

Article

CR-Selfdual Cubic Curves

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Abstract: We introduce a special class of cubic curves whose defining parameter satisfies a linear or quadratic equation provided by the values of a cross ratio. There are only seven such cubics and several properties of the real cubics in this class (some of them being elliptic curves) are discussed. Using the Möbius transformation, we extend this self-duality and obtain new families of remarkable complex cubics. In addition, we study (from the differential geometric viewpoint) the surface parameterized by all real cubic curves and we derive its curvature functions. As a by-product, we find a new classification of real Möbius transformations and some estimates for the number of vertices of real cubic curves.

Keywords: cubic curve; cross ratio; CR-selfdual cubic; elliptic curve; J -invariant; vertices of cubic curves; classification of Möbius transformations

MSC: 53A04; 53A05; 14H52

1. Introduction

The setting of this paper is provided by Chapter II of the following book [1], devoted to the theory of cubic curves. Let $E \subset \mathbb{C}\mathbb{P}_2$ be a fixed (smooth) cubic and let $E = E(\lambda)$ be its normal form (a.k.a. Legendre form [2]):

$$E(\lambda) : y^2 = x(x-1)(x-\lambda), \quad (1)$$

where λ is a complex number different from 0 and 1.

Denote by $CR = CR(\lambda)$ the set

$$C(ross)R(atio) := \left\{ \lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1} \right\}. \quad (2)$$

A main (classification) result of this theory appears in [1], on page 53: if $E(\lambda)$ and $E(\lambda')$ are cubic curves which are isomorphic as complex analytic manifolds, then the set of branch-point cross ratios for $E(\lambda)$ must equal the set for $E(\lambda')$, i.e., $\lambda' \in CR(\lambda)$. (The relation between λ and λ' is an equivalence relation on $\mathbb{C} \setminus \{0, 1\}$; cf. [1], p. 53).

This result allows us to introduce a particular (and finite) class of cubic curves through the following:

Definition 1. The cubic curve $E(\lambda)$ is called CR-selfdual if its cardinal $|CR(\lambda)| < 6$, and CR-generic otherwise.



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The first aim of the present work is to obtain and study all the CR -selfdual cubic curves. Fifteen algebraic equations (not all distinct) provide these curves; we also obtain as roots, formally, the values 0 and 1 (for which some previous ratios are undefined).

Understanding the properties of elliptic curves is fundamental, both theoretically and in applications (elliptic curve cryptography, integer factorization). There are many books that address the slightly broader topic of cubic curves, particularly through methods from topology, algebraic geometry, number theory, analytic geometry, and complex analysis [1–6]. Essential historical data and further references may be found in the following expository paper [7].

In order to look for finer classification results, we suggest a differential geometric approach [8]. The main idea is to associate new metric invariants, possibly related to curvature, with the elliptic curves in a canonical way, and to use the machinery of Riemannian geometry in order to extract information from these invariants.

In Section 2, we offer a case-by-case study of the CR -selfdual complex cubic curves. In addition to the classical tools of algebraic geometry, we use methods of differential geometry such as curvature functions. In particular, we consider a surface parametrized by all the real cubic curves (in Legendre form) and we calculate its Gaussian and its mean curvature functions; this new differential object contains much information about the whole set of real elliptic curves, which calls for “extraction procedures” with affine, metric, and/or differential tools.

Theorem 1 gives a classification of non-singular complex cubic curves $E(\lambda)$ with respect to the cardinal of their $CR(\lambda)$ set. The “generic” ones have $|CR(\lambda)| = 6$; the five “exceptional” ones are already well known in the algebraic theory of complex cubic curves, where they distinguish themselves in numerous other contexts.

In Section 3, we investigate some algebraic properties for the “exceptional” cubic curves: $E(-1)$, $E(\frac{1}{2})$, $E(2)$, $E(\frac{1+i\sqrt{3}}{2})$, and $E(\frac{1-i\sqrt{3}}{2})$ (non-singular); $E(0)$ and $E(1)$ (singular).

In Section 4, we begin a refinement of the preceding theory, in which we “perturb” notions and properties by means of a fixed arbitrary complex function f . We define an (f -dependent) weaker notion of “fixed point” and of “ CR -like selfduality”.

For the particular case when f is a Möbius transformation, we prove two characterizing results (Proposition 1 and Proposition 2). As a by-product, we explore numerically the following: the case when f and the j -invariant commute; and a plane curve parameterized by “all” the (real) j -invariants and a surface generated in a similar way. Theorem 2 provides a new classification of real Möbius transformations, expressed by the behavior of the J -invariant.

In Section 5, we derive new invariant surfaces that are related to the curvature of real cubic curves. As a by-product, we prove an estimation for the number of vertices admitted by a real cubic curve and, in particular, by the CR -selfdual ones (Proposition 4). This estimation allows an improved conjecture concerning this number, compared with a previous one in [8].

Conventions. All geometric objects are supposed to be smooth and acting on their maximal definition domain, unless otherwise stated. By “cubics”, we mean real or complex “cubic curves”. All the figures are original and were created using Maple.

2. The CR -Selfdual Cubic Curves

Let λ be a fixed complex number, different from 0 and 1. We make a case-by-case study when the cubic curve $E(\lambda)$ is CR -selfdual. So, we must characterize the property of the set $CR(\lambda)$ having a cardinality of less than six. Using Formula (2), we deduce that (at least) two of the six elements from the set $CR(\lambda)$ must coincide. There are (at least) $C_6^2 = 15$

possible cases; after studying them, we will see that it is no longer necessary to consider the other cases (when three or more elements of the set $CR(\lambda)$ coincide).

Case I. Suppose $\lambda = \frac{1}{\lambda}$. We obtain $\lambda = \pm 1$, corresponding to two cubic curves:

$$E(-1) : y^2 = x(x^2 - 1), \quad E(1) : y^2 = x(x - 1)^2. \tag{3}$$

(Remember that we supposed $\lambda \neq 0$ and $\lambda \neq 1$; so, we consider the cubic $E(1)$ only for completeness.)

Recall, after ([6], p. 192), that a cross ratio with the value -1 is called *harmonic*. Hence, we call $E(-1)$ the *harmonic cubic curve*. We have $CR(-1) = \{-1, \frac{1}{2}, 2\}$. We denote $x = u + iv$ and $y = t + iw$. In coordinates (u, v, w) , the cubic $E(-1)$ from (3) is the (real) algebraic surface given by the implicit equation

$$4w^4 + 4uw^2(-3v^2 + u^2 - 1) = v^2(3u^2 - v^2 - 1)^2$$

and is (partially) represented in Figure 1.

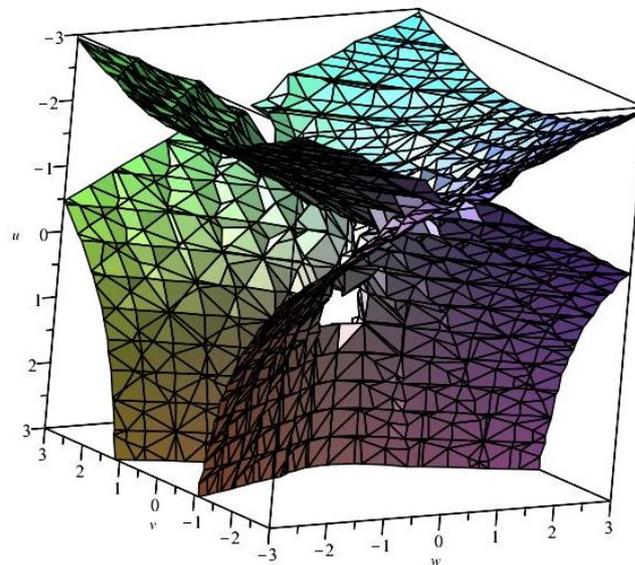


Figure 1. Picture of the complex cubic $E(-1)$ from (3).

By analogy, the (complex) cubic $E(1)$ is given by the implicit equation

$$4w^4 + 4uw^2[(u - 1)^2 - v^2 - 2v^2(u - 1)] = v^2[2u(u - 1) + (u - 1)^2 - v^2]^2$$

and is (partially) represented in Figure 2.

We can restrain the complex variables x and y to be real ones and we denote $z := x + iy$, $i = \sqrt{-1}$; in this case, the equation of the (real) harmonic cubic curve is written (in complex coordinates) as

$$E(-1) : 2(z - \bar{z})^2 = (z + \bar{z})(z + \bar{z} + 2)(2 - z - \bar{z}) \tag{4}$$

and is represented in Figure 3. Analogously, the real cubic curve $E(1)$ is represented in Figure 4.

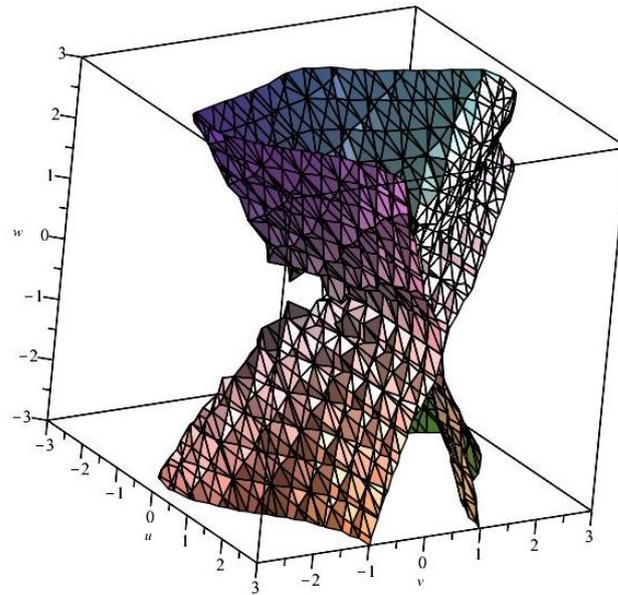


Figure 2. Picture of the complex cubic $E(1)$ from (3).

