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A cubic curve associated to a Menelaus-Ceva triple

Mircea Crasmareanu*

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ABSTRACT

This note aims to study the cubic curve naturally associated to a triple of strictly positive numbers satisfying the Menelaus-Ceva condition. Restricting to a fixed triangle Δ we discuss several main points and the Lemoine line of Δ in this unified framework and we point out the cases when the cubic curve is an elliptic one.

Keywords: *cubic curve; elliptic curve; triangle geometry.*

AMS Subject Classification (2020): *Primary: 51M04; Secondary: 14H52*

1. Introduction

One of the main ideas of [2] is the fact that the moduli space of acute triangles is closely related to another famous moduli space, namely that of *elliptic curves*.

The present study continues this relationship between triangles and curves by associating to a given triangle Δ (not necessary acute) a cubic curve $\mathcal{C}$. More precisely, this curve is associated to a triple of collinear points or to a triple of three concurrent lines, these points or lines living naturally in the geometry of Δ. This common approach is based on the similar technical nature of Menelaus and Ceva theorems which ask if the product of three ratios is $+1$. Hence, given a triple $T = (r_1, r_2, r_3)$ of strictly positive numbers we call it *a Menelaus-Ceva triple* if $r_1 r_2 r_3 = 1$ and we associate a cubic curve to such a triple T.

The contents are as follows. The next section concerns with the computations of the Weierstrass (or depressed) coefficients of the cubic $\mathcal{C}$ in terms of $k_1 = \ln r_1$ and $k_2 = \ln r_2$. As main assumption we have that $r_1 \geq 1$ and $r_2 \geq 2$ but this hypothesis is not a restrictive one since in the if r_1 (or r_2 is strictly lower that 1 then we consider the inverse triple $\left(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}\right)$. A special interest is in computing the discriminant D of $\mathcal{C}$ and in the analysis of its vanishing possibility: we obtain that $D = 0$ implies $r_1 = r_2$.

The following section concerns with concrete examples. So, the main points and the Lemoine line of Δ are treated in our common setting and the corresponding cubics are derived. Due to the computational complications in some cases we provide directly the numerical values for the Weierstrass coefficients and D.

The last section uses the Affine-Euclidean two-dimensional geometry to define a natural Menelaus-Ceva triple to a fixed triangle. The case of the right triangle is discussed.

We emphasis that this line of research, namely to associate cubic (particularly elliptic) curves to some remarkable geometric settings was previously realised in the papers [5], [6] and [7]. Also, we no that this paper is dedicated to Professor Dr. Dorin Andrica on the occasion of his 70th birthday.

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2. From Menelaus-Ceva configurations to cubic curves

The famous theorems of Menelaus and Ceva belong to the geometry of a fixed triangle $\Delta = \Delta ABC$ for which each of its sides are dissected by one of the points A', B', C'. More precisely, A' belongs to the line BC, B' is on the line CA while C' belongs to the line AB. A main assumption is that none of the involved points coincides with another. Hence, these theorems are as follows ([9, p. 148-149]):
I) (Menelaus, approximatively the year 100) If A', B' and C' are collinear then:

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = +1. \tag{2.1}$$

II) (Ceva, 1678) If the lines AA', BB', CC' are concurrent then:

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = +1. \tag{2.2}$$

It is worth to remark that the above segments are oriented and A', B', C' are different from A, B, C. The converses of Menelaus and Ceva theorems are also true. Their $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and Nil-variants are recently studied in [12] and [13] respectively.

We remark the common value of the product of ratios involved in these results and hence we introduce:

Definition 2.1. A *Menelaus-Ceva triple* is a triple $T = (r_1, r_2, r_3) \in (0, +\infty)^3$ satisfying $r_1 \cdot r_2 \cdot r_3 = 1$. The associated *logarithmic triple* is $\ln C = (k_1, k_2, k_3) := (\ln r_1, \ln r_2, \ln r_3)$.

In the following we fix the case $(k_1 \geq 0, k_2 \geq 0, k_3 \leq 0)$ since for the opposite case $(k_1 \geq 0, k_2 \leq 0, k_3 \leq 0)$ we replace T by its inverse $\left(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}\right)$.

The geometry of the triangle Δ contains several remarkable cubic curves defined as the locus of the most important points of Δ; see [11]. Our approach here is more analytical and algebraic, namely to define from the beginning a Weierstrass-expressed cubic curve. Therefore, the starting idea of this note is to associate to the given triple T the following cubic curve:

Definition 2.2. The *T-cubic curve* is the implicit defined plane curve:

$$\mathcal{C}(T) : y^2 = x^3 + px + q := (x - k_1)(x - k_2)(x - k_3) \tag{2.3}$$

with the well-known ([3]) *discriminant*: $D(\mathcal{C}(T)) := 4p^3 + 27q^2$.

