

Some remarks on the moment curves in \mathbb{R}^4 and \mathbb{R}^3

 M. Crasmareanu^{1*}, 
¹Faculty of Mathematics, University "Al. I. Cuza", Iasi, 700506, Romania, <http://www.math.uaic.ro/~mcrasm>

ABSTRACT

The moment curves and their normalizations are key tools in obtaining the famous Kac formula from the theory of random polynomials. We study here the normalized moment curves $\Gamma_n \in S^n$ in the low dimensions n where S^n is the Euclidean n -dimensional unit sphere; more precisely we consider $n = 3$ and $n = 2$. First, we compute the image of the normalized moment curve Γ_3 under the well-known Hopf fibre map and show that this remarkable map reduces the length of Γ_3 . Second, we analyze the curve Γ_2 using the theory of spherical Legendre curves. An image of Γ_2 is included.

Mathematics Subject Classification (2020): 53A04 53A45 53A55

Keywords: moment curve – Hopf map – spherical curve

1. INTRODUCTION

The setting of this paper is provided by the space \mathbb{R}^{n+1} , $n \in \mathbb{N}^* = \{1, 2, \dots\}$, which is an Euclidean vector space with respect to the canonical inner product:

$$\begin{cases} \langle u, v \rangle := u^1 v^1 + \dots + u^{n+1} v^{n+1}, u = (u^1, \dots, u^{n+1}), v = (v^1, \dots, v^{n+1}) \in \mathbb{R}^{n+1}, \\ 0 \leq \|u\|^2 := \langle u, u \rangle. \end{cases} \quad (1.1)$$

A special curve, called *moment curve*, is defined in (Edelman et al. 1995, p. 5-6) as:

$$\gamma_n : \mathbb{R} \rightarrow \mathbb{R}^{n+1}, \quad \gamma_n(t) := (1, t, \dots, t^n). \quad (1.2)$$

It is a regular curve since the norm of its derivative is strictly positive: $\|\gamma'_n(t)\| > 0$ for all $t \in \mathbb{R}$. Also, *the normalized moment curve* is:

$$\Gamma_n : \mathbb{R} \rightarrow S^n, \quad \Gamma_n(t) := \frac{\gamma_n(t)}{\|\gamma_n(t)\|}. \quad (1.3)$$

where S^n is the unit sphere of $\mathbb{E}^{n+1} := (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$. A main result of Edelman et al. (1995) is that the expected number of real zeros E_n of a random polynomial of degree n if the coefficients are independent and distributed normally is given by:

$$E_n = \frac{L(\Gamma_n)}{\pi} \quad (1.4)$$

where $L(\Gamma_n)$ is the Euclidean length of Γ_n . The first few values of E_n are: 1, 1.29702, 1.49276, 1.64049. Also, it is easy to express the image of the curve Γ_n through the stereographic projection φ_N from the North pole $N(0, \dots, 0, 1) \in S^n$:

$$\varphi_N(\Gamma_n(t)) = \frac{\gamma_{n-1}(t)}{\sqrt{1+t^2+\dots+t^{2n}-t^{2n}}} \in \mathbb{R}^n. \quad (1.5)$$

The present work concerns with the normalized moment curve Γ_n for the low values $n = 3$ and $n = 2$. More precisely, when $n = 3$ we use the well-known Hopf bundle and as result we obtain a lower length. The second case is treated in the framework of spherical Legendre curves since Γ_2 appears as a frontal curve in this theory. For both values of n we study the image of the normalized moment curve through the Veronese map. Other spherical curves in the case $n = 2$, as the Clelia curve and the spherical nephroid, are studied in Crasmareanu (2024).

Corresponding Author: M. Crasmareanu E-mail: mcrasm@uaic.ro

Submitted: 28.01.2025 • **Revision Requested:** 14.05.2025 • **Last Revision Received:** 16.05.2025 • **Accepted:** 00.00.0000 • **Published Online:** 00.00.0000



This article is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License (CC BY-NC 4.0)

2. THE MOMENT CURVE IN \mathbb{R}^4 AND THE HOPF MAP

The setting of this section is provided by $n = 3$ since the sphere $S^3 = SU(2)$ is the total space of the famous Hopf bundle $H : S^3 \subset \mathbb{C}^2 \rightarrow S^2 \left(\frac{1}{2}\right) \subset \mathbb{R} \times \mathbb{C}$:

$$H(z, w) = \left(\frac{1}{2}(|z|^2 - |w|^2), z\bar{w}\right). \tag{2.1}$$

We express the curve Γ_3 as:

$$\Gamma_3(t) = (z(t), w(t)), \quad z(t) = \frac{1+it}{\sqrt{(1+t^2)(1+t^4)}}, \quad w(t) = t^2z(t), \quad |z(t)|^2 = \frac{1}{1+t^4} \tag{2.2}$$

and then a straightforward computation gives:

$$H(\Gamma_3(t)) = \frac{1}{2} \left(\frac{1-t^4}{1+t^4}, \frac{2t^2}{1+t^4}, 0\right) \in S^2 \left(\frac{1}{2}\right) \subset \mathbb{R} \times \mathbb{R} \times \{0\}. \tag{2.3}$$