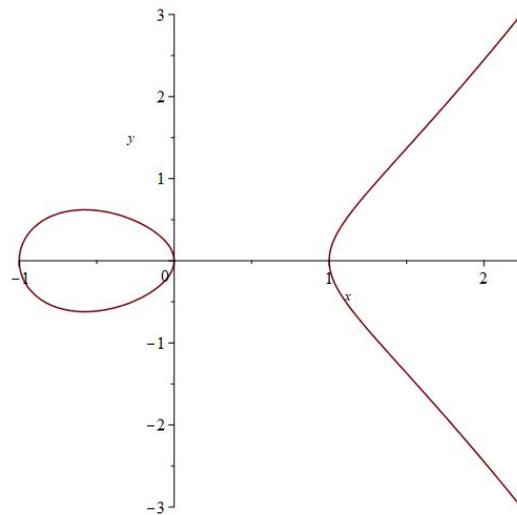


Figure 3. Picture of the real cubic $E(-1)$ from (4).

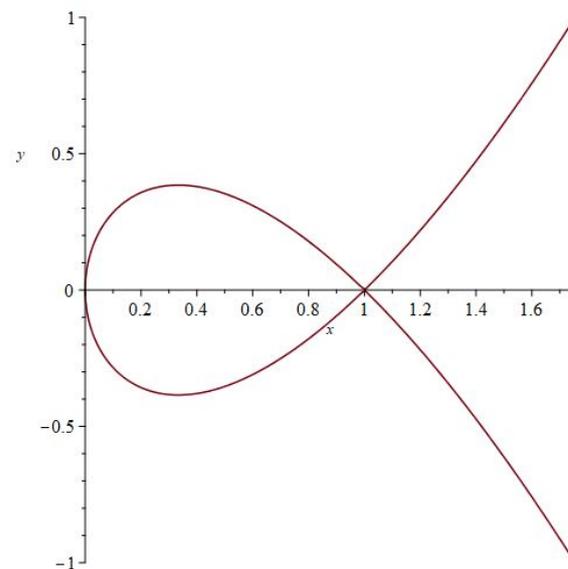


Figure 4. Picture of the real cubic $E(1)$.

Case II. Suppose $\lambda = 1 - \lambda$. We obtain the cubic curve

$$E\left(\frac{1}{2}\right) : y^2 = x(x-1)\left(x - \frac{1}{2}\right) \tag{5}$$

and $CR(-1) = CR(2) = CR(\frac{1}{2}) = \{-1, \frac{1}{2}, 2\}$. The complex cubic $E(\frac{1}{2})$ is given by the implicit equation

$$16w^4 + 16w^2[(u^2 - v^2 - u)(u - \frac{1}{2}) - v^2(2u - 1)] = v^2[2(u^2 - v^2 - u) + (2u - 1)^2]^2$$

and is (partially) represented in Figure 5.

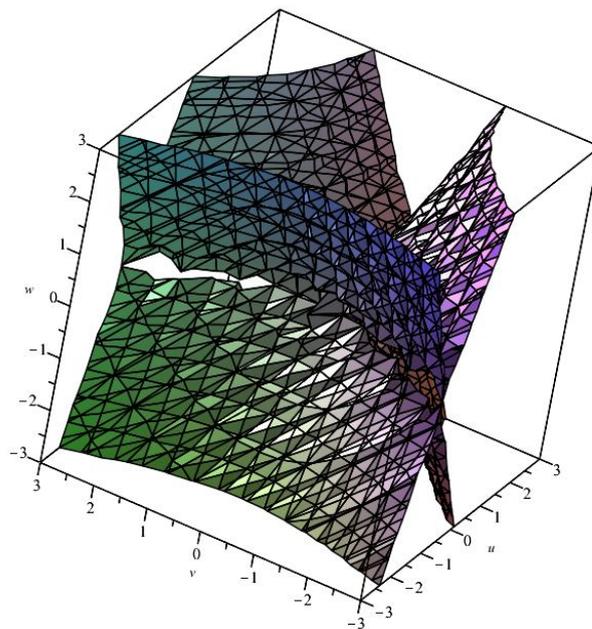


Figure 5. Picture of the complex cubic $E(\frac{1}{2})$ from (5).

If we restrain the complex variables x and y to be real ones, we denote $z := x + iy$, $i = \sqrt{-1}$; in this case, the equation of the real cubic curve $E(\frac{1}{2})$ is

$$2(z - \bar{z})^2 = (z + \bar{z})(z + \bar{z} - 1)(2 - z - \bar{z}) \tag{6}$$

and is represented in Figure 6.

Case III. Suppose $\lambda = \frac{1}{1-\lambda}$. We obtain as solutions $\lambda_{\pm} = \frac{1 \pm i\sqrt{3}}{2}$, the (non-real) third-order roots of -1 ; in fact, $\lambda_+ = e^{\frac{\pi i}{3}}$. The corresponding cubics are called *equianharmonic* and $CR(\frac{1 \pm i\sqrt{3}}{2}) = \{\frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2}\}$.

For $\epsilon = \pm 1$, the complex cubic $E(\frac{1+\epsilon\sqrt{3}i}{2})$ is given by the implicit equation

$$\begin{aligned} 16w^4 + 16w^2[(u^2 - v^2 - u)(u - \frac{1}{2}) - v(2u - 1)(v - \frac{\epsilon\sqrt{3}}{2})] = \\ = [(u^2 - v^2 - u)(2v - \epsilon\sqrt{3}) + v(2u - 1)^2]^2 \end{aligned}$$

and is (partially) represented in Figures 7 and 8.

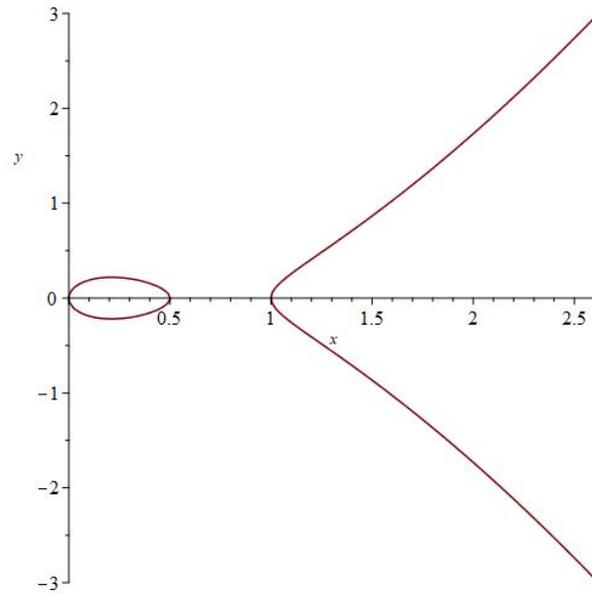


Figure 6. Picture of the real cubic $E(\frac{1}{2})$ from (6).

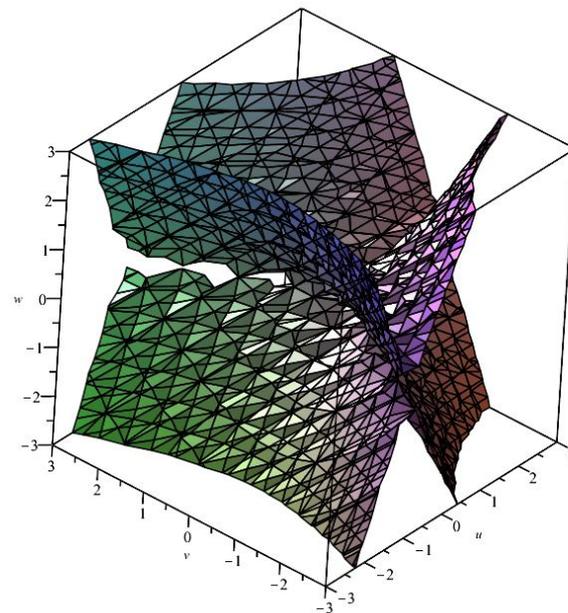


Figure 7. Picture of the complex cubic $E(\frac{1-i\sqrt{3}}{2})$.

Case IV. If $\lambda = \frac{\lambda-1}{\lambda}$, then we have the same solutions λ_{\pm} as in case III.

Case V. Suppose $\lambda = \frac{\lambda}{\lambda-1}$. We obtain the corresponding two cubic curves:

$$E(0) : y^2 = x^2(x - 1), \quad E(2) : y^2 = x(x - 1)(x - 2). \tag{7}$$

In fact, $E(2)$ is a translation of the harmonic curve $E(-1)$, since performing the translation $x \rightarrow \tilde{x} - 1$ in the equation of $E(-1)$ yields the equation of $E(2)$ with x replaced by \tilde{x} . We recover the previous formula $CR(2) = CR(-1) = CR(\frac{1}{2}) = \{-1, \frac{1}{2}, 2\}$.

The complex cubic $E(0)$ is given by the implicit equation

$$4w^4 + 4w^2[(u - 1)(u^2 - v^2) - 2uv^2] = v^2[3u^2 - v^2 - 2u]^2$$

and is (partially) represented in Figure 9. Its real restriction is represented in Figure 10.

