

Flow-selfdual curves in a geometric surface

Mircea Crasmareanu

Faculty of Mathematics

University "Al. I. Cuza"

Iasi, 700506

Romania

ORCID ID: 0000-0002-5230-2751

mcrasm@uaic.ro

Abstract. For a natural parametrization of a curve γ in an orientable two-dimensional Riemannian manifold, we compare two differential equations associated to γ . The main tool of our study is the geodesic curvature k of γ and when these equations coincide we call γ as being flow-selfdual since the second equation corresponds to the flow-curvature k_f of γ in the same manner as the first equation involves k . We obtain that these curves have a constant geodesic curvature and then we discuss four examples. Also, we generalize this type of differential equations to vector fields on Riemannian manifolds of arbitrary dimension.

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1. Flow-selfdual curves and tangential vector fields

The framework of this study is a *geometric surface* i.e. ([3]) a smooth, orientable two-dimensional Riemannian manifold (M^2, g) . Being orientable M supports an almost complex structure J ; in fact J is integrable and for an arbitrary point $p \in M$ we consider $J_p : T_p M \rightarrow T_p M$ as being the multiplication with the complex unit $i \in \mathbb{C}$. Let ∇ be the Levi-Civita connection of g .

Fix also a smooth curve $\gamma : I \subseteq \mathbb{R} \rightarrow M$ which we suppose to be *regular*: $\gamma'(t) \in T_{\gamma(t)} M \setminus \{0\}$. Let $\mathfrak{X}(\gamma)$ be the $C^\infty(I)$ -module of vector fields along γ i.e. smooth maps $X : I \rightarrow TM$ with $X(t) \in T_{\gamma(t)} M$ for all $t \in I$. It follows the *unit tangent vector field* $T \in \mathfrak{X}(\gamma)$ with:

$$T(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|} \quad (1.1)$$

where $\|\cdot\|$ denotes the norm induced by g on the tangent spaces. Therefore, the *Frenet frame* of γ is $\mathcal{F} := \begin{pmatrix} T \\ N := J(T) \end{pmatrix} \in \mathfrak{X}(\gamma) \times \mathfrak{X}(\gamma)$.

The Riemannian geometry of γ is described by its *geodesic curvature* $k : I \rightarrow \mathbb{R}$ provided by the *Frenet equations*:

$$\nabla_{T(t)}\mathcal{F}(t) = \begin{pmatrix} 0 & k(t) \\ -k(t) & 0 \end{pmatrix} \mathcal{F}(t) = k(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{F}(t) \quad (1.2)$$

which means:

$$k(t) := \frac{g(\nabla_{\gamma'(t)}T(t), N(t))}{\|\gamma'(t)\|} = \frac{g(\nabla_{\gamma'(t)}\gamma'(t), J(\gamma'(t)))}{\|\gamma'(t)\|^3}. \quad (1.3)$$

Recall also the pair (g, J) yields the *symplectic form* $\Omega(\cdot, \cdot) := g(\cdot, J\cdot)$ and whence:

$$k(t) := \frac{\Omega(\nabla_{\gamma'(t)}\gamma'(t), \gamma'(t))}{\|\gamma'(t)\|^3}. \quad (1.4)$$

The starting point of this short note is the remark that under the hypothesis of γ being parametrized by arc-length the second covariant derivative applied to the Frenet equations yields the following differential equation:

$$\mathcal{E} : (\nabla_{\gamma'}\nabla_{\gamma'}\gamma')(t) - \frac{k'(t)}{k(t)}(\nabla_{\gamma'}\gamma')(t) + k^2(t)\gamma'(t) = 0. \quad (1.5)$$

The pair (g, J) being a Kähler structure (since $\dim M = 2$) it follows that ∇ commutes with N and then N satisfies the same differential equation. For curves parametrized by arc-length the vector field $\nabla_{\gamma'}\gamma'$ is called *the curvature vector field* of γ .

In the very recent paper [3] we introduce a modification of the curvature k called *flow-curvature* and denoted k^f . For a general parametrization of γ it holds:

$$k^f(t) := k(t) - \frac{1}{\|\gamma'(t)\|} < k(t). \quad (1.6)$$

Since k^f is obtained exactly in the same manner as k i.e. through the Frenet equation of the flow-frame:

$$\mathcal{F}^f(t) := \begin{pmatrix} E_1^f \\ E_2^f \end{pmatrix} (t) = \text{Rotation}(t)\mathcal{F}(t) = \begin{pmatrix} \cos tT(t) - \sin tN(t) \\ \sin tT(t) + \cos tN(t) \end{pmatrix} \quad (1.7)$$

it follows a second differential equation of third order satisfied by non-flow-flat curves:

$$\mathcal{E}^f : (\nabla_{\gamma'}\nabla_{\gamma'}E_1^f)(t) - \frac{(k^f)'(t)}{k^f(t)}(\nabla_{\gamma'}E_1^f)(t) + (k^f)^2(t)E_1^f(t) = 0. \quad (1.8)$$

