant, i.e., $Tz(r; 1, 2) = Tz(r)$. Moreover, the expression (1.1) is also defined for plane curves. The main theoretical result of this note is that our new object (1.1) belongs to the centro-affine differential geometry of \mathbb{E}^n as it is shown in the following proposition.

Proposition 1.2 *The expression $Tz(\cdot; \alpha, \beta)$ is a centro-affine invariant of the set of smooth curves in \mathbb{E}^n .*

Proof We follow closely the proof of the Theorem 3.1 in [4]. First of all we recall that a centro-affine invariant is a quantity preserved by the action of the n -special linear group $SL(n, \mathbb{R})$ on the set \mathcal{C}_n of all smooth n -dimensional curves. Hence, let $\Gamma \in SL(n, \mathbb{R})$ be a special matrix that transforms the curve r into the curve \tilde{r} , where $\tilde{r} = \Gamma \cdot r$; here r and \tilde{r} are regarded as column matrices i.e., $r, \tilde{r} \in M_{n,1}(\mathbb{R})$. Since the Wronskian $W(r)$ is linear in its arguments, it may be shown that the Wronskian of the transformed curve remains constant under a centro-affine transformation, namely,

$$W(r) = \det(\Gamma^{-1})W(\tilde{r}) = W(\tilde{r}). \quad (1.2)$$

Additionally, a relation similar to (1.2) holds for the derivative curves r' and \tilde{r}' . Consequently, the claimed conclusion follows directly from the invariance of the Wronskians $W(r)$ and $W(r')$ under the centro-affine transformation defined by Γ . \square

The case $n = 2$. Since Tzitzeica's approach starts with the 3D case, we devote the rest of this section to the case of the plane curves $r(\cdot) = (x(\cdot), y(\cdot))$. If the component functions x and y of r are linearly independent, then they are solutions to a linear second-order differential equation that may be introduced by the vanishing of a third order Wronskian, namely,

$$\begin{cases} \begin{vmatrix} x & y & \theta \\ x' & y' & \theta' \\ x'' & y'' & \theta'' \end{vmatrix} = 0 \rightarrow \theta''(t) + p_1(t)\theta'(t) + p_2(t)\theta(t) = 0, \\ p_1 := -\frac{[W(r)]'}{W(r)}, \quad p_2 := \frac{W(r')}{W(r)} = Tz(r; \alpha = \beta = 1), \quad |p_2(t)| \leq \frac{\|r'(t)\| \|r''(t)\|}{|W(r)(t)|}. \end{cases} \quad (1.3)$$

By using the *the centro-affine paramater* $s := s(t)$ defined by

$$s(t) := \lambda \int e^{(-\int p_1(t)dt)} dt + \mu = \lambda \int W(r)(u) du + \mu, \quad \lambda, \mu \in \mathbb{R}, \quad (1.4)$$

the differential equation given in (1.3) becomes the second order homogeneous linear Schrödinger equation (see (6.40) in [8, p. 195]), i.e.,

$$\frac{d^2\theta(s)}{ds^2} + \frac{1}{k_c(s)}\theta(s) = 0, \quad (1.5)$$

where the function $k_c = k_c(s)$ is called *the centro-affine curvature* of the curve r . More specifically, the curves with $k_c = \pm C^2 \neq 0$, where C is a nonzero constant, are conics centered at the origin. We do not provide a general formula for the centro-affine curvature since we treat directly two particular examples where this curvature follows directly through the corresponding equation (1.5).

2 The power curve as first general example

In this section, we study *the power curve* that is defined by

$$r(t) = (t^{a_1}, \dots, t^{a_n}) \in \mathbb{E}^n, \quad (2.1)$$

where a_1, \dots, a_n are n fixed real numbers, referred to as the curve's parameters in this paper. A straightforward computation of the Wronskians $W(r)$ and $W(r')$ associated with the curve (2.1) and the condition that their ratio is a nonzero constant yields the following result.

Proposition 2.1 *If $\alpha \neq \beta$, then (2.1) is a generalized Tzitzeica curve of (α, β) -type iff*

$$\sum_{i=1}^n a_i = \frac{n[\beta(n-1) - \alpha(n+1)]}{2(\beta - \alpha)}. \quad (2.2)$$

Note that if α and β are integers such that $\beta - \alpha = \pm 1$, then the rational number $\sum_{i=1}^n a_i$ becomes an integer. Moreover, in the classical Tzitzeica's case (0.3), since $\beta = n + 1$ and $\alpha = n - 1$, the sum $\sum_{i=1}^n a_i$ vanishes. In this latter situation, in the (a_1, \dots, a_n) -space of the curve's parameters, the relation $\sum_{i=1}^n a_i = 0$ may be regarded as a hyperplane π_n that passes through the origin of \mathbb{R}^n .