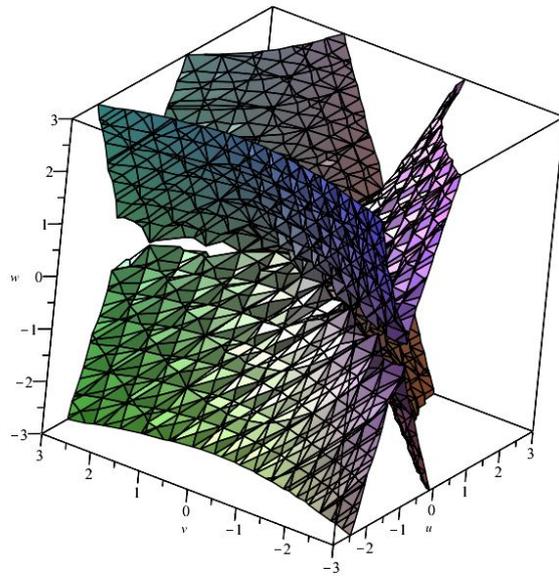


Figure 8. Picture of the complex cubic $E(\frac{1+i\sqrt{3}}{2})$.

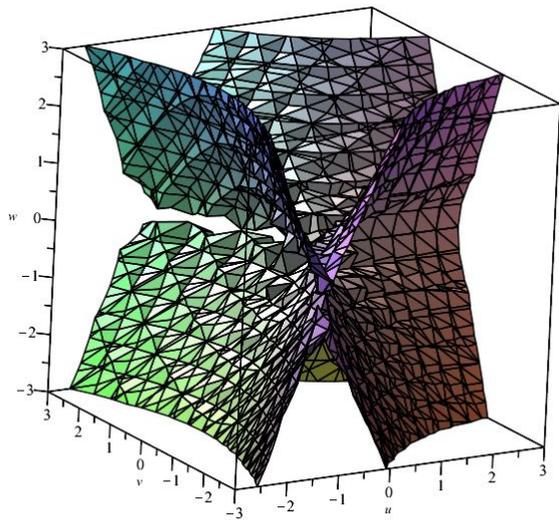


Figure 9. Picture of the complex cubic $E(0)$ from (7).

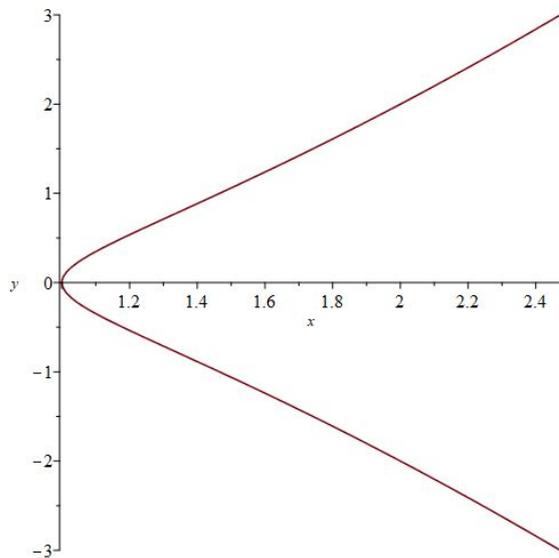


Figure 10. Picture of the real cubic $E(0)$ from (7).

- Case VI.** If $\frac{1}{\lambda} = 1 - \lambda$, we obtain the solution from Case III.
- Case VII.** If $\frac{1}{\lambda} = \frac{1}{1-\lambda}$, we obtain the solution from Case II.
- Case VIII.** If $\frac{1}{\lambda} = \frac{\lambda-1}{\lambda}$, we obtain the solution from Case V.
- Case IX.** If $\frac{1}{\lambda} = \frac{\lambda}{\lambda-1}$, we obtain the solution from Case IV.
- Case X.** If $1 - \lambda = \frac{1}{1-\lambda}$, we obtain the solution from Case V.
- Case XI.** If $1 - \lambda = \frac{\lambda-1}{\lambda}$, we obtain the solution from Case I.
- Case XII.** If $1 - \lambda = \frac{\lambda}{\lambda-1}$, we obtain the solution from Case III.
- Case XIII.** If $\frac{1}{1-\lambda} = \frac{\lambda-1}{\lambda}$, we obtain the solution from Case III.
- Case XIV.** If $\frac{1}{1-\lambda} = \frac{\lambda}{\lambda-1}$, we obtain the solution from Case I.
- Case XV.** If $\frac{\lambda-1}{\lambda} = \frac{\lambda}{\lambda-1}$, we obtain the solution from Case V.

The following result provides a classification of complex cubic curves.

Theorem 1. *The set of complex cubic curves $E(\lambda)$ (with λ different from 0 and 1) is the disjoint sum of the following sets:*

- (i) $\{ E(-1), E(2), E(\frac{1}{2}) \}$, with $|CR| = 3$;
- (ii) $\{ E(\frac{1-i\sqrt{3}}{2}), E(\frac{1+i\sqrt{3}}{2}) \}$, with $|CR| = 2$;
- (iii) *The set of the remaining CR-generic ones, with $|CR| = 6$.*

Proof. Let $\lambda \in \mathbb{C} \setminus \{0, 1\}$. If $|CR(\lambda)| = 6$, then $E(\lambda)$ is generic, so it belongs to (iii).

If $|CR(\lambda)| < 6$, then $E(\lambda)$ is CR-selfdual. The previous analysis of cases I-XV shows that $|CR(\lambda)| = 3$ or $|CR(\lambda)| = 2$, for exactly the values of λ present in (i) or (ii), respectively. Namely, this happens if and only if $\lambda \in \{-1, 2, \frac{1}{2}\}$ or $\lambda \in \{\frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2}\}$, respectively. \square

Remark 1. (i) *We remember the well-known identification of the symmetric group S_3 with the group of isometries $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ of the equilateral triangle, where*

$$f_1 = id_{\mathbb{C}}, f_2(z) = \frac{1}{z}, f_3(z) = 1 - z, f_4(z) = \frac{z}{z-1}, f_5(z) = \frac{1}{1-z}, f_6(z) = \frac{z-1}{z}$$

The maps f_1, f_5 , and f_6 are rotations while the functions f_2, f_3 , and f_4 are involutions. Hence, the orbit space $(\mathbb{C} \setminus \{0, 1\}) / S_3$ is the moduli space of elliptic curves; see ([4], p. 325).

The set of all the fixed points of $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ is $\{-1, 0, \frac{1}{2}, 1, 2\}$.

(ii) *The sets $CR(\lambda)$ are the orbits of the action of the group S_3 on $\mathbb{C} \setminus \{0, 1\}$.*

For a fixed set $CR(\lambda)$, we calculate the arithmetic mean $AM(\lambda) = \frac{1}{2}$, the (formal) “geometric mean” $GM(\lambda) = 1$, and the harmonic mean $HM(\lambda) = 2$. We obtain $AM(\lambda) = [HM(\lambda)]^{-1}$ and $AM(\lambda) \cdot HM(\lambda) = GM(\lambda)$ (which is just an instance of the general property of sets invariant under the inversion f_2). It is interesting that these three means of complex numbers are constant functions and, moreover, are rational numbers.

(iii) *Theorem 1 shows a distinct role that harmonic and equiharmonic cubic curves play among all other cubic curves. The remaining two cubics $E(0)$ and $E(1)$ also appear as special cases.*

(iv) *When we consider the real cubic (1), we may extract*

$$\lambda = x - \frac{y^2}{x(x-1)}$$

and represent the graph of the function $\lambda = \lambda(x, y)$, as in Figure 11. The split domain $\{(-\infty, 0) \cup (0, 1) \cup (1, \infty)\} \times \mathbb{R}$ gives rise to three disjoint surfaces; each of them prescribes a special behavior of the corresponding cubics. This graph can be interpreted as the moduli space of all cubics, endowed with a special parametrization.

We remark that the graph has a symmetry plane $y = 0$. There are no critical points of λ , and hence no extremum ones. The coordinate lines $y \rightarrow \lambda(x_0, y)$, for $x_0 \notin \{0, 1\}$, are parabolas with vertices on the line $(x, 0, x)$, convex for $x \in (0, 1)$ and concave otherwise.

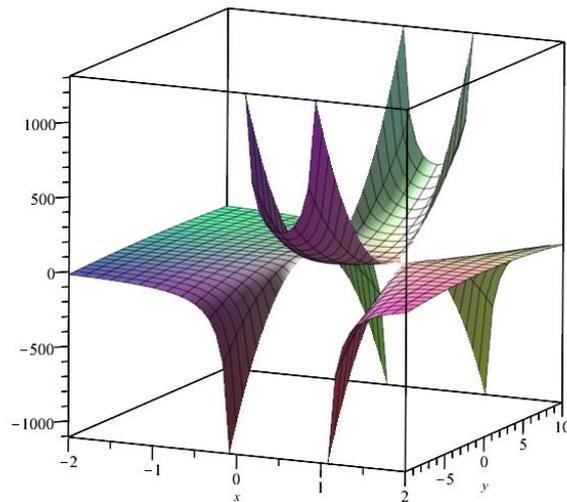


Figure 11. Graph of $\lambda = \lambda(x, y)$ as surface in \mathbb{R}^3 .

The coordinate curve $x \rightarrow \lambda(x, y_0)$, for $y_0 \neq 0$, has three connected components. The curve is strictly increasing for $x > 1$; it has a minimum point for $x \in (0, 1)$ and a maximum point for $x < 0$; and it admits the straight line (x, y_0, x) as asymptotic at both $-\infty$ and $+\infty$.

