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The Cardano-golden ratio and the associated curves

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Abstract The aim of this paper is to introduce and study the cubic real polynomials P having as Cardano resolvent exactly the quadratic equation providing the well-known golden ratio Φ. One obtains that these polynomials form a 1-parameter family and the unique positive root of the depressed case is called *Cardano-golden ratio*. We generalize this cubic depressed polynomial to arbitrary grade $n \geq 3$. Also for this depressed polynomial a cubic curve is naturally associated. A regular curve and a regular surface in $\mathbb{R}^3$, both called *golden*, are defined and studied from the point of view of differential geometry.

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1 Introduction

It is commonly recognized that Gerolamo Cardano, by using informations from Scipione del Ferro and Niccolo Fontana Tartaglia, published the first formula for solving cubic equations in his book *Ars Magna* from 1545. In this approach a given monic (and reduced) cubic polynomial P_d is solved through a monic quadratic polynomial $CR(P_d)$ which we call *the Cardano resolvent* of P_d; see, for example, the excellent paper [1] or the book [5].

The starting point of this note is to consider the quadratic polynomial $CR(P_d)$ as being exactly the polynomial P^2_{golden} defining the golden ratio (or mean) $\Phi = \frac{1}{2}(\sqrt{5}+1)$. We call P_d (as well as the general cubic polynomials P with P_d as depressed form) as being *Cardano-golden polynomial*. We express completely this family obtaining its 1-parameter dependence. We introduce also the cubic curve $\mathcal{C}(golden) : y^2 = P_d(x)$ with P_d the Cardano-golden depressed polynomial; we found $\mathcal{C}(golden)$ as elliptic curve on the database https://www.lmfdb.org/.

In the last section we define a regular curve and a regular surface in the Euclidean space $\mathbb{R}^3$, both called *golden* due to the relationship with the general Cardano-golden polynomial. These objects are studied from the point of view of differential geometry: for the golden curve we compute its curvature and torsion while for the golden surface we compute its Gaussian curvature.

2 The Cardano-golden cubic polynomial and a generalization

Fix a natural number $n \in \mathbb{N}^*$. The general setting of this work is provided by the n-dimensional real linear space of monic polynomials of degree n:

$$\mathbb{R}_n^{monic}[x] := \{P(x) = x^n + a_1 x^{n-1} + ... + a_n; a_1, ..., a_n \in \mathbb{R}\}.$$

Our study focuses on a fixed cubic polynomial $P \in \mathbb{R}_3^{monic}[x]$:

$$P(x) := x^3 + Ax^2 + Bx + C \tag{2.1}$$

which with the well-known Tschirnhausen transformation $x = y - \frac{A}{3}$ has the *reduced* form (here the subscript d means *depressed*):

$$P_d(y) := y^3 + py + q, \quad p = B - \frac{A^2}{3} \leq B, \quad q = C - \frac{AB}{3} + \frac{2A^3}{27}. \tag{2.2}$$

We recall very briefly the study of P_d. Being a cubic polynomial it has at least a real root y_0. We consider now an associated quadratic polynomial:

$$P_0(u) = u^2 - y_0 u - \frac{p}{3} \in \mathbb{R}_2^{monic}[u], \tag{2.3}$$

and hence its roots $\alpha, \beta \in \mathbb{C}$ satisfy:

$$\alpha + \beta = y_0, \quad \alpha\beta = -\frac{p}{3}. \tag{2.4}$$

From $P_d(y_0) = 0$ it follows immediately:

$$\alpha^3 + \beta^3 = -q, \quad \alpha^3\beta^3 = -\frac{p^3}{27}, \tag{2.5}$$

which means that α^3 and β^3 are the (complex) roots of the quadratic polynomial

$$CR(P_d)(z) := z^2 + qz - \frac{p^3}{27}. \tag{2.6}$$

Let us call the polynomial $CR(P_d) \in \mathbb{R}_2^{monic}[z]$ as being the *Cardano resolvent* of P_d since its roots:

$$z_{\pm} := -\frac{q}{2} \pm \sqrt{\frac{\Delta}{4 \cdot 27}}, \quad \Delta := 4p^3 + 27q^2 \tag{2.7}$$

give the well-known Cardano formula of y_0:

$$y_0 = \sqrt[3]{z_-} + \sqrt[3]{z_+}. \tag{2.8}$$

Remark 2.1. There exists also a *Lagrange resolvent* of P_d: $Z^2 + 27qZ - 27p^3 = 0$, but the map $Z = 27z$ transforms this resolvent into the Cardano resolvent.

A famous quadratic polynomial is ([4]):

$$P^2_{golden}(X) = X^2 - X - 1 \tag{2.9}$$

since its positive root is exactly *the golden ratio*:

$$\Phi := \frac{1+\sqrt{5}}{2} \simeq 1.618. \tag{2.10}$$

Inspired by this fact we introduce:

Definition 2.1. *The polynomial P (or more precisely P_d) is called Cardano-golden polynomial if its Cardano resolvent is*

$$CR(P_d) = P^2_{golden}. \tag{2.11}$$

We have directly the relations:

$$q = -1, \quad p^3 = 27, \tag{2.12}$$

which yield:

