The circular sum of points of an affine segment

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Abstract: This note introduces a sum for two interior points of an affine segment $[AB]$ based on a trigonometric approach. Some properties of this operation along with general examples (including squares of endpoints, the midpoint, and the centroid) are discussed. A main result is that the midpoint of a segment is the circular sum of the points dividing the segment with the ratios 4 and 9. In addition, we completely solve a Diophantine equation associated to the circular sum in order to identify all points with integer ratio whose circular sum is a point with integer ratio.

Key words: affine segment; barycentric coordinates; circular sum

1. Introduction

Being situated "between" the Euclidean and projective geometries, affine geometry is remarkable in several ways. In this note, we focus our attention on the simpler, one-dimensional, affine geometry.

More precisely, our framework is an affine line $\mathcal{L}$ having a fixed affine frame $\{A, B\} \in \mathcal{L} \times \mathcal{L}$ with $A \neq B$. We then concentrate on segment $[AB]$ defined by the points of the given affine frame and identifify this segment with the interval $\left(0, \frac{\pi}{2}\right)$ based on a trigonometric approach, i.e. by using (the square of) the circular functions cosine and sine. This identification yields a trigonometric sum on the initial affine segment and we study this operation mainly through examples. Concretely, the midpoint of the segment and some remarkable interior points of a triangle are discussed.

As is usual in one-dimensional geometry, we look into the simple ratio of three collinear points as well as the ratio in which a point divides the given segment. A carefully analysis performed in the second section expresses the midpoint of $[AB]$ as the circular sum of two points M_1, $M_2 \in [AB]$ where M_1 divides $[AB]$ with the ratio $k_1 = 4$ and $k_2 = 9$, respectively. These values of k_1 and k_2 are obtained as solutions of a Diophantine equation representing an equilateral hyperbola. It is interesting to note that this Diophantine equation appears also to be satisfied by the first Chern class of an arbitrary almost complex structure on the manifold $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

In section 3, we determine all points on $[AB]$ with integer ratios for which the ratio corresponding to the circular sum is also an integer. To do so, we solve yet another Diophantine equation resulting from the definition to the circular sum. Finally, in chapter 4, we define a circular sum for points on a quotient of the line $\mathcal{L}$ and show that, together with this operation, this quotient space is a group.

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2. The circular geometry of an affine segment

The setting of this short note is provided by an *affine line* $\mathcal{L}$, i.e. an 1-dimensional real affine space, in which we fix an affine frame $\mathcal{F} = \{A, B\}, A, B \in \mathcal{L}, A \neq B$. For a fixed point $M \in \mathcal{L}$, we consider its associated *barycentric coordinates withy respect to* $\mathcal{F}$, namely $x(M)$ and $y(M)$ defined as follows:

$$\overrightarrow{OM} := x(M)\overrightarrow{OA} + y(M)\overrightarrow{OB}, \quad x(M) + y(M) = 1, \tag{2.1}$$

where $O \in \mathcal{L}$ is an arbitrary point. A well-known *affine invariant* of the geometry of $\mathcal{L} \setminus \{A\}$ is *the simple ratio*:

$$r(A, B; M) := \frac{x(M)}{y(M)} \in \mathbb{R} \setminus \{-1\}. \tag{2.2}$$

We recall that $r(A, B; M)r(B, A; M) = 1$ and $r(A, B; M) + r(B, M; A) = -1$. A product of quaternionic type on the set of simple ratio is introduced and studied in [6]. The "multiplicative involution" $f : (0, +\infty) \to (0, +\infty)$, $f(x) = \frac{1}{x}$ has the unique fixed point $x_0 = 1$ while the "additive involution" $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = -x - 1$ has the unique fixed point $x_1 = -\frac{1}{2}$. Since any real linear fractional map $F(x) = \frac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$ can be associated with a matrix of the form

$$F = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(\mathbb{R}),$$

the maps we considered above will have associated matrices given by:

$$f = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad g = \left(\begin{array}{cc} -1 & -1 \\ 1 & 0 \end{array} \right).$$

Note that the matrix f is the well-known (almost) product structure of the 2-manifold $\mathbb{R}^2$.

