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ARTICLE

Euclidean-Cardano and Cantor-Cardano cubic polynomials

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Abstract

The aim of this paper is to introduce and study two classes of cubic real polynomials P having the same Euclidean norm as their Cardano resolvent respectively having the square of the Euclidean norm equal to the height of the Cardano resolvent. The polynomial P is giving in its reduced form which means without the cubic term. We find three one-parameter families of polynomials in the first class. For the second class of polynomials we discuss some cases.

Keywords Cubic real polynomial · Cardano resolvent · Euclidean-Cardano polynomial · Cubic curve · Cantor-Cardano polynomial

Mathematics Subject Classification 12D05 · 51N35 · 65H04

1 Introduction

It is commonly recognized that Gerolamo Cardano, by using informations from Scipione del Ferro and Niccolo Fontana Tartaglia, published the first formula for solving cubic equations in his book *Ars Magna* from 1545. In this approach a given monic (and reduced) cubic polynomial P_d is solved through a monic quadratic polynomial $CR(P_d)$ which we call *the Cardano resolvent* of P; see, for example, the excellent paper [1].

The starting point of this note is consider both polynomials P_d and $CR(P_d)$ as vectors in $\mathbb{E}^2 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle_{Euclidean})$ (through their coefficients) and hence to associate their Euclidean norms $\| \cdot \|$. It follows a special class of P_d, namely that for which $\|P_d\| = \|DR(P_d)\|$, and this set is the subject of our research under the name of *Euclidean-Cardano polynomials*. Here P_d is the depressed form of an initial (general) cubic polynomial P.

The defining condition of isometry of the Cardano map $P_d \to CR(P_d)$ is expressed as an algebraic equation in terms of only one coefficient of P_d and it is easy to solve. In this way, we found 3 one-parameter families of Euclidean-Cardano polynomials and we note that a particular one is obtained as fixed point of the Cardano map. It is worth to point out that the iteration of the Cardano map from $\mathbb{R}^2$ to $\mathbb{R}^2$ has a nice expression involving the real function $f(x) = -\frac{x^3}{27}$.

In the last section we introduce a new class of polynomials P, namely that having $\|P_d\|^2 = h(CR(P_d))$ with h being the Cantor height of the polynomial $CR(P_d)$. A discussion is then

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performed in terms of the coefficients of P_d. We finish this introduction pointing that these types of polynomials are studied for the degree 4 in the papers [4, 5] where the Lagrange respectively the Descartes resolvent is used.

2 Euclidean-Cardano cubic polynomials

Fix a natural number $n \in \mathbb{N}^*$. The general setting of this work is provided by the n-dimensional real linear space of monic polynomials of degree n:

$$\mathbb{R}_n^{monic}[x] := \{P(x) = x^n + a_1 x^{n-1} + \cdots + a_n; a_1, ..., a_n \in \mathbb{R}\}.$$

It is an Euclidean space with respect the usual scalar product of $\mathbb{R}^n$ and hence the square of the induced norm is the sum of the square of the coefficients:

$$\|P\|^2 := a_1^2 + \cdots + a_n^2. \tag{2.1}$$

Remark 2.1 It is well-known that in order to prove that the set of real algebraic numbers is countable Cantor defines the *height* of above P as the positive (real) number:

$$h(P) := n + 1 + |a_1| + \cdots + |a_n|.$$

It follows the inequality:

$$(h(P) - n - 1)^2 \leq n\|P\|^2$$

with equality only for the polynomial $\Phi_n(x) = \frac{x^{n+1}-1}{x-1} = x^n + x^{n-1} + \cdots + x + 1$. □

Our study focuses on a fixed cubic polynomial $P \in \mathbb{R}_3^{monic}[x]$:

$$P(x) := x^3 + Ax^2 + Bx + C$$

which with the well-known transformation $x = y - \frac{A}{3}$ have the *reduced* form (here the subscript d means *depressed*):

$$P_d(y) := y^3 + py + q, \quad p = B - \frac{A^2}{3}, \quad q = C - \frac{AB}{3} + \frac{2A^3}{27}. \tag{2.2}$$