Returning to the expectation numbers it is very easy to see that $E_1 = 1$. Indeed, for $\Gamma_1(t) = \frac{1}{\sqrt{1+t^2}}(1, t) \in S^1$ we consider the change of parameter $t = \tan \varphi$. It follows:

$$\Gamma_1(\varphi) = (\cos \varphi, \sin \varphi) \in S^1$$

and the condition $\cos \varphi = \frac{1}{\sqrt{1+t^2}} > 0$ yields the domain $\varphi \in (0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$. Then we have the length $L(\gamma_1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ and it results $E_1 = 1$.

For the case $n = 3$ we perform the change of parameter $t^2 = \tan \varphi \geq 0$ and then:

$$H(\Gamma_3(\varphi)) = \frac{1}{2}(\cos 2\varphi, \sin 2\varphi, 0) \in S^2 \left(\frac{1}{2}\right) \tag{2.4}$$

but now $\varphi \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$. It follows that $2\varphi \in (0, \pi) \cup (2\pi, 3\pi)$ which implies that:

$$L(H \circ \Gamma_3) = \frac{1}{2}(\pi + \pi) = \pi < L(\Gamma_3) \simeq 1.49\pi. \tag{2.5}$$

Hence, the first conclusion of this section is that the Hopf map reduces the length of the normalized moment curve Γ_3 .

Expressing $z = z_1 = q_1 + ip_1$, $w = z_2 = q_2 + ip_2$ the standard *contact form of S^3* is the restriction of the 1-form:

$$\lambda_0 := \frac{1}{2}(p_1dq_1 - q_1dp_1 + p_2dq_2 - q_2dp_2) \tag{2.6}$$

to S^3 . The tangent vector field of the normalized moment curve is:

$$\Gamma'_3(t) = \frac{1}{(1+t^2+t^4)^{\frac{3}{2}}}(-t-t^3, t, 2t+t^3+t^5, 3t^2+2t^4+2t^6) \tag{2.7}$$

and then:

$$\lambda_0(\Gamma'_3(t)) = \frac{-t(1+t+2t^3+t^5+t^7)}{(1+t^2+t^4)^2}. \tag{2.8}$$

As element in the Lie group $SU(2)$ the normalized moment curve Γ_3 is:

$$\Gamma_3(t) = \frac{1}{\sqrt{(1+t^2)(1+t^4)}} \begin{pmatrix} t^2(t+i) & t+i \\ -(t-i) & t^2(t-i) \end{pmatrix} \in SU(2), \quad Tr\Gamma_3(t) = 2Im(w(t)). \tag{2.9}$$

Secondly, we recall the complex Veronese map $V : S^3 \subset \mathbb{C}^2 \rightarrow S^5 \subset \mathbb{C}^3$:

$$\begin{aligned} V(z = x + iy, w = u + iv) &:= (z^2, \sqrt{2}zw, w^2) = \\ &= (x^2 - y^2, 2xy, \sqrt{2}(xu - yv), \sqrt{2}(xv + yu), u^2 - v^2, 2uv). \end{aligned} \tag{2.10}$$

We obtain the new spherical curve:

$$V \circ \Gamma_3(t) = \frac{1}{(1+t^2)(1+t^4)}(1-t^2, 2t, \sqrt{2}t^2(1-t), \sqrt{2}t^2(1+t), t^4(1-t^2), 2t^5) \in S^5. \tag{2.11}$$

3. THE MOMENT CURVE IN \mathbb{R}^3 AND THE SPHERICAL LEGENDRE CURVES

Now we consider $n = 2$ and recall that the unit spherical bundle is a compact 3-dimensional contact metric manifold given by:

$$T_1S^2 := \{(u, v) \in S^2 \times S^2; \langle u, v \rangle = 0\} \quad (3.1)$$

see for example [Crasmareanu \(2016\)](#). There is a natural action of $O(3)$ on T_1S^2 :

$$(A, (u, v)) \in O(3) \times T_1S^2 \rightarrow (Au, Av) \in T_1S^2, \quad \langle Au, Av \rangle = \langle u, v \rangle = 0. \quad (3.2)$$

For example, the complex Veronese map applied on a pair $(z, w) \in S^3$ with $|z| = |w| = \frac{1}{\sqrt{2}}$ gives the symmetric orthogonal matrix:

$$A = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \in \text{Sym}(3) \cap O^-(3),$$

where $O^-(3) := \{\Gamma \in O(3); \det \Gamma = -1\}$ and $\text{Sym}(3)$ is the linear subspace of $M_3(\mathbb{R})$ consisting in symmetric matrices. Considering this matrix as representing a conic in the plane \mathbb{E}^2 (see the formalism of [Crasmareanu \(2021\)](#)) it is a hyperbola.