The plane $\Pi : \lambda - x = 0$ lies under the graph of λ , for $x \in (0, 1)$, and over that graph for $x \in (-\infty, 0) \cup (1, \infty)$. Moreover,

$$\Pi \cap \text{Im } \lambda = \{(x, 0, x) \mid x \in \mathbb{R} \setminus \{0, 1\}\}.$$

The graph of λ is convex for $x \in (0, 1)$ and concave otherwise. Hence, the middle connected component is convex, and the two side components (“left–right”) are concave.

We calculate the Gaussian curvature function of the surface:

$$K(x, y) = \frac{-4y^2x^5(x - 1)^5}{[2x^4(x - 1)^4 + 2x^2y^2(x - 1)^2(2x + 1) + y^4(2x - 1)^2]^2}.$$

We remark that K is a rational function and (roughly speaking) “asymptotically” zero; moreover, $K(x, y) = 0$ if and only if $y = 0$; the sign of $K(x, y)$ is the sign of $x(1 - x)$. Its shape can be viewed in Figures 12–14, with its (obvious) symmetry w.r.t. the plane $y = 0$.

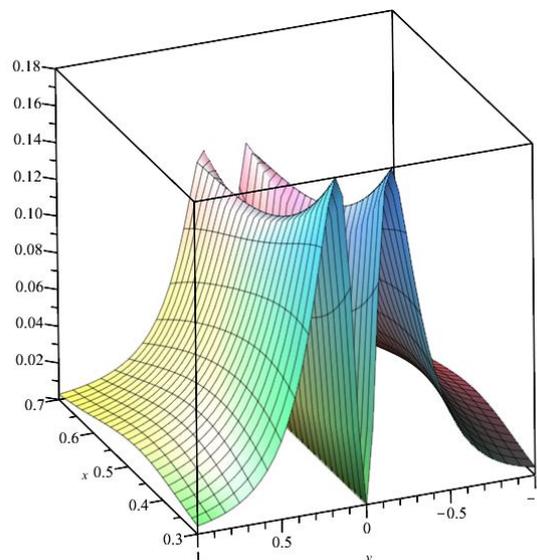


Figure 12. Graph of $K = K(x, y)$, for $x \in [0.3, 0.7]$.

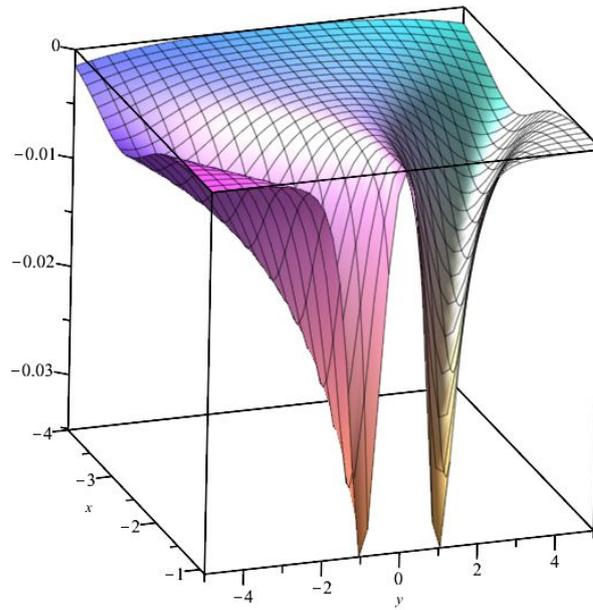


Figure 13. Graph of $K = K(x, y)$, for $x \in [-4, -0.5]$.

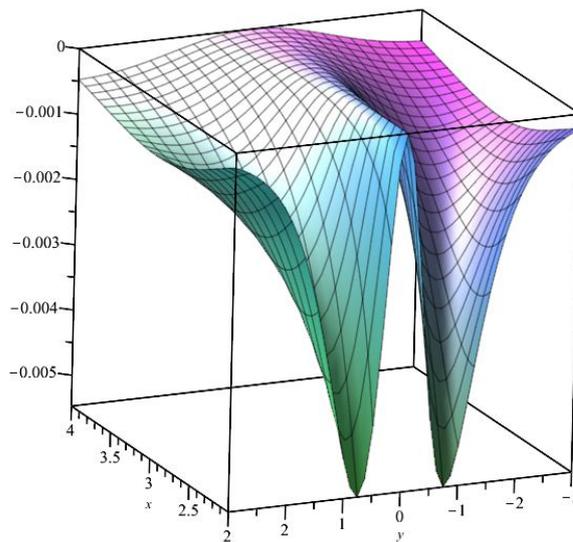


Figure 14. Graph of $K = K(x, y)$, for $x \in [2, 4]$.

We calculate the mean curvature function of the surface:

$$H(x, y) = \frac{-x(x - 1)[2x^4(x - 1)^4 + x^2y^2(x - 1)^2(3x^2 - 7x + 3) + y^4]}{[2x^4(x - 1)^4 + 2x^2y^2(x - 1)^2(2x + 1) + y^4(2x - 1)^2]^{\frac{3}{2}}}$$

We remark that H is not a rational function and is (roughly speaking) “asymptotically” zero; moreover, H never vanishes; the sign of $H(x, y)$ is the sign of $x(1 - x)$. Its shape can be viewed in Figures 15–17, with its (obvious) symmetry w.r.t. the plane $y = 0$.

(In general, for a graph surface defined by a rational function, the Gauss curvature function and the square of the mean curvature function are rational functions too.)

The eventual integer points (x, y, λ) correspond to points of both negative Gauss curvature and negative mean curvature, as $x(x - 1) > 0$.

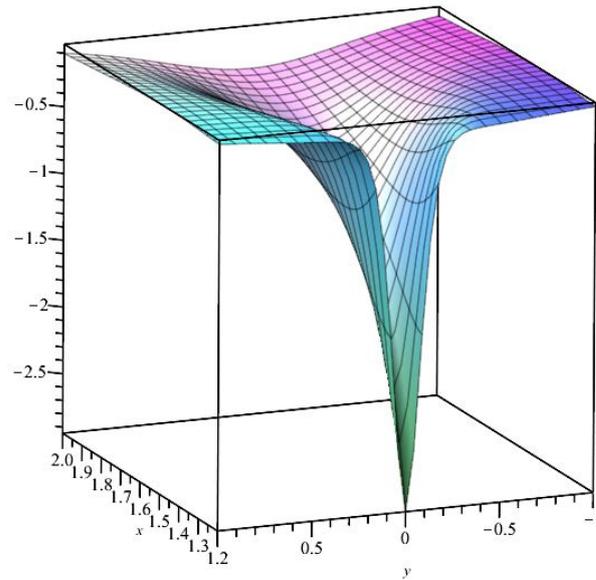


Figure 15. Graph of $H = H(x, y)$, for $x > 1$.

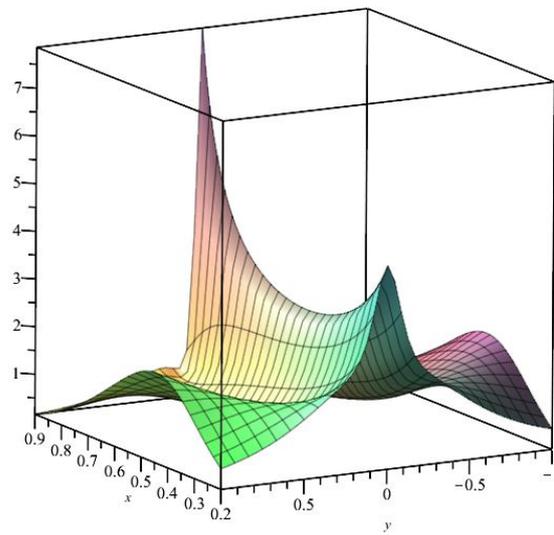


Figure 16. Graph of $H = H(x, y)$, for $x \in (0, 1)$.

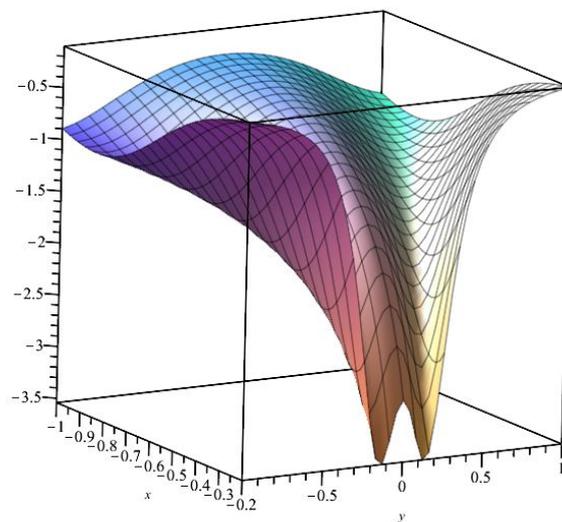


Figure 17. Graph of $H = H(x, y)$, for $x < 0$.

Starting from the well-known general inequality $H^2 \geq K$, we look for the locus of points where the principal curvatures are equal, i.e., for the solutions of the implicit equation $H^2 = K$. Numerical simulations suggest the inequality is strict (see Figure 18).