It is natural to connect the differential equations \mathcal{E} and \mathcal{E}^f and this leads to our new type of curves:

Definition 1.1 The non-flow-flat curve γ , parametrized by arc-length, is called *flow-selfdual* if it satisfies also the differential equation:

$$\mathcal{E}_f : (\nabla_{\gamma'} \nabla_{\gamma'} \gamma')(t) - \frac{(k^f)'(t)}{k^f(t)} (\nabla_{\gamma'} \gamma')(t) + (k^f)^2(t) \gamma'(t) = 0. \quad (1.9)$$

Our main theoretical result is the following:

Proposition 1.2 *The non-flow-flat curve γ is a flow-selfdual one if and only if $k = \frac{1}{2} = -k^f$ which means that all four unit vector fields γ', N, E_1^f, E_2^f satisfy the same differential equation of Schrödinger type:*

$$\mathcal{E} = \mathcal{E}^f = \mathcal{E}_f : (\nabla_{\gamma'} \nabla_{\gamma'} U)(t) + \frac{1}{4} U(t) = 0, \quad U \in \mathfrak{X}(\gamma). \quad (1.10)$$

Proof By comparing \mathcal{E} and \mathcal{E}_f it follows:

$$\frac{k'(t)}{k(t)(k(t) - 1)} (\nabla_{\gamma'} \gamma')(t) + (2k(t) - 1) \gamma'(t) = 0. \quad (1.11)$$

Due to the unit speed parametrization of γ the vector fields $\nabla_{\gamma'} \gamma'$ and γ' are orthogonal and then $2k - 1 = 0$. \square

Remarks 1.3 1) Let (Γ_{ij}^k) denote the Christoffel symbols of the metric g in a local chart of M in which $\gamma(t) = (x^i(t))$, $1 \leq i \leq 2$. Then the differential equation (1.10) becomes a scalar third-order one for a fixed $k \in \{1, 2\}$:

$$\frac{d}{dt} \left[\ddot{x}^k(t) + \Gamma_{ij}^k(\gamma(t)) \dot{x}^i(t) \dot{x}^j(t) \right] + \Gamma_{ij}^k(\gamma(t)) \dot{x}^i(t) \left[\ddot{x}^j(t) + \Gamma_{ab}^j(\gamma(t)) \dot{x}^a(t) \dot{x}^b(t) \right] + \frac{\dot{x}^k(t)}{4} = 0. \quad (1.12)$$

2) In the same paper [3] the flow-frame is generalized with an arbitrary (smooth) angle function $\Omega = \Omega(t)$ obtaining the Ω -curvature:

$$k^\Omega(t) := k(t) - \frac{\Omega'(t)}{\|\gamma'(t)\|}. \quad (1.13)$$

Hence, an arc-length parametrized curve with $k^\Omega \neq 0$ will be called Ω -flow-selfdual if the differential equation (1.9) holds with k^f replaced by Ω . The characterization of the proposition 1.2 reads now:

$$k(t) = \frac{\Omega'(t)}{2} = -k^\Omega(t). \quad (1.14)$$

\square

A second suitable generalization of our notion works at the level of vector fields $\xi \in \mathfrak{X}(M)$ = the Lie algebra of vector fields on M . Fix a unit ξ ; then we call ξ as being a *tangential vector field* if there exists a strictly positive $k \in C^\infty(M)$ (which we call *the curvature* of ξ) such that:

$$\nabla_\xi \nabla_\xi \xi - \xi(\ln k) \nabla_\xi \xi + k^2 \xi = 0. \quad (1.15)$$

Making the g -product of the left-hand-side term above with ξ gives, as is expected, that:

$$\|\nabla_\xi \xi\| = k > 0. \quad (1.16)$$

We remark that is not necessary to work in the initial dimension two. An example of tangential vector field is provided within the theory of *torse-forming vector fields*. Recall, after [2], that a fixed $V \in \mathfrak{X}(M)$ is called *torse-forming* if for all $X \in \mathfrak{X}(M)$ we have:

$$\nabla_X V = fX + \omega(X)V \quad (1.17)$$

for a smooth function $f \in C^\infty(M)$ and a 1-form $\omega \in \Omega^1(M)$. Now, suppose that $\nabla_\xi \xi$ is a *torse-forming* vector field with the data:

$$f = -k^2, \quad \omega = d(\ln k) \quad (1.18)$$

for a given strictly positive function k . It follows:

$$\nabla_X \nabla_\xi \xi - X(\ln k) \nabla_\xi \xi + k^2 X = 0 \quad (1.19)$$

and then for $X = \xi$ it results the definition (1.15).