Intriguingly, the expression (2.2) has a "projective flavor" with respect to the (α, β) pair. More precisely, by expressing its right-hand side in terms of β/α , we obtain the Möbius transformation of the projective line \mathbb{P}^1 given by

$$\mathcal{T}_n : \mathbb{P}^1 \rightarrow \mathbb{P}^1, x = \left[\frac{\beta}{\alpha} \right] \rightarrow \left[\frac{n(n-1)x - n(n+1)}{2(x-1)} \right] = \left[\frac{n(n-1)}{2} - \frac{n}{x-1} \right], \quad (2.3)$$

whose associated coefficient matrix T_n belongs to the general linear group, namely,

$$T_n = \begin{pmatrix} n(n-1) & -n(n+1) \\ 2 & -2 \end{pmatrix} \in GL(2, \mathbb{R}), \text{Tr}(T_n) = n^2 - n - 2 \geq 0, \det(T_n) = 4n \geq 8. \quad (2.4)$$

In particular, for $n = 2$, it can be seen that T_2 is a trace-free matrix and, hence, it belongs to the special linear Lie algebra $sl(2, \mathbb{R})$. Moreover, the equation $T_n(x) = x$ that defines the fixed points of the transformation T_n simplifies to the quadratic equation

$$2x^2 - (2 + n^2 - n)x + (n^2 + n) = 0 \quad (2.5)$$

whose discriminant $\Delta(n) = n^4 - 2n^3 - 3n^2 - 12n + 4$ has the roots $n_1 \simeq 0.305$, $n_2 \simeq 3.644$, and $n_{3,4} \in \mathbb{C} \setminus \mathbb{R}$. Since n is natural and $n \geq 2$, we observe that the first convenient case is $n = 4$; this means that for $n \geq 4$ the homothety T_n is hyperbolic. Note that the fixed points of T_4 are positive integers, i.e., $x_1 = [\beta_1/\alpha_1] = 2$ and $x_2 = [\beta_2/\alpha_2] = 5$. The eigenvalues of T_n are $\lambda_{1,2} = (n^2 - n - 2 \pm \sqrt{\Delta(n)})/2$ which for $n = 4$ gives $\lambda_1 = 2 = x_1$ and $\lambda_2 = 8$.

2.1 The case $n = 2$

Let us relabel $a_1 = a$ and $a_2 = b$. Assume $r(t) = (t^a, t^b)$ is a generalized Tzitzeica curve of (α, β) -type. Thus, in this case, the Tzitzeica condition

(2.2) and the Tzitzeica constant of the curve r are given by

$$a + b = \frac{\beta - 3\alpha}{\beta - \alpha} \quad \text{and} \quad Tz(r; \alpha, \beta) = \frac{(ab)^\alpha}{(b - a)^{\beta - \alpha}}. \quad (2.6)$$

In particular, for the initial Tzitzeica case of $\alpha = 1$ and $\beta = 2$, we have

$$a + b = -1 \quad \text{and} \quad Tz(r; 1, 2) = \frac{ab}{b - a} = \frac{a(a + 1)}{2a + 1}. \quad (2.7)$$

Moreover, by denoting $Tz(r; 1, 2) = C$, where $C \in \mathbb{R}^*$, we obtain a quadratic equation for a whose solutions are

$$a_{\pm}(C) = \frac{2C - 1 \pm \sqrt{1 + 4C^2}}{2}. \quad (2.8)$$

The values $a_{\pm}(C)$ generate two (1, 2)-type plane Tzitzeica curves. Surprisingly, for $C = 1$, the constant a_+ is equal to the golden ratio $\Phi = (1 + \sqrt{5})/2$.