We recall very briefly the study of P_d. Being a cubic polynomial it has a real root y_0. We consider now an associated polynomial of degree two:

$$P_0(u) = u^2 - y_0 u - \frac{p}{3} \in \mathbb{R}_2^{monic}[u] \tag{2.3}$$

and hence its roots $\alpha, \beta \in \mathbb{C}$ satisfy:

$$\alpha + \beta = y_0, \quad \alpha\beta = -\frac{p}{3}. \tag{2.4}$$

From $P_d(y_0) = 0$ it follows immediately:

$$\alpha^3 + \beta^3 = -q, \quad \alpha^3\beta^3 = -\frac{p^3}{27} \tag{2.5}$$

which means that α^3 and β^3 are the (complex) roots of the quadratic polynomial:

$$CR(P_d)(z) := z^2 + qz - \frac{p^3}{27}. \tag{2.6}$$

Let us call the polynomial $CR(P_d) \in \mathbb{R}_2^{monic}[z]$ as being the *Cardano resolvent* of P_d since its roots:

$$z_{\pm} := -\frac{q}{2} \pm \sqrt{\frac{\Delta}{4 \cdot 27}}, \quad \Delta := 4p^3 + 27q^2 \tag{2.7}$$

gives the well-known Cardano formula of y_0:

$$y_0 = \sqrt[3]{z_-} + \sqrt[3]{z_+}. \tag{2.8}$$

Remark 2.2 There exists also a *Lagrange resolvent* of P_d: $Z^2 + 27qZ - 27p^3 = 0$, but the map $Z = 27z$ transforms this resolvent into the Cardano resolvent. □

Inspired by [2] we introduce:

Definition 2.3 P (or more precisely P_d) is a *Euclidean-Cardano polynomial* if it preserves the Euclidean norm with respect to Cardano transformation CR:

$$\|P_d\| = \|CR(P_d)\|. \tag{2.9}$$

Hence, the restriction of CR to the set of Euclidean-Cardano polynomials is an Euclidean isometry.

We have immediately:

Proposition 2.4 *P_d is an Euclidean-Cardano polynomial if and only if q is arbitrary and:*

$$p \in \{-3\sqrt{3}, 0, 3\sqrt{3}\}. \tag{2.10}$$

The vanishing of p means that $3B = A^2$ for the initial cubic polynomial P.

Proof The square of the equation (2.9) reads as:

$$p^2 + q^2 = \frac{p^6}{27^2} + q^2 \tag{2.11}$$

with the solutions ($p = 0, \pm 3\sqrt{3}, q \in \mathbb{R}$). Let us remark that these three distinct values defines the elliptic curve $y^2 = (x - 0)(x + 3\sqrt{3})(x - 3\sqrt{3})$ i.e.:

$$y^2 = x^3 - 27x \tag{2.12}$$

which is a CM one; see all its main properties at https://www.lmfdb.org/EllipticCurve/Q/576/a/1. □

Corollary 2.5 *There exist three* 1*-parameter families of reduced Euclidean-Cardano cubic polynomials:*

1) $P_d^1(y; q) = y^3 + q, \quad CR(P_d^1)(z) = z^2 + qz = z(z + q), \quad \Delta^1(q) = 27q^2 \geq 0,$
2) $P_d^2(y; q) = y^3 - 3\sqrt{3}y + q, \quad CR(P_d^2)(z) = z^2 + qz + 3\sqrt{3}, \Delta^2(q) = 27(q^2 - 12\sqrt{3}) \geq -324\sqrt{3},$
3) $P_d^3(y; q) = y^3 + 3\sqrt{3}y + q, \quad CR(P_d^3)(z) = z^2 + qz - 3\sqrt{3}, \quad \Delta^3(q) = 27(q^2 + 12\sqrt{3}) \geq 324\sqrt{3}.$