In the following, we concentrate on the *affine segment* $[AB]$ consisting of the points $M \in \mathcal{L}$ with nonnegative $x(M)$ and $y(M)$. Inspired by the cover map $exp : \mathbb{R} \to S^1$, $exp(s) = e^{2\pi is}$, we start by identifying the given point $M \in [AB]$ with a complex number $z(M)$ given by:

$$\left\{ \begin{array}{l} z(M) := x(M) + iy(M) = \cos^2 t + i \sin^2 t = z(t) \in \mathbb{C}, \quad t \in [0, \frac{\pi}{2}] \\ |z(M)|^2 = 1 - \frac{1}{2}[\sin(2t)]^2 = 1 - \frac{2\tan^2 t}{(1 - \tan^2 t)^2}, \quad r(A, B; M) = (\cot t)^2. \end{array} \right. \tag{2.3}$$

In the following, we use the notation $M(x(M), y(M); t)$ or simply $M(t)$ for points $M \in [AB]$. Let us remark that for all points $M \in [AB] \setminus \{A\}$, the complex number $z(M)$ belongs to the upper half plane $\mathbb{C}_+ := \{z = x + iy \in \mathbb{C}; y > 0\}$ which is the support space for the Poincaré model of plane hyperbolic geometry.

In order to understand the geometry of the affine segment $[AB]$ we introduce a sum for any two of its points as follows:

Definition 2.1 i) *The circular sum* in $[AB]$ is the symmetric map $\oplus : [AB] \times [AB] \to [AB]$ given by

$$M_1 \oplus M_2 := z^{-1}(z(t_1 + t_2)) \tag{2.4}$$

where the sum in the right-hand-side is considered modulo $[0, \frac{\pi}{2}]$ in order to assure the existence of the inverse map z^{-1}.

ii) *The complementary point* $M^c \in [AB]$ of M is provided by relation (1.3), with t replaced by $\frac{\pi}{2} - t$.

Remark 2.2

(i) Suppose that the complex number associated to M is defined alternatively as $z(M) := \cos^2\left(\frac{u\pi}{2}\right) + i\sin^2\left(\frac{u\pi}{2}\right)$, $u \in [0,1]$, and that we use the notation $M(u) \in [AB]$. Fix the point $M = M(\alpha) \in [AB]$. The ergodic properties of the map $f_\alpha(u) = u + \alpha(mod\ 1)$ are considered in [8, p. 81]: such a map is ergodic but not mixing when α is an irrational number and the considered measure is the restriction of the Lebesgue measure of $\mathbb{R}$.

(ii) Concerning the complementary point we have:

$$r(B,A;M) = r(A,B;M^c), \quad z(M) + z(M^c) = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}. \tag{2.5}$$

In the following, we study this new product through some examples.

Example 2.3 We have $A(1,0;0)$, $B\left(0,1;\frac{\pi}{2}\right)$ and then B is the complementary point of A with:

$$r(A,B;B) = 0, \quad A \oplus A = A, \quad A \oplus B = B = B \oplus B. \tag{2.6}$$

Moreover, A is the neutral element of the circular sum: $A \oplus M = M$ for all points $M \in [AB]$.

Example 2.4 The midpoint of $[AB]$ is $mid(A,B)\left(\frac{1}{2},\frac{1}{2};\frac{\pi}{4}\right)$ and then $mid(A,B)$ is self-complementary with:

$$r(A,B;mid(A,B)) = 1, \quad mid(A,B)_\oplus^2 = B, \quad B \oplus mid(A,B) = mid(A,B). \tag{2.7}$$

More generally, the square of $z(t)$ in the usual complex algebra $\mathbb{C}$ is:

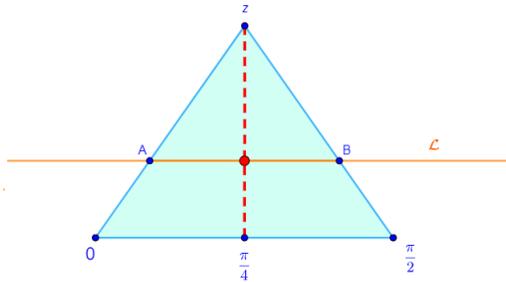$$z(t)^2 = -\cos(2t) + \frac{i}{2}[\sin(2t)]^2. \tag{2.8}$$