Hence, the main theoretical result is the expression of D as function of the first two parameters k_1, k_2:

Theorem 2.1. *The values of the Weierstrass coefficients and the discriminant are:*

$$\begin{cases} p = -(k_1^2 + k_1k_2 + k_2^2) \leq 0, \quad q = k_1k_2(k_1 + k_2) \geq 0, \\ D(\mathcal{C}(T)) = -4(k_1^2 + k_2^2)^2(k_1^2 + 3k_1k_2 + k_2^2) + 5k_1^2k_2^2(3k_1^2 + 10k_1k_2 + 3k_2^2). \end{cases} \tag{2.4}$$

Proof Replacing the value of $k_3 = -k_1 - k_2$ in expressions $p = k_1k_2 + k_2k_3 + k_3k_1$ and $q = -k_1k_2k_3$ yields the claimed relations. It is worth to remark that the AM-MG inequality $0 \leq 2k_1k_2 \leq k_1^2 + k_2^2$ yields some bounds for $D(\mathcal{C}(T))$:

$$16k_1^3k_2^3 - 4(k_1^2 + k_2^2)^2(k_1^2 + 3k_1k_2 + k_2^2) \leq D(\mathcal{C}(T)) \leq -20k_1k_2(k_1^2 + k_2^2)^2 + 5k_1^2k_2^2(3k_1^2 + 10k_1k_2 + 3k_2^2). \tag{2.5}$$

□

It follows that we can define a special class of triples provided by the vanishing of $D(\mathcal{C}(T))$:

Definition 2.3. T is called *singular-cubic triple* if:

$$4(k_1^2 + k_2^2)^2(k_1^2 + 3k_1k_2 + k_2^2) = 5k_1^2k_2^2(3k_1^2 + 10k_1k_2 + 3k_2^2). \tag{2.6}$$

Proposition 2.1. *If T is a singular-cubic triple then $k_1 = k_2$ i.e. $r_1 = r_2$.*

Proof The relation (2.6) can be written:

$$\frac{3k_1^2+10k_1k_2+3k_2^2}{k_1^2+3k_1k_2+k_2^2}=\frac{4}{5}\cdot\left(\frac{k_1^2+k_2}{k_1k_2}\right)^2 \tag{2.7}$$

and the AM-GM inequality gives that the right-hand-side is $\geq \frac{16}{5}$. Hence, we have:

$$5(3k_1^2+10k_1k_2+3k_2^2)\geq 16(k_1^2+3k_1k_2+k_2^2) \tag{2.8}$$

which reduces to $2k_1k_2 \geq k_1^2+k_2^2$. Therefore, we must have the equality in the AM-GM inequality for the pair (k_1, k_2).

A second proof is obtained by remarking that the left-hand-side and the right-hand-side of (2.6) are both homogeneous polynomials of order 6. Replacing $k_1=0$ in (2.6) yields $k_2=0$. If $k_1>0$ we use the substitution $k_2=\lambda k_1$ to obtain that (2.6) means:

$$4(1+\lambda^2)^2(1+3\lambda+\lambda^2)-5\lambda^2(3+10\lambda+\lambda^2)=0. \tag{2.9}$$

But the left-hand-side polynomial above is $(\lambda-1)^2(\lambda+2)^2(2\lambda+1)^2$ and hence the only strictly positive solution is (the double solution) $\lambda=1$. □

Remark 2.1. i) We point out that the double roots above $\left(\lambda=1,\lambda=-2,\lambda=-\frac{1}{2}\right)$ are the opposite of the values $\left(-1,2,\frac{1}{2}\right)$ yielding elliptic curves in the case i) of Theorem 1 from [8].
ii) The case of equality $k_1=k_2(=k)$ provides the 1-parameter family of singular cubic curves:

$$\mathcal{C}(k): y^2=(x-k)^2(x+2k)=x^3-3k^2x+2k^3. \tag{2.10}$$

It is amazing that the particular cubic curve associated to $T=(e,e,e^{-2})$:

$$\mathcal{C}(k=1): y^2=x^3-3x+2 \tag{2.11}$$

has 12 integral points $(x,y)\in\mathbb{Z}\times\mathbb{Z}$: $(-2,0)$, $(1,0)$, $(-1,\pm 2)$, $(2,\pm 2)$, $(7,\pm 18)$, $(14,\pm 52)$, $(23,\pm 110)$. More generally than (2.11) the 1-parametric cubic curve:

$$\mathcal{C}_t: y^2=x^3-3x+2-\frac{27t^2}{4} \tag{2.12}$$

being elliptic for $t\neq 0$ is studied in the paper [10] from the point of view of CM-multiplication.
iii) For the general cubic polynomial $P\in\mathbb{R}_3^{monic}[x]$:

$$P(x):=x^3+Ax^2+Bx+C \tag{2.13}$$

the well-known Tschirnhausen transformation $x=y-\frac{A}{3}$ yields *reduced* form (here the subscript d means *depressed*):

$$P_d(y):=y^3+py+q,\quad p=B-\frac{A^2}{3}\leq B,\quad q=C-\frac{AB}{3}+\frac{2A^3}{27}. \tag{2.14}$$