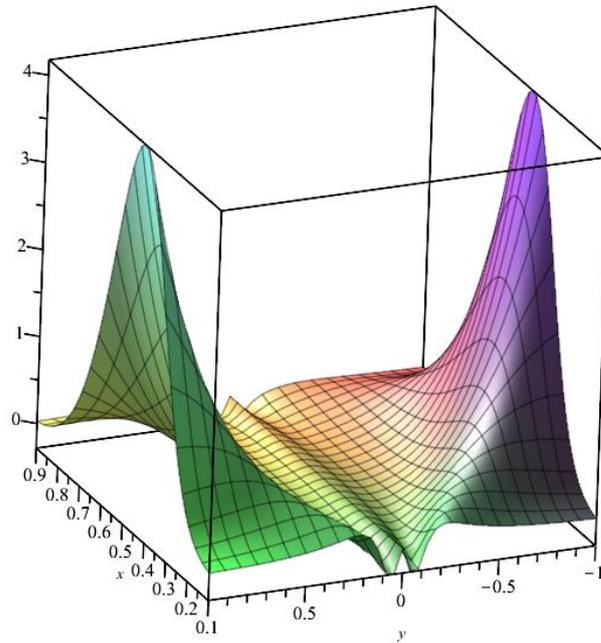


Figure 18. Graph of $H^2 - K$.

The curves of constant Gaussian curvature or constant mean curvature can be obtained in implicit form; for example, in Figure 19, we picture the locus $\{(x, y) \mid K(x, y) = 0.1\}$.

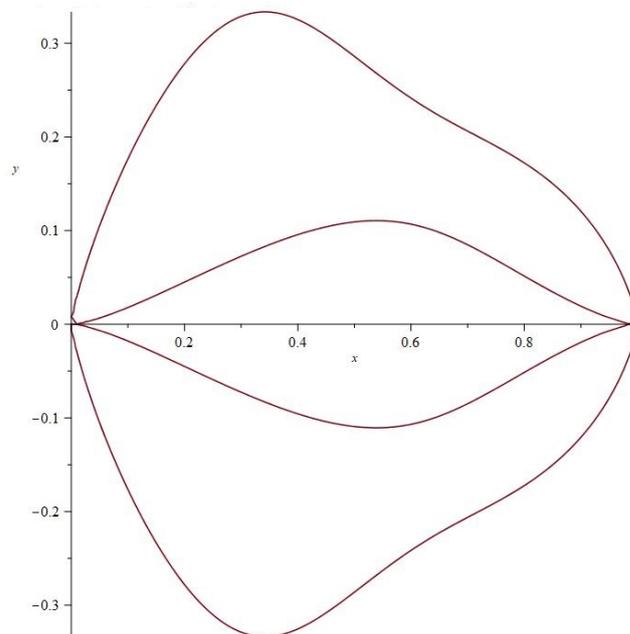


Figure 19. The locus of the points (x, y) where $K(x, y) = 0.1$.

In addition to the previous properties of the graph of λ , we point out one more speculative: an “almost” symmetry w.r.t. the plane $x = \frac{1}{2}$, which is more evident at the level of curvature functions H and K . It is possible that this tiny “symmetry breaking” may hide a more deep mathematical fact. Are there some connections with other “similar” algebraic objects, such as the quartic surface

$$y^2 = x(x - 1)(x^2 - x - \lambda),$$

which is symmetric also w.r.t. $x = \frac{1}{2}$?

(v) The complex analogue of the preceding remark is more complicated. We denote the complex coordinates $x = u^1 + i \cdot u^2, y = u^3 + i \cdot u^4$. Let $u := (u^1, u^2, u^3, u^4)$, with $(u^1, u^2) \notin \{(0, 0), (1, 0)\}$. We re-write Formula (1) as

$$\operatorname{Re} \lambda(u) = u^1 - A(u) \cdot \left\{ [(u^3)^2 - (u^4)^2] \cdot [u^1(u^1 - 1) - (u^2)^2] - 2u^2u^3u^4(2u^1 - 1) \right\},$$

$$\operatorname{Im} \lambda(u) = u^2 + A(u) \cdot \left\{ -[(u^3)^2 - (u^4)^2]u^2(2u^1 - 1) + 2u^3u^4[u^1(u^1 - 1) - (u^2)^2] \right\},$$

where

$$A(u) := \frac{1}{[(u^1)^2 + (u^2)^2] \cdot [(u^1 - 1)^2 + (u^2)^2]}.$$

The graph of λ is a complex surface in \mathbb{C}^3 , but can be studied also as a four-dimensional real submanifold (of codimension two) in \mathbb{R}^6 . This study is beyond the scope of our paper.

(vi) The five complex numbers $-1, \frac{1}{2}, 2, \frac{1-i\sqrt{3}}{2}$, and $\frac{1+i\sqrt{3}}{2}$ are the vertices and the center of a “calisson” rhombus, which is the reunion of two equilateral triangles in the complex plane.

We can point out two interesting properties of these numbers. First, consider $g_2 = g_2(\lambda)$ and $g_3 = g_3(\lambda)$, the modular invariants associated with an elliptic curve in the form (1) (see [5], p. 76; for an example, compare with Remark 2 below). Then, $-1, \frac{1}{2}$, and 2 are precisely the critical values of g_3 and $\frac{1-i\sqrt{3}}{2}$ and $\frac{1+i\sqrt{3}}{2}$ are precisely the critical values of g_2 .

The second property involves the $J = J(\lambda)$ invariant associated with the complex cubic $E(\lambda)$, namely,

$$J(\lambda) := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 \cdot (\lambda - 1)^2}.$$

One knows ([2], page 49) that J is a six-to-one function except above $J = 0$ (when it is two-to-one and $\lambda \in \{\frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2}\}$) and above $J = 1728$ (when it is three-to-one and $\lambda \in \{-1, \frac{1}{2}, 2\}$). These three different behaviors of the function J correspond to the three cases in Theorem 1, respectively.

We restrict the function $J = J(\lambda)$ to real numbers λ . The graph of J is pictured in Figure 20. If we replace $\lambda = \lambda(x, y)$ and re-write $J = J(x, y)$, then its graph is a parameterized surface in \mathbb{R}^3 , as pictured in Figure 21. The six areas where the values of J go to ∞ are around the points $(0, 0)$ and $(1, 0)$ and the those belonging to the real cubic curves $E(0)$ and $E(1)$.

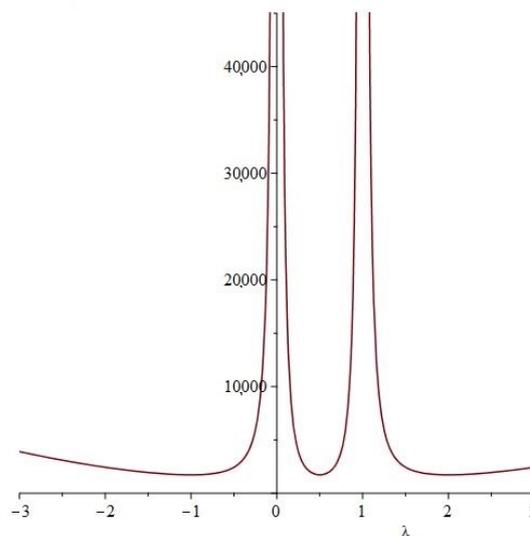


Figure 20. Graph of function $J = J(\lambda)$.

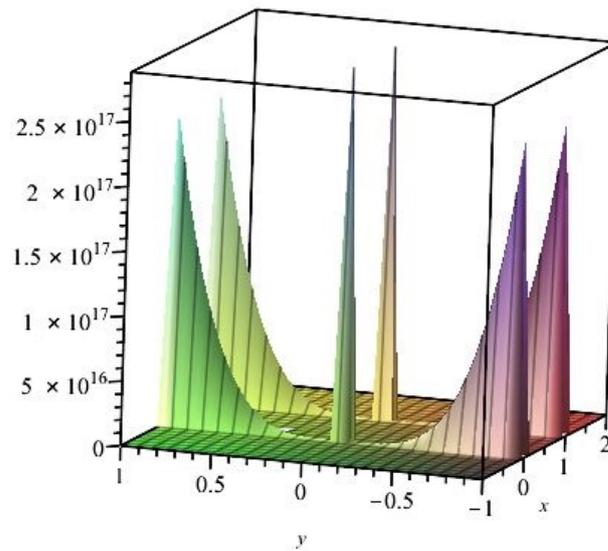


Figure 21. Graph of function $J = J(x, y)$.

3. Properties of CR-Selfdual Cubics

The conclusion of the first section is the existence of seven CR-selfdual cubic curves. We will now provide some of their properties. In fact, we will restrict our study to the five real CR-selfdual cubics $E(-1)$, $E(1)$, $E(\frac{1}{2})$, $E(0)$, and $E(2)$; a constant interest is in their integer (or lattice) points. From the differential geometry point of view, we compute the curvature of some cubics for which we know a regular (i.e., with nonzero velocity) parametrization.

3.1. The cubic $E(-1)$

First of all, we point out that, since the roots of the cubic polynomial defining this cubic curve are distinct, the harmonic cubic is an *elliptic curve* and its remarkable properties are detailed at the database: <https://www.lmfdb.org/EllipticCurve/Q/32/a/3>. Also, it is worth pointing out that this elliptic curve is the subject of several studies in algebraic number theory; see, for example, [9].