Figure 1. The middle point

Example 2.5 In two-dimensional affine geometry, a frame consists of a nondegenerate triangle. So, let us consider the nondegenerate triangle $\Delta = \Delta ABC$ in the Euclidean plane and the median AA'. This way, $A' = mid(B,C)$. Then *the centroid* (or *the center of gravity*) G of Δ is located on $[AA']$ to two-thirds of the distance from A. Hence, working on the segment $[AA']$, we have $G\left(\frac{1}{3},\frac{2}{3};t_G\right)$ with $r(A,B;G) = \frac{1}{2}$ and:

$$\cos t_G = \frac{1}{\sqrt{3}}, \quad \sin t_G = \frac{\sqrt{2}}{\sqrt{3}}, \quad t_G = 54.74°. \tag{2.9}$$

This angle $t_G = 54.74°$ has been studied in [7] and has been referred to as the "magical angle."

Related to this example is another remarkable fact from the geometry of the given triangle: the orthocenter H, the centroid G and the circumcenter O are collinear. More precisely, G lies between H and O with $\overrightarrow{HG} = 2\overrightarrow{GO}$. The line containing these points is *the Euler line* of the triangle and the nine-point center N is the mid-point of HO. A remarkable theorem of Euler proves that the side-lengths of a nonequilateral triangle are fully determined by its incenter I and any two of these four centers on the Euler line.

Example 2.6 Let $b > 0$ and $c > 0$ be respectively the length of sides AC and AB in Δ; also let $A_0 \in [BC]$ be the foot of the bisector of the angle $\hat{A} = \angle BAC$. The bisector theorem says that:

$$\frac{BA_0}{A_0C} = \frac{c}{b} \tag{2.10}$$

and hence on $[BC]$ we have: $x(A_0) = \frac{b}{b+c}$, $y(A_0) = \frac{c}{b+c}$ and $r(B, C; A_0) = \frac{b}{c}$.

3. Dividing the segment into a given ratio

The last two examples from the previous section suggest studying a general point M dividing $[AB]$ in the ratio $k \in \mathbb{R} \setminus \{-1\}$ i.e.:

$$\overrightarrow{AM} = k\overrightarrow{MB}. \tag{3.1}$$

It is well-known that we have:

$$x(M) = \frac{1}{1+k}, \quad y(M) = \frac{k}{1+k}, \quad r(A, B; M) = \frac{1}{k} \tag{3.2}$$

which means that the corresponding angle for M (when $k > 0$) is provided by:

$$\cos t = \frac{1}{\sqrt{1+k}}, \quad \sin t = \frac{\sqrt{k}}{\sqrt{1+k}}. \tag{3.3}$$

Note that for points $M \in [AB], k$ above is in $[0, 1]$. Now, for two points $M_1, M_2 \in [AB]$ with associated $t-$values satisfying t_1 and t_2 with $t_1 + t_2 \leq \pi/2$ and associated $k-$values given by k_1 and k_2, a straightforward computation gives the sum point:

$$M_1 \oplus M_2 \left(\frac{(1 - \sqrt{k_1 k_2})^2}{(1 + k_1)(1 + k_2)}, \frac{(\sqrt{k_1} + \sqrt{k_2})^2}{(1 + k_1)(1 + k_2)}; t_1 + t_2 \right). \tag{3.4}$$

Its corresponding k can be shown to equal:

$$k = \left(\frac{\sqrt{k_1} + \sqrt{k_2}}{1 - \sqrt{k_1 k_2}} \right)^2$$

On the other hand, if $t_1 + t_2 > \pi/2$, then the corresponding k-value for the circular sum point is given by:

$$k = \left(\frac{1 - \sqrt{k_1 k_2}}{\sqrt{k_1} + \sqrt{k_2}} \right)^2. \tag{3.5}$$

Note that $t_1 + t_2 = \pi/2$ corresponds to $k_1 k_2 = 1$.