Another interesting particular case of the condition (2.6) is $a = 1$ that for $\alpha \neq \pm\beta$ implies

$$b = \frac{2\alpha}{\alpha - \beta} \quad \text{and} \quad Tz(r; \alpha, \beta) = \frac{(2\alpha)^\alpha (\alpha - \beta)^{\beta - 2\alpha}}{(\alpha + \beta)^{\beta - \alpha}}. \quad (2.9)$$

The corresponding Tzitzeica curve $r(t) = \left(t, t^{\frac{2\alpha}{\alpha - \beta}}\right)$ is, in fact, the graph of the smooth real power function

$$f : (0, +\infty) \rightarrow \mathbb{R}, \quad f(t) = t^{\frac{2\alpha}{\alpha - \beta}}. \quad (2.10)$$

Its implicit equation is given in Cartesian coordinates by

$$x^{2\alpha} y^{\beta - \alpha} = 1 \quad (2.11)$$

or in the $[X : Y : Z]$ projective coordinates as

$$X^{2\alpha} Y^{\beta - \alpha} = Z^{\alpha + \beta}. \quad (2.12)$$

At the end of this subsection, we determine the ordinary differential equation (1.3) related to the Tzitzeica curve (2.11). It can be shown that

$$W(r)(t) = \frac{\alpha + \beta}{\alpha - \beta} t^{\frac{2\alpha}{\alpha - \beta}} \quad \text{and} \quad [W(r)]'(t) = \frac{2\alpha(\alpha + \beta)}{(\alpha - \beta)^2} t^{\frac{\alpha + \beta}{\alpha - \beta}}. \quad (2.13)$$

From here, for $\beta \neq 3\alpha$, we get

$$p_1(t) = \frac{2\alpha}{\beta - \alpha} t^{-1}, \quad s(t) = t^{\frac{\beta - 3\alpha}{\beta - \alpha}}, \quad \lambda = \frac{\beta - 3\alpha}{\beta - \alpha}, \quad \text{and} \quad \mu = 0. \quad (2.14)$$

Hence, the new parametrization of the given Tzitzeica curve (2.11) is also a power curve

$$\tilde{r}(s) = \left(s^{\frac{\beta-\alpha}{\beta-3\alpha}}, s^{\frac{-2\alpha}{\beta-3\alpha}} \right), \quad W(\tilde{r}) = \frac{\alpha + \beta}{3\alpha - \beta} = \text{constant} (\neq 1) \rightarrow \tilde{p}_1 = 0 \quad (2.15)$$

whose centro-affine curvature is given in both parametrizations by

$$k_c(s) = \frac{(\beta - 3\alpha)^2}{2\alpha(\alpha - \beta)} s^2 \quad \text{and} \quad k_c(t) = \frac{(\beta - 3\alpha)^2}{2\alpha(\alpha - \beta)} t^{\frac{2(\beta-3\alpha)}{\beta-\alpha}}. \quad (2.16)$$

It is worth to remark that for a given constant $C \neq 0$ the corresponding reduced linear Schrödinger equation $\theta'' + \frac{\theta}{Cs^2} = 0$ has a variational nature being the Euler-Lagrange equation of the Lagrangian

$$L(s, \theta, \theta') = \frac{1}{2}(\theta')^2 - \frac{\theta^2}{2Cs^2}.$$

The coefficient of θ in (1.3) becomes

$$p_2(t) = \frac{2\alpha}{\alpha - \beta} t^{-2}. \quad (2.17)$$

Moreover, if $\beta = 2\alpha$, we obtain $p_1(t) = \frac{2}{t}$ and $p_2(t) = -\frac{2}{t^2}$. In this case, the ordinary differential equation (1.3) turns into the following Sturm-Liouville equation

$$(t^2\theta')' - 2\theta = 0. \quad (2.18)$$

On the other hand, the case $\beta = 3\alpha$ corresponds to the equilateral hyperbola $\tilde{r}(s) = (e^s, e^{-s})$ for which the Tzitzeica constant and centro-affine curvature are constant, namely, we have $Tz(r) = (-1/4)^\alpha$ and $k_c = -1$.

2.2 The case $n = 3$

Denote a_1, a_2 , and a_3 with a, b , and c , respectively. The Tziteica condition and the Tzitzeica constant related to r are as follows

$$\begin{cases} a + b + c = \frac{3(\beta-2\alpha)}{\beta-\alpha}, \\ Tz(r; \alpha, \beta) = \frac{(abc[(b-1)(c-1)(c-b)+(c-1)(a-1)(a-c)+(a-1)(b-1)(b-a)])^\alpha}{[bc(c-b)+ca(a-c)+ab(b-a)]^\beta}. \end{cases} \quad (2.19)$$

In particular, the classical case $\alpha = 1, \beta = 2$ corresponds to the Tziteica condition $a + b + c = 0$. A well-known example is obtained for

$$a = -1, \quad b = -2, \quad c = 3, \quad W(r)(t) = -\frac{20}{t^3}, \quad \text{and} \quad Tz(r) = -\frac{3}{10}, \quad (2.20)$$

and the associated curve belongs to the algebraic surface of equation $xyz = 1$ which is a Tzitzeica surface with positive Gaussian curvature.