The cubic $E(-1)$ has only three integer points $(\pm 1, 0)$ and $(0, 0)$. This cubic appears as a special case concerning the framing theory of cubic curves; see page 94 of the cited book. We can say something also about the rational points of $E(-1)$ by applying the Theorem 4.5 of ([5], p. 84): the sets $V_1 = \{(a, b, c) \in \mathbb{Q}^3; a^2 + b^2 = c^2, ab = 2\}$ and $V_2 = \{(x, y) \in E(-1); xy \neq 0\}$ are in an one-to-one correspondence through the following maps:

$$f_1 : V_1 \rightarrow V_2, f_1(a, b, c) = \left(\frac{b}{c-a}, \frac{2}{c-a} \right), \tag{8}$$

$$f_2 : V_2 \rightarrow V_1, f_2(x, y) = \left(\frac{x^2 - 1}{y}, \frac{2x}{y}, \frac{x^2 + 1}{y} \right).$$

By direct computation, or by using the two functions from (8), we see that both sets V_1 and V_2 do not contain integer points.

Recall also that, given the triple $e_1 < e_2 < e_3$ and its associated elliptic curve $E(e_1, e_2, e_3) : y^2 = (x - e_1)(x - e_2)(x - e_3)$, its *real period* is the following positive number:

$$\Omega(E(e_1, e_2, e_3)) := 4 \int_{e_3}^{+\infty} \frac{dx}{y} = \frac{2\pi}{AGM(\sqrt{e_3 - e_1}, \sqrt{e_2 - e_1})}$$

with $AGM(u > 0, v > 0)$ being the arithmetic-geometric mean of the positive numbers u and v . Hence, the harmonic elliptic curve has

$$\Omega(E(-1)) = \frac{2\pi}{AGM(\sqrt{2}, 1)}, \quad AGM(\sqrt{2}, 1) = 1.1981\dots$$

3.2. The cubic $E(1)$

The cubic curve $E(1)$ has a countable infinity of integer points $(n^2, \pm n(n^2 - 1)), n \in \mathbb{N}$. This fact suggests the following parametrization:

$$E(1) : r_1(t) = (t^2, t(t^2 - 1)), \quad t \in \mathbb{R}.$$

Next, we can compute the curvature of $E(1)$ as regular Euclidean plane curve:

$$k_1(t) := \frac{x'(t)y''(t) - y'(t)x''(t)}{\|r_1'(t)\|^3} = \frac{6t^2 + 2}{[9t^4 - 2t^2 + 1]^{\frac{3}{2}}} > 0 \quad .$$

Here, $\|r_1'\|$ means the Euclidean norm of the derivative of the vectorial function r_1 . We note that the paper [8] studies the number of vertices of an elliptic curve expressed in the (reduced) Weierstrass form. We remark that the harmonic cubic $E(-1)$ is in the reduced Weierstrass form.

In order to find the Weierstrass form of $E(1)$, we make the Cardano substitution ([10], p. 157) $x = u + \frac{2}{3}$ and then we derive the following:

$$E(1) : y^2 = u^3 - \frac{u}{3} + \frac{2}{27} = u^3 + pu + q.$$

Hence, its *discriminant* is as follows:

$$\Delta(E(1)) := \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = \left(-\frac{1}{9}\right)^3 + \left(\frac{1}{27}\right)^2 = 0.$$

The discriminant of the elliptic curve $E(-1)$ is $\Delta(E(-1)) = -\frac{1}{27} < 0$.

3.3. The cubic $E\left(\frac{1}{2}\right)$

The elliptic curve $E\left(\frac{1}{2}\right)$ has only two integer points, $(1, 0)$ and $(0, 0)$, and the following real period:

$$\Omega\left(E\left(\frac{1}{2}\right)\right) = \frac{2\pi}{AGM\left(1, \frac{1}{\sqrt{2}}\right)}.$$

In addition, it has the rational point $\left(\frac{1}{2}, 0\right)$. Its Cardano substitution $x = u + \frac{1}{2}$ yields the Weierstrass form as follows:

$$E\left(\frac{1}{2}\right) : y^2 = u^3 - \frac{u}{4}, \quad \Delta\left(E\left(\frac{1}{2}\right)\right) = -\frac{1}{12^3} < 0.$$

There exists a relationship between the elliptic curves $E(-1)$ and $E\left(\frac{1}{2}\right)$ provided by the function $J = J(\lambda)$ (called the J -invariant) associated to the general cubic $E(\lambda)$ ($\lambda \notin \{0, 1\}$) through the following:

$$J(\lambda) := 2^8 \cdot \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2} = J\left(\frac{1}{\lambda}\right) = J(1 - \lambda).$$

Indeed, we have

$$J(-1) = 2^6 \cdot 3^3 = J(2) = J\left(\frac{1}{2}\right), \quad J\left(\frac{1+i\sqrt{3}}{2}\right) = J\left(\frac{1-i\sqrt{3}}{2}\right) = 0.$$

Remark 2. On page 76 of the book [5], the perimeter of the canonical ellipse $E(a > b > 0) : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ of eccentricity $e := \sqrt{1 - \frac{b^2}{a^2}} \in (0, 1)$ is computed by means of an elliptic integral involving the following functions of $\lambda = 1 - e^2$:

$$g_2(\lambda) := \frac{\sqrt[3]{4}}{3}(\lambda^2 + \lambda - 1),$$

$$g_3(\lambda) := -\frac{1}{27}(2\lambda^3 + 3\lambda^2 - 3\lambda - 2) = -\frac{2}{27}(\lambda - 1)(\lambda + 1)\left(\lambda + \frac{1}{2}\right).$$

The ellipse with $e = \frac{1}{\sqrt{2}}$ yielding our $\lambda = \frac{1}{2}$ is called self-complementary and its geometry is studied in [11].

3.4. The cubic $E(0)$

The cubic curve $E(0) \setminus \{(0, 0)\}$ has a countable infinity of integer points $(n^2 + 1, \pm n(n^2 + 1))$, $n \in \mathbb{N}$. Again, we have a parametrization ([1], p. 77):

$$E(0) \setminus \{(0, 0)\} : r_0(t) = (t^2 + 1, t^3 + t), \quad t \in \mathbb{R}.$$

Its corresponding curvature is as follows:

$$k_0(t) = \frac{6t^2 - 2}{[9t^4 + 10t^2 + 1]^{\frac{3}{2}}}.$$

The inflection points (as points of vanishing curvature) of $E(0)$ follow: $r_0\left(-\frac{1}{\sqrt{3}}\right) = \left(\frac{4}{3}, -\frac{4}{3\sqrt{3}}\right)$ and $r_0\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{4}{3}, \frac{4}{3\sqrt{3}}\right)$.

Its Cardano substitution $x = u + \frac{1}{3}$ provides the Weierstrass form:

$$E(0) : y^2 = u^3 - \frac{u}{3} - \frac{2}{27}, \quad \Delta(E(0)) = 0.$$

3.5. The cubic $E(2)$

The elliptic curve $E(2)$ has only three integer points, $(0, 0)$, $(1, 0)$, and $(2, 0)$, as translations of the integer points of $E(-1)$.

4. Refining the Selfduality

Let $D \subseteq \mathbb{C}$, $f : D \rightarrow \mathbb{C}$ be a function and $\lambda \in D$ be a fixed complex number.

Definition 2. The number λ is a CR-fixed point of f if $f(\lambda) \in CR(\lambda)$.

If $\bar{\lambda}$ is a fixed point for f , then λ is also a CR-fixed point of f . Hence, this new notion extends the property of a point to be a fixed one. (Intuitively, it is a kind of “fixed point property”, modulo the action of S_3 .)

If $f \in S_3$, then any $\lambda \in D$ is a CR-fixed point of f .

Definition 3. The cubic $E(\lambda)$ is CRf-selfdual if $E(f(\lambda))$ is CR-selfdual.

When f is the identity function, any CR-selfdual cubic is (tautologically) CRf-selfdual too.

Remark 3. As the previous context is too large, we shall restrain ourselves first to the study of a specific family of complex functions: the Möbius functions.

Consider f a fixed Möbius transformation of the (extended) complex plane $\hat{\mathbb{C}}$, given by

$$f(z) = \frac{az + b}{cz + d} \quad , \tag{9}$$

with a, b, c , and d (fixed) complex numbers with $ad - bc \neq 0$. By convention, $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$.

To each cubic $E(\lambda)$, we associate a cubic $E(f(\lambda))$, defining an action of the Möbius group onto the set of complex cubics in normal form. For completeness, we consider the cubic “at infinity” $E(\infty)$ as being the plane “at infinity”.

The general properties of this action (the orbits, the stabilizers, transitivity, effectiveness, freedom, etc.) are identical with those for the action of the Möbius group on $\hat{\mathbb{C}}$.

The six numbers in the $CR(\lambda)$ set in (2) are the values of the six specific Möbius transformations in S_3 (which is a finite subgroup of the Möbius group), calculated in λ . It would be quite natural to replace S_3 by another (finite or not) subgroup of the Möbius group and try to find the relevance of its orbits, similar to the role played by $CR(\lambda)$ in the classification of complex cubics or of other algebraic curves.

Proposition 1. With the notation given in the previous remark, the complex number λ is a CR-fixed point of f if and only if

$$\lambda \neq -\frac{b}{a} \quad , \quad \lambda \neq \frac{d-b}{a-c} \tag{10}$$

and λ is a fixed point for one of the following Möbius functions: $f, \frac{1}{f}, 1-f, 1-\frac{1}{f}, \frac{1}{1-f}, \frac{f}{f-1}$.

Proof. From the context in Section 2, we must have $f(\lambda) \notin \{0, 1\}$; hence, we obtain Formula (10).