Let $\mathbb{R}_+^2 = \{(u, v \in \mathbb{R}^2; u > 0, v > 0\}$ be the first quadrant and let H^1 the unit hyperbola $\{(u, v) \in \mathbb{R}_+^2; uv = 1\}$. The two expressions for k above suggests that we introduce two functions, $F : \mathbb{R}_+^2 \setminus H^1 \to (0, +\infty)$ given by:

$$F(u, v) := \left(\frac{\sqrt{u} + \sqrt{v}}{1 - \sqrt{uv}} \right)^2 \tag{3.6}$$

and G which is the reciprocal of F. The corresponding diagonal function of F, for example is

$$F(u, u) = \frac{4u}{(1 - u)^2}, \quad F(\tan^2 t, \tan^2 t) = 2(1 - |z(M)|^2). \tag{3.7}$$

The $1-$level curve associated to both F and G consists of two pieces:

$$C(\pm) : \sqrt{u} + \sqrt{v} = \pm(1 - \sqrt{uv}). \tag{3.8}$$

With the new variables $x = \sqrt{u}$, $y = \sqrt{v}$ it results the positive component of the hyperbola:

$$C(\pm) : xy \pm x \pm y - 1 = 0 \tag{3.9}$$

having the eccentricity $e = \sqrt{2}$; hence $C(\pm)$ is an equilateral hyperbola, see [4, p. 141]. On $C(-)$ there exists two integer points, $(2, 3)$ and $(3, 2)$, and hence on $F(u, v) = 1$ we have two integer points $(4, 9)$ and $(9, 4)$ which yields:

Theorem 3.1 *The midpoint $mid(A, B)$ is the circular sum of the points $M_1(k_1 = 4)$ and $M_2(k_2 = 9)$ having the corresponding angles*:

$$\cos t_1 = \frac{1}{\sqrt{5}}, \quad \sin t_1 = \frac{2}{\sqrt{5}}, \quad \cos t_2 = \frac{1}{\sqrt{10}}, \quad \sin t_2 = \frac{3}{\sqrt{10}}. \tag{3.10}$$

Concretely, $t_1 = 63.43°$ and $t_2 = 71.57°$ with the sum $t_1 + t_2 = 135° = 90° + 45°$ and indeed $t_1 + t_2 = \frac{\pi}{4} \left(mod \ \frac{\pi}{2} \right)$. We consider as a first open problem if this equality can be alternatively considered as being the clock equality $1 \equiv 4 + 9 \ (mod \ 12)$; namely, we have in $\mathbb{Z}_{12}$ the "Pythagorean theorem" $1^2 = 2^2 + 3^2$ from the more general Pythagorean theorem is $1^2 = n^2 + (n + 1)^2 \ (mod \ 2n(n + 1))$ for any $n \geq 1$. The pair $(x = 3, y = 2)$ is also a nontrivial solution of the Pell equation $x^2 - 2y^2 = 1$ representing again a hyperbola having the eccentricity $e = \sqrt{\frac{3}{2}}$.

Remark 3.2

(i) In the relation (2.24) of the paper [5] is defined a noninternal composition law on $(0, 1)$ through:

$$u_1 \odot u_2 := \sqrt{\frac{1 - (u_1 u_2)^2}{(u_1)^2 + (u_2)^2}} \tag{3.11}$$

where the noninternal means that $u_1 \odot u_2 \notin (0, 1)$; for example $1/2 \odot 1/2 = \sqrt{\frac{3}{2}} > 1$. A comparison with (2.5) gives a relationship between these composition laws:

$$k_1 \oplus k_2 = (\sqrt[4]{k_1} \odot \sqrt[4]{k_2})^{-4} \tag{3.12}$$

when $k_1, k_2 \in (0, 1)$.

(ii) The canonical form of the equilateral hyperbola $C(-)$ is:

$$\begin{cases} C(-): X^2 - Y^2 - 8 = 0, & (X_0, Y_0) = (3, 1) \in C(-) \\ X := x + y - 2, & Y := x - y, \quad x = \frac{1}{2}(X + Y) + 1, \quad y = \frac{1}{2}(X - Y) + 1. \end{cases} \tag{3.13}$$

The integral point $(X_0, Y_0) \in C(-)$ provides the integral point $(x_0, y_0) = (3, 2)$ through the affine formulae above. An amazing fact holds now: if J is an almost complex structure on the (real 4-dimensional but nonorientable) manifold $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, then its first Chern class has the expression $c_1(J) = (X, Y)$ with (X, Y) satisfying exactly the equation (2.14) of $C(-)$; see [1, p. 2153]. The standard almost complex structure J_0 on $\mathbb{C}P^2$ has $c_1(J_0) = 3$.