2.3 Back to the general n

A way to manage the general relation (2.2) is by imposing other constraints to the parameters a_i . Being a single equation we search for the elementary arithmetic progression: $a_2 = a_1 + 1, \dots, a_n = a_1 + (n - 1)$. It results

$$a_1 = \frac{\alpha}{\alpha - \beta} \quad (2.21)$$

which for the general case must satisfies the initial conditions $W(r) \neq 0$ and $W(r') \neq 0$. More specifically, for a non-zero $W(r')$ we have the constraint

$$\frac{\alpha}{\alpha - \beta} \notin \{1, 2, \dots, n - 1\}. \quad (2.22)$$

A remarkable example of a power curve whose parameters form an arithmetic sequence is the moment curve ([7, p. 5])

$$v(t) = (1, t, \dots, t^{n-1}). \quad (2.23)$$

As aforementioned, (2.23) has $W(v') = 0$ and, hence, we seek a conformal deformation

$$r(t) = \varphi(t)(1, t, \dots, t^{n-1}) \quad (2.24)$$

of the curve by using a smooth unknown function φ . For finding concrete examples, we restrict again to the plane case $n = 2$ for which the Tzitzeica condition (1.1) with the constant $Tz(r) = C \in \mathbb{R} \setminus \{0\}$ reads as a second order differential equation

$$\varphi'' = \frac{2(\varphi')^2}{\varphi} - C \frac{1}{\alpha} \varphi^{\frac{2\beta}{\alpha} - 1}. \quad (2.25)$$

The differential equation (1.3) satisfied by the plane curve (2.24) is

$$\varphi^2(t)\theta''(t) - 2\varphi(t)\varphi'(t)\theta'(t) + [2(\varphi'(t))^2 - \varphi(t)\varphi''(t)]\theta(t) = 0. \quad (2.26)$$

2.4 A special case: $\beta = \alpha + 2$

One of the referees suggest as possible new interesting particular case for the general (1.1) the case $\beta = \alpha + 2$. This subsection deal with this case, treated directly on the example of power curve. Specifically, the general condition (2.2) becomes

$$\sum_{i=1}^n \alpha_i = \frac{n}{2}(n - \alpha - 1) \quad (2.27)$$

with the corresponding particular cases

$$\begin{cases} n = 2 \rightarrow a + b = 1 - \alpha, & Tz(r; \alpha, \beta = \alpha + 2) = \frac{(-\alpha)^\alpha}{(\alpha+1)^2} \\ n = 3 \rightarrow a + b + c = \frac{3}{2}(2 - \alpha). \end{cases} \quad (2.28)$$

The first relation above implies for the plane case ($n = 2$) the inequality $\alpha \neq -1$ and a suitable example is the power curve

$$C_\alpha : r_\alpha(t) = (t, t^{-\alpha}) \rightarrow C_\alpha : y = x^{-\alpha} \quad (2.29)$$

having the curvature:

$$k_\alpha(t) = \frac{\alpha(\alpha + 1)t^{-(\alpha+2)}}{[1 + \alpha^2 t^{-2(\alpha+1)}]^{\frac{3}{2}}}. \quad (2.30)$$

The SODE (1.3) satisfied by C_α is

$$\theta''(t) + \alpha \left[\frac{\theta'(t)}{t} - \frac{\theta(t)}{t^2} \right] = 0. \quad (2.31)$$

3 The exponential curve as second general example

In the previous subsection 2.1, we have obtained an exponential parametrization of the unit equilateral hyperbola, and that result led us to expand our study to a new curve called the *exponential curve* defined as

$$r(t) = (e^{a_1 t}, \dots, e^{a_n t}) \in \mathbb{E}^n, \quad (3.1)$$

where a_1, \dots, a_n are fixed nonzero real numbers. The main result follows again directly.