Let $\lambda \in \mathbb{C}$ be a CR-fixed point of f in (9). It follows that $f(\lambda) \in CR(\lambda)$. A short calculation leads to the six conditions. \square

Proposition 2. With the notation given in the previous remark, the complex cubic $E(\lambda)$ is a CRf-selfdual if and only if

$$\lambda \neq -\frac{b}{a} \quad , \quad \lambda \neq \frac{d-b}{a-c} \tag{11}$$

and

$$\lambda \in \left\{ -\frac{b+d}{a+c}, \frac{2d-b}{a-2c}, \frac{d-2b}{2a-c}, \frac{d(1+i\sqrt{3})-2b}{2a-c(1+i\sqrt{3})}, \frac{d(1-i\sqrt{3})-2b}{2a-c(1-i\sqrt{3})} \right\}. \tag{12}$$

The proof is similar to that of the previous proposition and will be omitted.

Remark 4. (i) For any complex number λ , there exists a Möbius transformation h such that $E(\lambda)$ is a CRh-selfdual cubic. Indeed, if $\lambda = 0$, put $h(z) = \frac{z+1}{z-1}$. For $\lambda \neq 0$, choose $h(z) = \frac{2z-\lambda}{z-2\lambda}$. (We see that h is not unique.)

(ii) The classification of fixed points of Möbius transformations (elliptic/circular, hyperbolic, parabolic) is one of the main tools for understanding their behavior. The type of the CR-fixed points of f follows from the type of the fixed points of $f_i \circ f$, for $i = \overline{1, 6}$.

(iii) For each fixed Möbius transformation f , the number of CRf -selfdual cubics is finite (and equal to two or three, cf. Theorem 1). Instead, given a complex number λ , there exists an infinity of Möbius transformations g such that λ is CRg -selfdual; thus, the Möbius group acts as a “fiber bundle”-like set over \mathbb{C} .

(iv) Suppose the restrictions in (11) are fulfilled. The five-point set in Formula (12) is the image of the “calisson” rhombus vertices from Remark 1,iv through the Möbius transformation f^{-1} —i.e.,

$$f^{-1}\left(\left\{-1, \frac{1}{2}, 2, \frac{1-i\sqrt{3}}{2}, \frac{1+i\sqrt{3}}{2}\right\}\right) .$$

(v) For a fixed Möbius transformation f , we define the new complex functions $\alpha_f(z) := f(J(z))$ and $\beta_f(z) := J(f(z))$, explicitly written as follows:

$$\alpha_f(z) = \frac{2^8 a(z^2 - z + 1)^3 + bz^2(z - 1)^2}{2^8 c(z^2 - z + 1)^3 + dz^2(z - 1)^2} ,$$

$$\beta_f(z) = 2^8 \cdot \frac{[(az + b)^2 - (az + b)(cz + d) + (cz + d)^2]^3}{(az + b)^2 \cdot (cz + d)^2 \cdot (az + b - cz - d)^2} .$$

The graphs of their restrictions to \mathbb{R} are pictured in Figures 22 and 23 for the particular case $a = 2, b = 5, c = 1, d = 3$.

An interesting problem is to determine the complex (or the real) numbers z , such that $J(f(z)) = f(J(z))$. When f is the identity function, the locus of these points is the full domain of definition of f . We prove this property is characteristic, at least for real Möbius transformations.

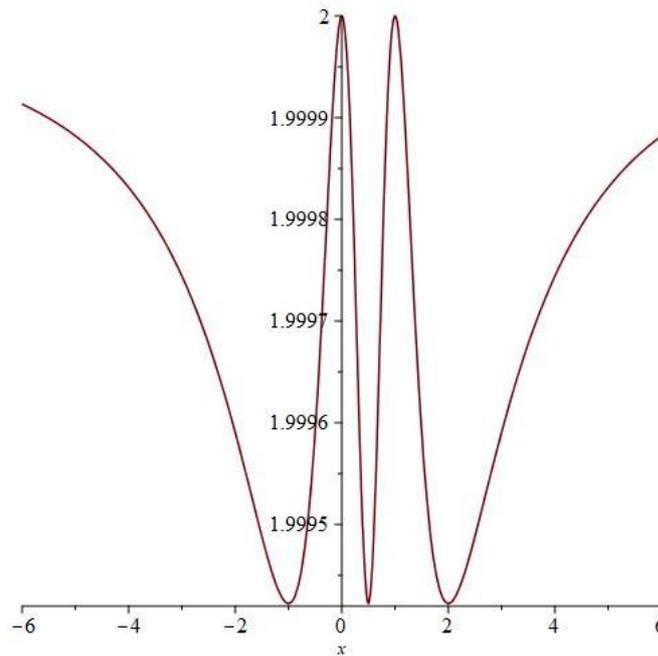


Figure 22. Graph of (real) function α_f for $a = 2, b = 5, c = 1, d = 3$.

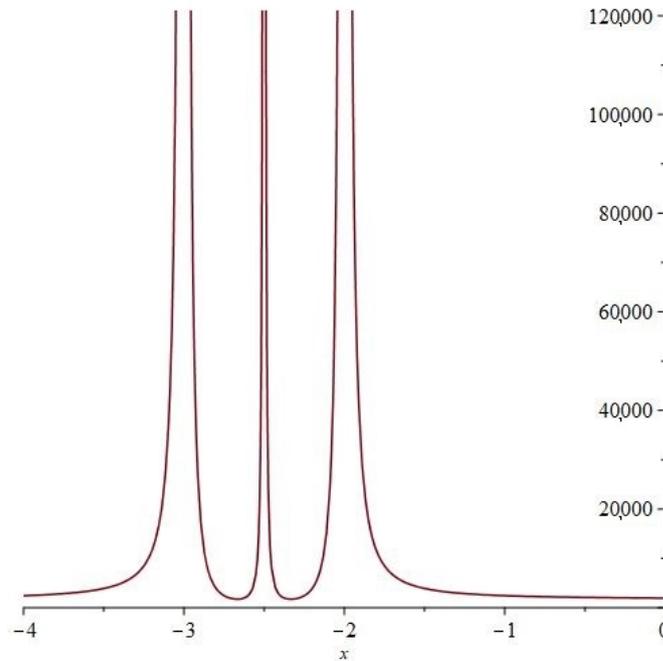


Figure 23. Graph of (real) function β_f for $a = 2, b = 5, c = 1, d = 3$.

Theorem 2. Let f be a fixed Möbius transformation given by (9). Suppose the constants a, b, c , and d are real. Then, the set $M_f := \{z \in \mathbb{C} \mid \alpha_f(z) := \beta_f(z)\}$

- (i) is the full domain of definition of f , if f is the identity function;
- (ii) contains at most 12 elements, otherwise.

Proof. The hypothesis $\alpha_f(z) := \beta_f(z)$ is

$$\frac{2^8 a(z^2 - z + 1)^3 + bz^2(z - 1)^2}{2^8 c(z^2 - z + 1)^3 + dz^2(z - 1)^2} = 2^8 \cdot \frac{[(az + b)^2 - (az + b)(cz + d) + (cz + d)^2]^3}{(az + b)^2 \cdot (cz + d)^2 \cdot (az + b - cz - d)^2}$$

and can be re-written as a polynomial equation $F(z) = 0$ with the complex variable z , where

$$F(z) := -65536 \cdot (a^6 - \frac{769}{256}ca^5 + \frac{769}{128}c^2a^4 - \frac{1793}{256}c^3a^3 + 6c^4a^2 - 3c^5a + c^6) \cdot c \cdot z^{12} + \dots \text{ terms of lower degree .} \tag{13}$$

If the polynomial F is not identically zero, then its degree is at least 1 and at most 12. Hence, we obtain conclusion (ii). The cardinality of the set M_f is determined by the multiplicities of the roots of F and by the degree of the polynomial F .

Suppose the polynomial F is identically zero. We shall prove that $a = d$ and $b = c = 0$, thus justifying conclusion (i).

Suppose, ad absurdum, that $c \neq 0$. Then, we must have

$$a^6 - \frac{769}{256}ca^5 + \frac{769}{128}c^2a^4 - \frac{1793}{256}c^3a^3 + 6c^4a^2 - 3c^5a + c^6 = 0 .$$

We denote $w := \frac{a}{c}$. The equation

$$w^6 - \frac{769}{256}w^5 + \frac{769}{128}w^4 - \frac{1793}{256}w^3 + 6w^2 - 3w + 1 = 0$$

does not admit real roots; hence, we obtain a contradiction. It follows that $c = 0$.

Replacing $c = 0$ in (13), we obtain a simplified expression of F :

$$F(z) = -256a^5(a - d)z^{10} + 512a^4(a^2 - 3ab + 2bd - d^2)z^9 + \dots \text{ terms of lower degree} . \quad (14)$$

The generic condition $ad - bc \neq 0$ implies $a \neq 0$. Due to the fact that F must be identically zero, we obtain that the coefficient of z^{10} is 0; hence, $a = d$. We use this equality in (14) and we obtain that the coefficient of z^9 is $-512a^5b$. But this coefficient must be null; hence, $b = 0$. \square

Remark 5. (i) As a by-product, from Theorem 2 we obtain a new classification of real Möbius transformations: the (set with unique element the) identity transformation and 12 disjoint sets

$$MOB_i := \{f \text{ Möbius transformation} \mid |M_f| = i\} ,$$

for $i = \overline{1, 12}$.

We can refine this classification as follows: each set MOB_i can be split into subsets, according to the degree of the associated polynomial F from the proof of Theorem 2 and/or the multiplicities of its roots. For example, we can consider the family of real Möbius transformations h , such that $|M_h| = 12$, $\deg F = 12$ and all the roots of the associated polynomials F are real (and distinct).