(iii) The notion of Mazur-Ulam space is generalized in [2] by replacing the midpoint with the point M with $x(M) = \lambda \in (0, 1)$. A second open problem is if the $\frac{4}{5}$-Mazur-Ulam space and/or the $\frac{9}{10}$-Mazur-Ulam space have some special features.

(iv) Three-dimensional Penrose tilings using two rhombohedra of equal sides but with angles t_1 and $\pi - t_1$ are obtained in [3]. A third open problem is if the triangles with the angles $(A = \frac{\pi}{4}, B = t_1, C = t_2)$ has some remarkable properties. Supposing that its circumcircle is exactly the unit circle S^1 then its sides are irrational numbers:

$$a = \sqrt{2} < b = \frac{4}{\sqrt{5}} < c = \frac{6}{\sqrt{10}}; \quad 3ab = 4c \tag{3.14}$$

while its area is a rational number: $S = \frac{6}{5}$. The barycentric coordinates of the points $D \in (BA)$, $E \in (CA)$ from the figure below are:

$$x(D) = \frac{4}{9} < y(D) = \frac{5}{9}, \quad x(E) = \frac{3}{8} < y(E) = \frac{5}{8}. \tag{3.15}$$

It is worth to remark the unitary complex number:

$$z_A = \frac{1}{5}(3 + 4i) = \cos\left(2t_3 - \frac{\pi}{2}\right) + i\sin\left(2t_3 - \frac{\pi}{2}\right), \quad 2t_3 - \frac{\pi}{2} = 53.14 \tag{2.17}$$

and its relationship with the Pythagorean triple $(3, 4, 5)$.

Also, it is easy to find the barycentric coordinates of the point O with respect to the nondegerate triangle ABC; namely, since:

$$S(OBC) = \frac{1}{2}, \quad S(OCA) = \frac{2}{5}, \quad S(OAB) = \frac{3}{10} \tag{3.16}$$

we have:

$$\begin{cases} x_{ABC}(O) := \frac{S(OBC)}{S(ABC)} = \frac{5}{12}, \\ y_{ABC}(O) := \frac{S(OCA)}{S(ABC)} = \frac{1}{3} = \frac{4}{12}, \\ z_{ABC}(O) := \frac{S(OAB)}{S(ABC)} = \frac{1}{4} = \frac{3}{12}. \end{cases} \tag{3.17}$$

Since the barycentric coordinates of O can be considered as a probability vector we compute its *entropy*:

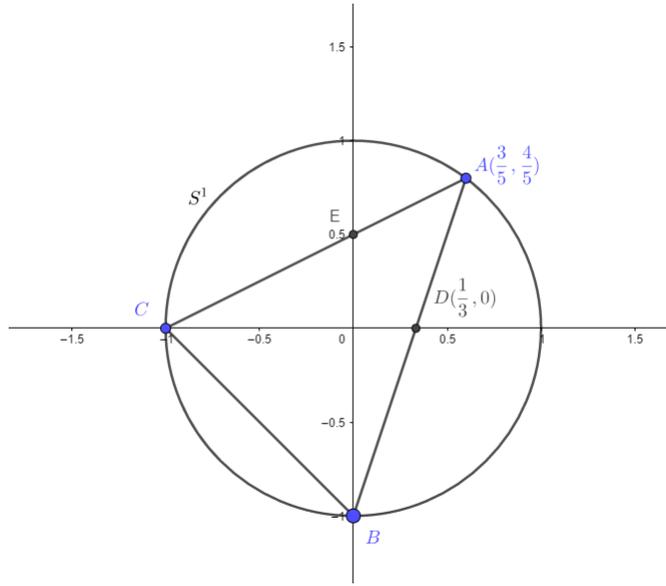$$H_{ABC}(O) = \frac{5}{12}\ln(12/5) + \frac{1}{4}\ln(4) + \frac{1}{3}\ln(3) \simeq 1.0775.$$