Proposition 3.1 *The exponential curve (3.1) has the Wronskian $W(r) \neq 0$ only for distinct values of the real parameters a_i , $i = 1, \dots, n$. In this case, r is a Tzitzeica curve if and only if $\alpha = \beta$, and, moreover, the Tzitzeica constant is given by:*

$$Tz(r; \alpha, \alpha) = (a_1 a_2 \dots a_n)^\alpha = \left(\frac{k \|r'\|^2}{d(O; r)} \right)^\alpha. \quad (3.2)$$

Here, the last ratio works for the dimension $n = 2$: k is the curvature while $d(O; r)$ is the distance from the origin $O(0, 0)$ to the tangent line to r .

Remark 3.2 In the (a_1, \dots, a_n) -space of curve's parameters, the hypersurface H^{n-1} defined by $a_1 \dots a_n = 1$ is exactly a Tzitzeica hypersurface. The Tzitzeica constant related to H^{n-1} , denoted here by $Tz(H^{n-1})$, has the value $Tz(H^{n-1}) = n^{-n}$. □

Recall that for $n = 2$, we use the notation $a_1 = a$ and $a_2 = b$. Since

$$W(r)(t) = (b-a)e^{(a+b)t}, \quad [W(r)]'(t) = (b^2 - a^2)e^{(a+b)t}, \quad p_1 = -(a+b), \quad \text{and } p_2 = ab, \quad (3.3)$$

in order to find the centro-affine parameter, we assume $a + b \neq 0$. Therefore, we have

$$s(t) = e^{(a+b)t}, \quad \lambda = \frac{a+b}{a-b} (\neq 1), \quad \text{and } \mu = 1. \quad (3.4)$$

From here, we obtain the following power parametrization

$$\begin{cases} \tilde{r}(s) = \left(s^{\frac{a}{a+b}}, s^{\frac{b}{a+b}} \right), & W(\tilde{r}) = \frac{b-a}{b+a} = \text{constant} (\neq 1) \rightarrow \tilde{p}_1 = 0, \\ W(\tilde{r}')(s) = \frac{ab(b-a)}{(a+b)^3 s^2}, & k_c(s) = -\frac{(a+b)^2 s^2}{ab}. \end{cases} \quad (3.5)$$

The implicit equation of the curve \tilde{r} is given by

$$\tilde{r} : x^b = y^a. \quad (3.6)$$

The equilateral hyperbola $r(t) = (e^{at}, e^{-at})$ has its associated Wronskians $W(r)$ and $W(r')$ constant, namely, $W(r) = -2a$ and $W(r') = 2a^3$. The centro-affine curvature of the curve r is equal to $k_c = -\frac{1}{a^2} < 0$.

4 Conclusions

The current paper generalizes the nonlinear differential equation of the Tzitzeica curves by introducing the Tzitzeica function and Tzitzeica curves of type (α, β) . In order to find a geometrical interpretation of the Tzitzeica function (1.1), recall that the curvature k and the torsion τ of a smooth, regular curve r in 3D are expressed as

$$k = \frac{\|r' \times r''\|}{\|r'\|^3} \quad \text{and} \quad \tau = \frac{W(r')}{\|r' \times r''\|^2}. \quad (4.1)$$

Besides these functions, let us consider the distance from the origin $O(0, 0)$ to the osculating plane at an arbitrary point of the curve r that is given by

$$d = \frac{|W(r)|}{\|r' \times r''\|}. \quad (4.2)$$

By using (4.1) and (4.2), the expression (1.1) may be rewritten in the form

$$Tz(r; \alpha, \beta) = \frac{\tau^\alpha k^{2\alpha-\beta} \|r'\|^{6\alpha-3\beta}}{(\pm d)^\beta}. \quad (4.3)$$

Therefore, the Tzitzeica function $Tz(r; \alpha, \beta)$ associated with a curve r depends on its speed, torsion, τ , curvature k , and the distance d defined in (4.2). For $\alpha = 1$ and $\beta = 2\gamma$, where γ is a rational number, this simplifies to the generalization

$$Tz(r; 1, 2\gamma) = \frac{\tau}{d^{2\gamma} \|r' \times r''\|^{2\gamma-2}} \quad (4.4)$$

of $Tz(r)$ given by (0.2) that has been recently proposed in the paper [4]; in particular for $\gamma = 1$ we have exactly the function (0.1), hence the classical Tzitzeica curves. We are confident that this generalization will lead to interesting explicit (α, β) -type Tzitzeica curves which may consequently

be used in the context of the theory of skew curves and its applications to various research areas, such as computer graphics, image processing, and animation.

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