Recall also that an important tool in dynamics on curves is the Fermi-Walker derivative, which is the map ([4]) $\nabla_\gamma^{FW} : \mathfrak{X}(\gamma) \rightarrow \mathfrak{X}(\gamma)$:

$$\nabla_\gamma^{FW}(X) := \frac{d}{dt}X + \|r'(\cdot)\|k[\langle X, N \rangle T - \langle X, T \rangle N] \rightarrow \nabla_\gamma^{FW}(T) = \nabla_\gamma^{FW}(N) = 0. \quad (1.20)$$

Hence, we generalize this derivative as follows: the Fermi-Walker derivative generated by a tangential vector field ξ is the map $\nabla^\xi : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ given by:

$$\nabla^\xi(X) := \nabla_\xi X + g(X, \nabla_\xi \xi)\xi - g(X, \xi)\nabla_\xi \xi. \quad (1.21)$$

From the equation (1.14) it results that ξ and $\nabla_\xi \xi$ are eigenvector fields of ∇^ξ :

$$\nabla^\xi(\xi) = 0, \quad \nabla^\xi(\nabla_\xi \xi) = \xi(\ln k)\nabla_\xi \xi. \quad (1.22)$$

2. Examples of flow-selfdual curves

Example 2.1 As is expected the plane Euclidean geometry $\mathbb{E}^2 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle_{can})$ is the simplest case. The circle $\mathcal{C}(O, R = 2)$ is the "generic" Euclidean flow-orthogonal curve; it has the arc-length parametrization and Frenet data:

$$\gamma(t) = 2 \left(\cos \frac{t}{2}, \sin \frac{t}{2} \right), \quad \gamma'(t) = \left(-\sin \frac{t}{2}, \cos \frac{t}{2} \right), \quad N(t) = -\exp \left(i \frac{t}{2} \right) = -\frac{1}{2}\gamma(t) \quad (2.1)$$

and then the flow-frame is:

$$E_1^f(t) = \left(\sin \frac{t}{2}, \cos \frac{t}{2} \right), \quad E_2^f(t) = \left(-\cos \frac{t}{2}, \sin \frac{t}{2} \right) = \overline{N(t)} \quad (2.2)$$

where we use the complex conjugate; the ordinary differential equation (1.10) is: $u'' + \frac{1}{4}u = 0$. An associated interesting problem is if there exists a Riemannian metric on (an open subset of) \mathbb{R}^2 having as geodesics the Euclidean circles; for the case of Finslerian metric this problem is already solved in [1]. \square

Example 2.2 Fix (M, g) a rotationally symmetric surface i.e., conform [6], M is the product $\mathbb{S}^1 \times I$ with \mathbb{S}^1 the Euclidean unit circle and $I \subseteq \mathbb{R}$, endowed with the warped product metric:

$$g = dr^2 + f(r)^2 d\varphi^2, \quad r \in I, \quad \varphi \in \mathbb{S}^1. \quad (2.3)$$

This surface is oriented by the 2-form $dr \wedge d\varphi$ and then:

$$J \left(\frac{\partial}{\partial r} \right) = \frac{1}{f(r)} \frac{\partial}{\partial \varphi}, \quad J \left(\frac{\partial}{\partial \varphi} \right) = -f(r) \frac{\partial}{\partial r}. \quad (2.4)$$

Fix now the curve $\gamma(t) := (r(t), \varphi(t))$ parameterized by the arc-length t . Let $\sigma = \sigma(t)$ be the *structural angle* of γ i.e. the oriented angle between $\frac{\partial}{\partial r}$ and T . It follows the Frenet frame:

$$T(t) = \cos \sigma(t) \frac{\partial}{\partial r} \Big|_t + \frac{\sin \sigma(t)}{f(r(t))} \frac{\partial}{\partial \varphi} \Big|_t, \quad N(t) = -\sin \sigma(t) \frac{\partial}{\partial r} \Big|_t + \frac{\cos \sigma(t)}{f(r(t))} \frac{\partial}{\partial \varphi} \Big|_t. \quad (2.5)$$