Figure 2. The coordinates of the vertices

4. Finding points and sums with integer ratios

The purpose of this section is to find all triples of natural numbers (k_1, k_2, k) satisfying the relation $k_1 \oplus k_2 = k$. We begin with the following:

Lemma 4.1 *If the natural numbers k_1, k_2, k satisfy the equation:*

$$k_1 \oplus k_2 = k$$

then $\sqrt{k_1 k_2}$, $\sqrt{k k_1}$ and $\sqrt{k k_2}$ are natural numbers as well.

Proof Since k_1 and k_2 are nonnegative integers, we must have $k_1 k_2 > 1$ unless $k_1 = k_2 = 1$. However if $k_1 = k_2 = 1$, then $k_1 \oplus k_2$ is not defined. With $k_1 k_2 > 1$, we have:

$$\left(\frac{1 - \sqrt{k_1 k_2}}{\sqrt{k_1} + \sqrt{k_2}} \right)^2 = k.$$

As $1 - \sqrt{k_1 k_2} < 0$ the relation above implies:

$$\frac{1 - \sqrt{k_1 k_2}}{\sqrt{k_1} + \sqrt{k_2}} = -\sqrt{k}$$

which is equivalent to:

$$\sqrt{k_1 k_2} - 1 = \sqrt{k}(\sqrt{k_1} + \sqrt{k_2}). \tag{4.1}$$

Squaring both sides in the relation above and rearranging, we obtain:

$$\sqrt{k_1 k_2} = \frac{1 + k_1 k_2 - k(k_1 + k_2)}{2(k + 1)} \in \mathbb{Q}.$$

Thus, we must have that $\sqrt{k_1 k_2}$ is an integer. Clearly, by relation (4.1), this shows that $\sqrt{kk_1} + \sqrt{kk_2}$ must also be an integer. But then, as is well known, we must have that $\sqrt{kk_1}$ and $\sqrt{kk_2}$ are integers.

Now, if we let $m_1 = \sqrt{kk_1}$ and $m_2 = \sqrt{kk_2}, m_1, m_2 \in \mathbb{N}$, then we have $\sqrt{k_1} = m_1/\sqrt{k}$ and $\sqrt{k_2} = m_2/\sqrt{k}$. This way, relation (4.1) becomes:

$$\frac{m_1 m_2}{k} - 1 = \sqrt{k}\frac{m_1 + m_2}{\sqrt{k}} \iff k = \frac{m_1 m_2}{m_1 + m_2 + 1} \in \mathbb{N}.$$

Note that, since $m_1^2 = kk_1$ and $m_2^2 = kk_2$, we have that $k = \frac{m_1 m_2}{m_1 + m_2 + 1}$ is a factor of both m_1^2 and m_2^2.

Lemma 4.2 If m_1, m_2 are natural numbers such that $\frac{m_1 m_2}{m_1 + m_2 + 1}$ is an integer that divides both m_1^2 and m_2^2, then $\frac{m_1 m_2}{m_1 + m_2 + 1}$ is a perfect square.

Proof If $d = (m_1, m_2), d \geq 1$, then $m_i = dn_i, n_i \in \mathbb{N}, i = 1, 2, (n_1, n_2) = 1$. The fact that m_i^2 is a multiple of $\frac{m_1 m_2}{m_1 + m_2 + 1}$ can be written as

$$n_i^2 d^2 = \alpha_i \frac{d^2 n_1 n_2}{d(n_1 + n_2) + 1},$$

for some $\alpha_i \in \mathbb{N}, i = 1, 2$. This is equivalent to:

$$n_i[d(n_1 + n_2) + 1] = \alpha_i n_j, i \neq j.$$

Since $(n_1, n_2) = 1$, the relation above implies that $d(n_1 + n_2) + 1$ is a multiple of both n_1 and n_2. Hence, due again to the fact that n_1 and n_2 are relatively prime, we obtain that $d(n_1 + n_2 + 1)$ is a multiple of $n_1 n_2$.