(ii) We conjecture that Theorem 2 is true for complex Möbius transformations too. In the complex case, the calculations become much more complicated and a simpler qualitative proof would be very useful.

5. The Curvature and the Vertices of the Real Cubic Curves in Legendre Form

In [8], the vertices of the real elliptic curves in Weierstrass form are studied. We revisit parts of this study here for real cubic curves in Legendre form (1). The proofs are simpler than in [8] and the results are similar but do not overlap completely.

Let $E(\lambda)$ be a fixed cubic curve, for an arbitrary real number λ . In a regular point (x, y) of $E(\lambda)$, we calculate the value of the curvature function k_λ of $E(\lambda)$ as

$$k_\lambda(x, y) = \frac{8y^2(-3x + \lambda + 1) + 2[3x^2 - 2(\lambda + 1)x + \lambda]^2}{\{[3x^2 - 2(\lambda + 1)x + \lambda]^2 + 4y^2\}^{\frac{3}{2}}} . \quad (15)$$

Using (1), we can express

$$\lambda = x - \frac{y^2}{x(x - 1)}$$

and replace it in (15). We obtain a parametrized surface $\tilde{k} = \tilde{k}(x, y)$, with

$$\tilde{k}(x, y) = \frac{2x(x - 1)[x^4(x - 1)^4 + y^4 - 2x^2(x - 1)^2(2x - 1)y^2]}{\{[x^2(x - 1)^2 + (2x - 1)y^2]^2 + 4x^2(x - 1)^2y^2\}^{\frac{3}{2}}} ,$$

which “measures” the curvature of all real cubic curves simultaneously. The graph of \tilde{k} is pictured in Figure 24.

We can also replace y^2 in (15), and we obtain another parametrized surface $\hat{k} = \hat{k}(x, \lambda)$,

$$\hat{k}(x, \lambda) = \frac{2\{-3x^4 + 4(\lambda + 1)x^3 - 6\lambda x^2 + \lambda^2\}}{\{9x^4 - 4(3\lambda + 2)x^3 + 2\lambda(2\lambda + 5)x^2 - 4\lambda^2x + \lambda^2\}^{\frac{3}{2}}} , \quad (16)$$

which “measures” the curvature of all real cubic curves simultaneously. The graph of \hat{k} is pictured in Figure 25.

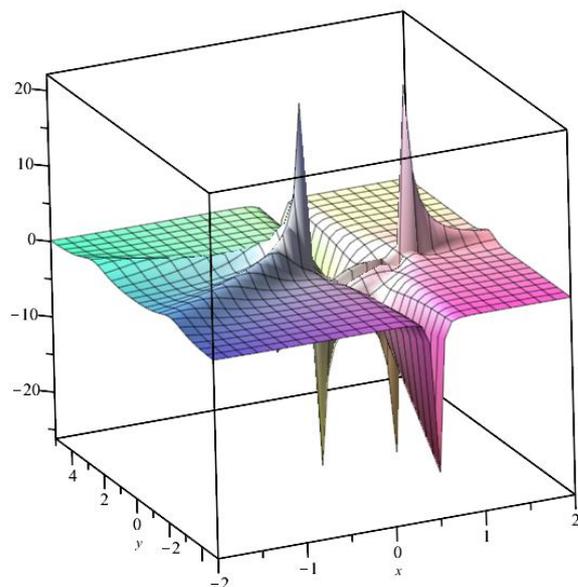


Figure 24. Graph of function \tilde{k} for $x \in [-2, 2]$ and $y \in [-5, 5]$.

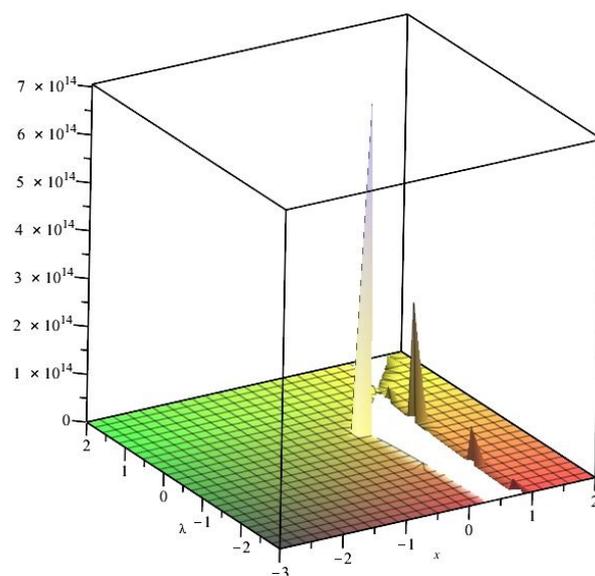


Figure 25. Graph of function \hat{k} for $x \in [-3, 2]$ and $\lambda \in [-3, 2]$.

Both parameterized surfaces \tilde{k} and \hat{k} are “absolute” differential invariants associated with the whole family of real cubic curves. From this viewpoint, they are similar with the parameterized surface in Remark 1 (iv). A similar study of their Gaussian and mean curvature functions merits a separate paper.

For the study of vertices, one can consider another parametrization of real cubic curves. Let $E(\lambda)$ be a real cubic curve and define

$$c_\lambda(t) := \left(t, \sqrt{t(t-1)(t-\lambda)} \right)$$

as a parametrization of the positive branch of $E(\lambda)$. We calculate the curvature of c_λ in a regular point, as $\check{k}_\lambda(t) = \hat{k}(t, \lambda)$, as expected.

The vertices of c_λ are provided by the real solutions of the equation $(\check{k}_\lambda)'(t) = 0$. The denominator of $(\check{k}_\lambda)'$ is a polynomial function of degree seven; thus, it has at least one and at most seven real roots:

$$9t^7 - (21\lambda + 20)t^6 + (10\lambda^2 + 51\lambda + 8)t^5 - (25\lambda^2 + 20\lambda)t^4 - 5\lambda^2t^3 + (13\lambda^3 + 8\lambda^2)t^2 - (2\lambda^4 + 7\lambda^3)t + \lambda^4 = 0 \quad (17)$$

Proposition 3. *A real cubic curve has at least 4 and at most 16 vertices.*

Proof. For $E(0)$ and $E(1)$, the first coordinates of the vertices are (approximately) $t \in \{0.52, 1.69\}$ and $t \in \{-0.62, 0.17\}$, respectively. Hence, $E(0)$ and $E(1)$ have four vertices each. In this case, there exists a third root of Equation (17) (namely, 0 and 1, respectively), but the points on the curves $((0,0)$ on $E(0)$ and $(1,0)$ on $E(1)$, respectively) are not regular.

Suppose $\lambda \notin \{0, 1\}$. The real roots t of Equation (17) provides pairs

$$\left(t, -\sqrt{t(t-1)(t-\lambda)}\right), \left(t, +\sqrt{t(t-1)(t-\lambda)}\right)$$

of vertices of $E(\lambda)$. The number of the real roots is between 1 and 7, so the number of the vertices on the positive and on the negative branches of the cubic curve is between 2 and 14.

The lower limit two can be improved: $E(\lambda)$ has a component which is a closed simple curve (as in Figures 3 and 6). The theorem of the four vertices tells us that $E(\lambda)$ admits at least four vertices.

The upper limit 14 could, eventually, exceed 14, because any of the three points of intersection of $E(\lambda)$ with the first axis of coordinates, namely, $(0,0)$, $(1,0)$, and $(\lambda,0)$, might be a vertex.

If $t = 0$ is a root of (17), then $\lambda = 0$, which is impossible, as the point $(0,0)$ is not regular on $E(0)$. For the points $(1,0)$ and $(\lambda,0)$, we did not find such contradictions, so our best estimation for the maximum number of vertices is 16. \square

In particular, we obtain the following:

Proposition 4. *The real (non-singular) CR-selfdual cubic curves $E(-1)$, $E(\frac{1}{2})$, and $E(2)$ have ten vertices.*

Proof. Direct approximate calculations gives the first coordinate of the following vertices:

For $E(-1)$, $t \in \{-1.9, -0.67, -0.17, 0.52, 1.97\}$;

For $E(\frac{1}{2})$, $t \in \{-0.42, 0.09, 0.36, 0.74, 1.5\}$;

For $E(2)$, $t \in \{-0.9, 0.32, 0.82, 1.52, 2.97\}$. \square

Proposition 4 and our numerical simulations support the conjecture that 10 is the maximal number of vertices, for any real cubic curve. More surprisingly, it seems that 10 is (generically) THE number of the respective vertices.

6. Conclusions

We defined and studied the CR-selfdual complex elliptic curves written in normal (Legendre) form (1), which shed a new light on some special cases, namely, for $\lambda \in \{-1, 0, \frac{1}{2}, 1, 2\}$. New properties of these special elliptic curves were pointed out in Section 3.

The differential geometry of the surface of all the real elliptic curves was studied in Section 2; its Gaussian and its mean curvature functions were determined, together with symmetries and an intriguing “almost symmetry” w.r.t. the plane $x = \frac{1}{2}$.

The “absolute” self-duality from Section 1 was extended in Section 4 to a weaker “relative” self-duality, depending on a fixed Möbius transformation. Proposition 1 and Proposition 2 characterize this extended self-duality and a weaker notion of fixed point. As a by-product, Theorem 2 provides a new classification of real Möbius transformations; its complex counterpart remains an open and far more challenging problem.

In Proposition 4, we determined the vertices of the real CR -selfdual cubic curves, as a particular case of a weaker general estimation in Proposition 3.

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