The first derivative of T is then:

$$(\nabla_{\gamma'} \gamma')(t) = \left(\sigma'(t) + \frac{f'(r)}{f(r)}(t) \sin \sigma(t) \right) N(t) \quad (2.6)$$

which provides the expression of the geodesic curvature for γ :

$$k(t) = \sigma'(t) + \frac{f'(r)}{f(r)}(t) \sin \sigma(t). \quad (2.7)$$

The Proposition 1.1 of the cited paper [6] (or [7, p. 89]) offers a conservation law along γ , which for our constant $k = \frac{1}{2}$ reads as follows:

Proposition 2.3 *The smooth function:*

$$t \in [0, L(\gamma)) \rightarrow \mathcal{F}(t) := f(r(t)) \sin \sigma(t) - \frac{1}{2} \int_{r(0)}^{r(t)} f(\xi) d\xi \quad (2.8)$$

is constant along a given flow-selfdual curve γ .

The Euclidean plane geometry means $f(r) = r$ and the circle $\mathcal{C}(O, R > 0)$ gives $r = \text{constant} = R$, $\varphi(t) = \frac{t}{R}$, $t \in [0, 2\pi R]$ and $\sigma = \text{constant} = \frac{\pi}{2}$. \square

Example 2.4 For the hyperbolic plane geometry we use the Poincaré model of [5, p. 103]: $\mathbb{H}^2 := \left(\mathbb{R}_{y>0}^2; g = \frac{1}{y^2}(dx^2 + dy^2) \right)$. Fix a curve $\gamma : t \in [0, +\infty) \rightarrow (x(t), y(t)) \in \mathbb{H}^2$ parametrized by arc-length. With the computation of the

geodesic curvature from the cited book it follows that γ is a flow-selfdual curve if and only if the following differential system is satisfied:

$$[x'(t)]^2 + [y'(t)]^2 = [y(t)]^2, \quad x'(t) \left(\frac{1}{y(t)} + \frac{y''(t)}{y(t)^2} \right) - x''(t) \frac{y'(t)}{y(t)^2} = \frac{1}{2}. \quad (2.9)$$

A straightforward computation gives a single second order differential equation, which is written in a more simple form as:

$$\ddot{y} - \frac{2}{y}\dot{y} + y = \frac{\sqrt{y^2 - \dot{y}^2}}{2} \quad (2.10)$$

Unfortunately, being a nonlinear differential equation we cannot solve explicitly. In fact, we know that the types of hyperbolic curves with constant geodesic curvature k are as follows: a) circles, for $k > 1$; b) horocycles, with $k = 1$; c) equidistant curves (i.e. curves of finite distance from a hyperbolic geodesic), for $k \in (0, 1)$. Hence, the hyperbolic flow-selfdual curves are a family of equidistant curves. \square

Example 2.5 Let γ be a arc-length parametrized curve in the unit sphere $\mathbb{S}^2 := (S^2 \subset \mathbb{R}^3, g = (\langle \cdot, \cdot \rangle_{can})|_{S^2})$. Its usual Frenet curvature and torsion as space curve are $k^F > 0$ and τ^F . In fact, from the relationship:

$$k^F = \sqrt{k^2 + 1} \geq 1 \quad (2.11)$$

it follows that a flow-selfdual curve on \mathbb{S}^2 have a constant Frenet curvature $k^F = \frac{\sqrt{5}}{2}$. As concrete example we have the horizontal circle:

$$\gamma(t) = \frac{2}{\sqrt{5}} \left(\cos \left(\frac{\sqrt{5}}{2} t \right), \sin \left(\frac{\sqrt{5}}{2} t \right), \frac{1}{2} \right), t \in \mathbb{R}, \quad \tau^F \equiv 0. \quad (2.12)$$

More generally, recall that for the given arc-length parametrized curve γ on the regular surface $S \subset \mathbb{R}^3$ its geodesic curvature satisfies:

$$k = k^F \sin \theta \quad (2.13)$$

with θ the oriented angle between the normal N_γ of the curve and the normal N_S of S . For a flow-selfdual curve on $S = S^2$ it results the angle θ provided by:

$$\sin \theta = \frac{1}{\sqrt{5}}, \quad \cos \theta = \pm \frac{2}{\sqrt{5}}. \quad (2.14)$$

\square

3. Conclusions

This note concerns with a particular class of curves in an orientable geometric surface (M^2, g) . The curves in this class have a constant geodesic curvature, and hence they are remarkable objects for the differential geometry of the given pair (M^2, g) . Four examples illustrate the significance of these curves in some important geometries. From a dynamical point of view we generalise the Fermi-Walker derivative and we hope this operator to become more suitable in some future works.

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