However, because:

$$\frac{m_1 m_2}{m_1 + m_2 + 1} = \frac{d^2 n_1 n_2}{d(n_1 + n_2) + 1}$$

is an integer, due to the fact that $(d^2, d(n_1 + n_2) + 1) = 1$, it follows that $n_1 n_2$ is a multiple of $d(n_1 + n_2) + 1$. By the assertion above, this implies that $n_1 n_2 = d(n_1 + n_2) + 1$. This way

$$\frac{mn}{m + n + 1} = \frac{d^2 n_1 n_2}{d(n_1 + n_2) + 1} = d^2.$$

Based on the results above we have:

Theorem 4.3 *If (k_1, k_2, k) is a triple of natural numbers satisfying:*

$$k_1 \oplus k_2 = k,$$

then k_1, k_2 and k are perfect squares.

Proof With the same notations as above, by the previous lemma, we have that k must be a perfect square. Since $m_i^2 = (\sqrt{k})^2 k_i, i = 1, 2$, it follows that k_1 and k_2 must also be perfect squares.

Next, we characterize all triples (k_1, k_2, k) as above.

Theorem 4.4 *The triple of natural numbers (k_1, k_2, k) satisfies the equation:*

$$k_1 \oplus k_2 = k$$

if and only if:

$$k_1 = m^2, k_2 = \left(\frac{m^2+1}{r} - m \right)^2, k = (m-r)^2,$$

where $m \geq 2$ is some natural number and r is a divisor of $m^2 + 1$.

Proof By the previous theorem, we may assume $k_1 = m^2, k_2 = n^2, k = p^2$, for some $m, n, p \in \mathbb{N}^*$. It is easy to see that $m \geq 2$ since otherwise we would have $p = \frac{n-1}{n+1} < 1$. Equation 4.1 can now be written as

$$mn - 1 = p(m+n) \iff m^2 + 1 = (m+n)(m-p).$$

By letting $r = m - p$, we have that r is a factor of $m^2 + 1$ and $p = m - r > 0$. This way,

$$n = \frac{m^2+1}{r} - m$$

and the formulas in the theorem follow.

Remark 4.5 The fact that the midpoint of segment $[AB]$ can be uniquely written as the circular sum of points $M_1(k_1 = 9)$ and $M_2(k_2 = 4)$ can also be ascertained based on the theorem above. Indeed if $m - r = 1$ and $r = m - 1$ is a factor of $m^2 + 1$, then $m - 1$ is a factor of $2 = m^2 + 1 - (m-1)(m+1)$, which implies that $m = 3$ or $m = 2$. This way, $k_1 = 9, k_2 = 10/2 - 1 = 4, k = 1$ or $k_1 = 4, k_2 = 9, k = 1$.

Remark 4.6 Note that it is impossible to have $k_1 = k_2$ since this would lead to:

$$2\sqrt{k} = \frac{k_1 - 1}{\sqrt{k_1}} = \sqrt{k_1} - \frac{1}{\sqrt{k_1}},$$

where the left side is an integer while the right side is not.

Remark 4.7 Triples of integers satisfying the equation $k_1 \oplus k_2 = k$ can be found rather easily based on the last theorem. For example, choosing $m = 4$ gives $m^2 + 1 = 17$, so the only choice for r is $r = 1$. Thus, $n = (4^2 + 1)/1 - 4 = 13$ and $k_1 = 16, k_2 = 169, k = (4-1)^2 = 9$.

Remark 4.8 In specific cases, we can identify the triples (k_1, k_2, k) with a specified k. For example, if $k = 4$ then $m - r = \pm 2$. In order for $r = m \pm 2$ to be a factor of $m^2 + 1 = (m^2 - 4) + 5$, we must have that $m \pm 2$ is a factor of 5. For $r = m - 2$, we have $m - 2 = 1$ or $m - 2 = 5$, resulting in the following triples: $(9, 49, 4)$ and $(49, 9, 4)$. For $r = m + 2$, we have $m + 2 = 5$, resulting in the $(9, 49, 4)$ triple again. In general, for a random perfect square k, r needs to be a factor of $k + 1$.

Conflicts of interest The authors declare no conflict of interest.

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