Remark 2.6 i) It is well-known that P_d is a *strictly hyperbolic polynomial* (i.e. all its roots are real and distinct, [3]) if and only if $\Delta < 0$ which means that $z_-, z_+ \in \mathbb{C}\backslash\mathbb{R}$; for this *casus irreducibilis* see the chapter 7 of the book [6]. Then only P_d^2 satisfies this condition with:

$$|q| < q_0 = 2 \cdot 3^{\frac{3}{4}} \simeq 4.559. \tag{2.13}$$

Hence we have a 1-parameter family of elliptic curves defined by the parameter $q \in (-q_0, q_0)$:

$$E(q) : y^2 = P_d^2(x; q), \quad \Delta \in (-324\sqrt{3}, 0). \tag{2.14}$$

ii) Recall after [2] that the elliptic case of $\Delta < 0$ admits a trigonometrical approach. Namely, the formula (3.16) of the cited paper gives an angle θ provided by:

$$\cos\theta := \frac{3\sqrt{3}q}{2p\sqrt{-p}} \tag{2.15}$$

In our setting this means:

$$\cos\theta = -\frac{q}{q_0} \tag{2.16}$$

which gives the roots of P_d as:

$$y_k = 2\sqrt[4]{3}\cos\left(\theta + \frac{2k\pi}{3}\right), \quad k = 0, 1, 2. \tag{2.17}$$

3 The Cardano map

Let us consider now the *Cardano map*:

$$CR : (p, q) \in \mathbb{R}^2 \to \left(q, -\frac{p^3}{27}\right) \in \mathbb{R}^2 \tag{3.1}$$

which is a bijective map with:

$$CR^{-1}(P, Q) = (-3\sqrt[3]{Q}, P). \tag{3.2}$$

A direct analysis yields:

Proposition 3.1 *The Cardano map* (3.1) *has only one fixed point* $(0, 0)$ *which corresponds to* $P_d^1(y; q = 0)$. *The cubic curve* $y^2 = P_d^1(x; q = 0)$ *is the semicubical parabola* $y^2 = x^3$. *In terms of data introduced in the previous section we have:* $y_0 = \alpha = \beta = 0$.

Remark 3.2 The restriction of the map CR to the class of Euclidean-Cardano polynomials acts as:

$$(0, q) \to (q, 0), \quad (\pm 3\sqrt{3}, q) \to (q, \mp 3\sqrt{3}). \tag{3.3}$$

□

The iteration of the Cardano map has an interesting expression:

$$CR \circ CR(p, q) = -\frac{1}{27}(p^3, q^3) = (f(p), f(q)), \quad f : \mathbb{R} \to \mathbb{R}, \quad f(x) = -\frac{x^3}{27}. \tag{3.4}$$

We finish this section with another remarkable property of an Euclidean-Cardano polynomial P_d different to $P^1_d(\cdot; q = 0)$. The Euclidean angle φ between P_d and its Cardano resolvent is provided through:

$$\cos\varphi = \frac{\langle (p, q), \left(q, -p^3/27\right)\rangle_{Euclidean}}{p^2 + q^2} = \frac{pq}{p^2 + q^2}\left(1 - \frac{p^2}{27}\right) \tag{3.5}$$

meaning that P_d and its $CR(P_d)$ are orthogonal. This means, for example, that the pairs:

$$\frac{1}{\sqrt{q^2 + 27}}(P^2_d, CR(P^2_d)), \quad \frac{1}{\sqrt{q^2 + 27}}(P^3_d, CR(P^3_d))$$

are pairs of orthogonal rays in the unit circle S^1 of the Euclidean plane $\mathbb{E}^2$.

4 Cantor-Cardano cubic polynomials

We introduce now a new class of cubic polynomial in the same setting above but now considering the Cantor height.

Definition 4.1 P (or equivalently P_d) is a *Cantor-Cardano polynomial* if:

$$\|P_d\|^2 = h(CR(P_d)) \tag{4.1}$$

which means:

$$p^2 + q^2 = 3 + |q| + \frac{|p|^3}{27}. \tag{4.2}$$

We analyze some interesting cases according to the value of p.

Case I) For $p = 0$ it follows the characterization:

$$q^2 = 3 + 2|q| \tag{4.3}$$

and hence there are two subcases: $q > 0$ gives $q_+ = \frac{\sqrt{13}+1}{2} \simeq 2.3028$ while $q < 0$ implies $q_- = -q_+$.

Case II) $p = \pm 3\sqrt{3}$ implies the equation:

$$q^2 + 24 = |q| + 3\sqrt{3} \tag{4.4}$$

which has complex roots.

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Declarations

Conflict of interest The author declares no conflict of interest.

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