The general notion of Legendre curves associated to a contact form is well-known, but we will work directly in our framework using the approach of [Takahashi \(2016\)](#) (see also [CrasmareanuChapter \(2024\)](#)):

Definition 3.1 The smooth map $LC := (\gamma, \nu) : I \subseteq \mathbb{R} \rightarrow T_1S^2$, $t \in I \rightarrow (\gamma(t), \nu(t))$ is a *spherical Legendre curve* if $\langle \gamma'(t), \nu(t) \rangle = 0$ for all t in the open interval I . The map γ is called *the frontal* and ν is *the dual of γ* .

Since \mathbb{R}^3 is endowed also with the cross product \times we define $\mu = \gamma \times \nu$ and hence the triple $\mathcal{F} := \{\gamma, \nu, \mu\}^t$ is an positive oriented *moving frame* along the frontal γ ; here t means the transposition, so \mathcal{F} is a column matrix. Its moving equation is provided by the Proposition 2.2. of ?:

$$\frac{d}{dt} \mathcal{F}(t) = \begin{pmatrix} 0 & 0 & k_1(t) \\ 0 & 0 & k_2(t) \\ -k_1(t) & -k_2(t) & 0 \end{pmatrix} \mathcal{F}(t), \quad \begin{pmatrix} 0 & 0 & k_1(t) \\ 0 & 0 & k_2(t) \\ -k_1(t) & -k_2(t) & 0 \end{pmatrix} \in o(3). \quad (3.3)$$

The pair of smooth functions (k_1, k_2) is called *the curvature* of the spherical Legendre curve $LC = (\gamma, \nu)$. Sometimes, it is more useful to denote a given LC with all its elements as $LC = (\gamma, \nu; \mu)$.

Let us consider now the Example 2.8. of [Takahashi \(2016\)](#); equivalently the Example 2.6 of [CrasmareanuChapter \(2024\)](#). Starting with the natural numbers (k, m, n) satisfying $m = k + n$ the LC is defined as:

$$\gamma(t) = \frac{1}{\sqrt{1+t^{2n}+t^{2m}}}(1, t^n, t^m), \quad \nu(t) = \frac{1}{\sqrt{n^2+m^2t^{2k}+k^2t^{2m}}}(kt^m, -mt^k, n), \quad t \in \mathbb{R} \quad (3.4)$$

with the associated curvature pair:

$$k_1(t) = -\frac{t^{n-1}\sqrt{n^2+m^2t^{2k}+k^2t^{2m}}}{1+t^{2n}+t^{2m}}, \quad k_2(t) = \frac{kmnt^{k-1}\sqrt{1+t^{2n}+t^{2m}}}{n^2+m^2t^{2k}+k^2t^{2m}}. \quad (3.5)$$

In fact, we have:

$$\nu = \frac{\gamma \times \gamma'}{\|\gamma \times \gamma'\|} = \frac{\gamma \times \gamma'}{\|\gamma'\|} = \gamma \times T \quad (3.6)$$

where $\{T, N, B\}$ is the Frenet frame of the γ as bi-regular space curve. We point out that the general normalized moment curve Γ_n is exactly the tangent vector field T of the curve:

$$t \in \mathbb{R} \rightarrow \left(\int \gamma_n \right)(t) = \left(t, \frac{t^2}{2}, \dots, \frac{t^{n+1}}{n+1} \right).$$

Returning to our normalized moment curve Γ_2 it results that it corresponds exactly to the curve γ for $k = n = 1 < m = 2$; therefore ν can be called *the Legendre dual* of the normalized moment curve Γ_3 . We have:

$$\Gamma_2'(t) = \frac{1}{(1+t^2+t^4)^{\frac{3}{2}}}(-t+2t^3, 1-t^4, 2t+t^3), \quad \nu(t) = \frac{1}{\sqrt{1+4t^2+t^4}}(t^2, -2t, 1) \quad (3.7)$$

and the curvature pair:

$$k_1(t) = -\frac{\sqrt{1+4t^2+t^4}}{1+t^2+t^4} < 0, \quad k_2(t) = \frac{2\sqrt{1+t^2+t^4}}{1+4t^2+t^4} > 0. \quad (3.8)$$

The length of the curve Γ_2 is approximately $4.07472 < 2\pi$.