Proposition 2.1. *i) The Cardano-golden depressed polynomial*

$$P_d(x) = x^3 + 3x - 1 \tag{2.13}$$

has one real root denoted Φ^C and two complex roots:

$$\Phi^C = \sqrt[3]{\Phi} - \frac{1}{\sqrt[3]{\Phi}} \simeq 0.32219 > 0, \quad z_{\pm} = -0.1611 \pm 1.7544i \in \mathbb{C} \setminus \mathbb{R}. \tag{2.14}$$

ii) The general class of Cardano-golden cubic polynomials P is a 1-parameter family provided by:

$$P_A(x) = x^3 + Ax^2 + \frac{A^2+9}{3}x + \frac{A^3}{27} + A - 1 = \left(x + \frac{A}{3}\right)^3 + 3x + A - 1, \quad A \in \mathbb{R}. \tag{2.15}$$

Definition 2.2. *Let us call Φ^C as Cardano-golden ratio.*

Remark 2.2. i) The continued fraction expression of Φ^C is:

$$\Phi^C = [0; 3, 9, 1, 1, 1, 2, 1, 2, 3, 6, \ldots]. \tag{2.16}$$

The following matrix belongs to the Lie group $SL(3, \mathbb{Z})$ and has the characteristic polynomial $-P_d$:

$$\Gamma_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{pmatrix}.$$

ii) A general cubic curve:

$$\mathcal{C} : y^2 = P_d(x) = x^3 + px + q \tag{2.17}$$

has the discriminant Δ defined in the relation (2.7); see also [2] or [3]. Hence we define the *Cardano-golden cubic curve*:

$$\mathcal{C}(golden) : y^2 = x^3 + 3x - 1, \quad \Delta(golden) = 27 \cdot 5 = 135 > 0 \tag{2.18}$$

which is the elliptic curve https://www.lmfdb.org/EllipticCurve/Q/2160/x/2.

Also, the expressions of P^2_{golden} (from (2.9)) and P_d (from (2.13)) suggests to generalize them:

Definition 2.3. *If $n \geq 2$ then the n-golden polynomial is the monic polynomial:*

$$P^n_{golden}(x) := x^n + (-1)^{n-1}\frac{n(n-1)}{2}x - 1. \tag{2.19}$$

Let us study the low values of n:

a) $P^4_{golden}(x) = x^4 - 6x - 1$ has two complex roots and two real roots of opposite signs; the strictly positive root is approximately 1.8696. Its discriminant is $\Delta = 256 + 27 \cdot 6^4 = 35248 = 2^4 \cdot 2203$.

   The following matrix belongs to the Lie algebra $sl(4, \mathbb{Z})$ and has as characteristic polynomial exactly P^4_{golden}:

$$\Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

b) $P^5_{golden}(x) = x^5 + 10x - 1$ has four complex roots and a strictly positive root which is approximately 0.09999.

It is easy to prove that for an odd n the polynomial P^n_{golden} has a unique strictly positive root. Indeed, its derivative is

$$(P^n_{golden})'(x) = n\left[x^{n-1} + (-1)^{n-1}\frac{n-1}{2}\right], \tag{2.20}$$

and hence for odd n the function $x \in \mathbb{R} \to P^n_{golden}(x) \in \mathbb{R}$ is strictly increasing. This fact together with the inequality:

$$P^n_{golden}(0) = -1 < 0 < \lim_{x\to+\infty} P^n_{golden}(x) = +\infty \tag{2.21}$$

gives the claimed conclusion.

If n is even, then the derivative polynomial $\left(P^n_{golden}\right)'$ has two roots:

$$x^n_{\pm} = \pm\sqrt[n-1]{\frac{n-1}{2}}, \tag{2.22}$$

and hence the unique strictly positive root of P^n_{golden} is strictly greater than x^n_+. Therefore, if n is even then the minimum of P^n_{golden} is:

$$P^n_{golden}(x^n_+) = -1 - \left(\frac{n}{2} + 1\right)\frac{n-1}{2}x^n_+. \tag{2.23}$$

3 The golden space curve and the golden surface

The expression of the coefficients B and C suggests the map $A \in \mathbb{R} \to \mathcal{C}_{golden} : r(A) = (x(A), y(A), z(A)) = \left(A, \frac{A^2}{3} + 3, \frac{A^3}{27} + A - 1\right) \in \mathbb{R}^3$. This map parametrizes a space curve, which we can call *the golden 3D curve*, and is a skew (cubical) parabola. It belongs to the regular surface:

$$S(golden) : z = \frac{x}{3}\left(\frac{y}{3} - 1\right) + x - 1. \tag{3.1}$$

It follows that this surface can be called *the golden surface*.

The curvature and torsion functions of the golden $3D$ curve are:

$$k(A) = \frac{54\sqrt{162 - 9A^2 + A^4}}{(162 + 54A^2 + A^4)^{\frac{3}{2}}}, \quad \tau(A) = \frac{27}{162 - 9A^2 + A^4}. \tag{3.2}$$

Since $A^4 - 9A^2 + 162 = \frac{1}{4}(2A^2 - 9)^2 + \frac{567}{4}$, we get two inequalities:

$$k(A) \geq \frac{243\sqrt{7}}{(162 + 54A^2 + A^4)^{\frac{3}{2}}}, \quad 0 < \tau \leq \frac{4}{21}. \tag{3.3}$$

The Gaussian curvature of the golden surface is:

$$K(x, y) = \frac{-81}{[81 + x^2 + (y + 6)^2]^2} \in [-1/81, 0). \tag{3.4}$$

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