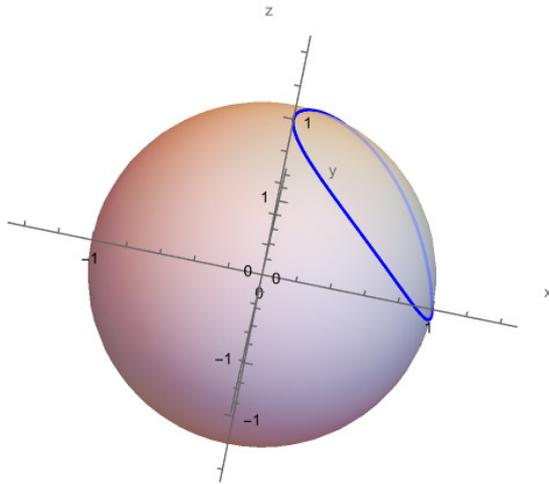


Figure 1. The curve Γ_2 .

Recall the standard parametrization of S^2 as regular surface:

$$S^2 : \bar{r}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u), \quad u \in (0, 2\pi), \quad v \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \quad (3.9)$$

The normalized moment curve $\Gamma_2 \in S^2$ is given then by $u = u(t), v = v(t)$ with:

$$\begin{cases} \cos u(t) = \frac{\sqrt{1+t^2}}{\sqrt{1+t^2+t^4}} > 0, & \sin u(t) = \frac{t^2}{\sqrt{1+t^2+t^4}} \geq 0, \\ \cos v(t) = \frac{1}{\sqrt{1+t^2}} > 0, & \sin v(t) = \frac{t}{\sqrt{1+t^2}}. \end{cases} \quad (3.10)$$

With the change of parameter $t = \tan \varphi$ we have also:

$$\begin{cases} \Gamma_2(\varphi) = \frac{1}{\sqrt{\cos^2 \varphi + \sin^4 \varphi}} (\cos^2 \varphi, \sin \varphi \cos \varphi, \sin^2 \varphi), \\ \nu(\varphi) = \frac{1}{\sqrt{1+2\cos^2 \varphi \sin^2 \varphi}} (\sin^2 \varphi, -\sin 2\varphi, \cos^2 \varphi). \end{cases} \quad (3.11)$$

We point out that Γ_2 is not a Viviani curve being the intersection of the sphere S^2 not with a cylinder but with the elliptic cone $EC : xz = y^2$.

Moreover, the above spherical curves can be studied through the Veronese map:

$$V : S^2 \subset \mathbb{R}^3 \rightarrow S^4 \subset \mathbb{R}^5, V(u, v, w) := \left(\sqrt{3}vw, \sqrt{3}wu, \sqrt{3}uv, \frac{\sqrt{3}}{2}(u^2 - v^2), w^2 - \frac{u^2 + v^2}{2} \right). \quad (3.12)$$

Then:

$$\begin{cases} V \circ \Gamma_2(t) = \frac{1}{1+t^2+t^4} \left(\sqrt{3}t^3, \sqrt{3}t^2, \sqrt{3}t, \frac{\sqrt{3}}{2}(1-t^2), t^4 - \frac{1+t^2}{2} \right), \\ V \circ \nu(t) = \frac{1}{1+2t^2+t^4} \left(-2\sqrt{3}t, \sqrt{3}t^2, -2\sqrt{3}t^3, \frac{\sqrt{3}t^2}{2}(t^2-4), 1 - \frac{t^4+4t^2}{2} \right). \end{cases} \quad (3.13)$$

An important remark is that the new curves, $V \circ \Gamma_2$ and $V \circ \nu$, are not orthogonal in \mathbb{E}^6 .

ACKNOWLEDGEMENTS

The author is grateful to two anonymous referees for the careful reading of an initial version of the paper and very useful suggestions.

LIST OF AUTHOR ORCIDS

M. Crasmareanu <https://orcid.org/0000-0002-5230-2751>

REFERENCES

Crasmareanu, M., 2016, *Chin. Ann. Math., Ser. B*, 37, no. 3, 419-426.

Crasmareanu, M., 2021, *Mat. Vesn.*, 73, no. 4, 256-267.

Crasmareanu, M., 2024, *Turkish J. Math.*, 48, no. 4, 749-756.

Crasmareanu, M., 2024, Chapter in: Bang-Yen Chen et al (Eds.), "Geometry of Submanifolds and Applications", Infosys Science Foundation Series, Springer, 197-207.

Edelman, A., Kostlan, E., 1995, *Bull. Am. Math. Soc.*, 32, no. 1, 1-37.

Takahashi, M., 2016, In: Nabarro, Ana Claudia et al (Eds.), "Real and complex singularities", *AMS Contemporary Mathematics* 675, 337-355.