By asking P_d to be the right-hand-side of the cubic curve (2.11) we obtain:

$$B=\frac{A^2}{3}-3,\quad C=\frac{A^3}{27}-A+2 \tag{2.15}$$

which means the 1-parameter family of cubic polynomials:

$$P_A(x)=\left(x+\frac{A}{3}\right)^3-3x+2-A. \tag{2.16}$$

The 1-parameter family of cubic curves $\mathcal{C}_A: y^2=P_A(x)$ has as envelope the pair of horizontal lines:

$$Envelope(\mathcal{C}_A): y_{\pm}=\pm\sqrt{2-\frac{8}{3\sqrt{3}}}\simeq\pm 0.6785. \tag{2.17}$$

□

3. Concrete examples

In the following we return to the initial triangle Δ with the sides $c \leq b \leq a$ and we study some of its remarkable points P and lines l. We denote directly $D(P)$ respectively $D(l)$ the corresponding discriminant.

Example 3.1. The centroid G is the first natural example with the same Ceva triple $T = (1, 1, 1)$ which is a singular-cubic triple. The associated cubic curve is the *semicubical parabola*: $\mathcal{C}(G) : y^2 = x^3$. $\square$

Example 3.2. The next important case is of the incenter I provided by the Ceva triple $T = \left(r_1 = \frac{b}{c} \geq 1, r_2 = \frac{a}{b} \geq 1, r_3 = \frac{c}{a} \leq 1\right)$. Then the Weierstrass coefficients are:

$$\begin{cases} p = \ln a \ln b + \ln b \ln c + \ln c \ln a - (\ln a)^2 - (\ln b)^2 - (\ln c)^2, \\ q = (\ln a - \ln c)[\ln a \ln b + \ln b \ln c - \ln c \ln a - (\ln b)^2]. \end{cases} \tag{3.1}$$

A remarkable particular case is when Δ is a *proper isosceles triangle*: $a = b > c$. Hence $\mathcal{C}(I)$ is an elliptic curve with:

$$p = -\left(\ln \frac{a}{c}\right)^2 < 0, \quad q = 0, \quad \mathcal{C}(I) : y^2 = x\left(x - \ln \frac{a}{c}\right)\left(x - \ln \frac{c}{a}\right). \tag{3.2}$$

Since $\frac{a}{c} = \frac{1}{2\cos A}$ we derive a new formula for this case of p:

$$p = -(\ln(2\cos A))^2 \tag{3.3}$$

and the concrete example $A = B = \frac{\pi}{4}$ gives the transcendental number $p = -\frac{(\ln 2)^2}{4} \simeq -0.1201$.

If Δ is the right-angled triangle with $c = 3$, $b = 4$ and $a = 5$ then $p \simeq -0.196$, $q \simeq 0.032$ and $\mathcal{C}(I)$ is again an elliptic curve with $D \simeq -0.024 < 0$. We invite the interested reader to check the general case of Pythagorean triples, using the well-known formulae of [4]. $\square$

Example 3.3. The orthocenter H of an acute Δ corresponds to the Ceva triple:

$$T = \left(r_1 = \frac{c\cos B}{b\cos C} = \frac{\tan C}{\tan B}, r_2 = \frac{a\cos C}{c\cos A} = \frac{\tan A}{\tan C}, r_3 = \frac{b\cos A}{a\cos B} = \frac{\tan B}{\tan A}\right) \tag{3.3}$$

and then:

$$p = [\ln(\tan C) - \ln(\tan B)][\ln(\tan A) - \ln(\tan C)] + [\ln(\tan A) - \ln(\tan C)][\ln(\tan B) - \ln(\tan A)] +$$
$$+ [\ln(\tan B) - \ln(\tan A)][\ln(\tan C) - \ln(\tan B)], \tag{2.4}$$
$$q = -[\ln(\tan C) - \ln(\tan B)][\ln(\tan A) - \ln(\tan C)][\ln(\tan B) - \ln(\tan A)]. \tag{3.5}$$

For the example of $C = 40°$, $B = 60°$ and $A = 80°$ we have the elliptic curve with $p \simeq -2.791$, $q \simeq -1.6425$ and $D(H) \simeq -14.167$. $\square$

Example 3.4. The circumcenter O of an acute triangle have the Ceva triple:

$$T = \left(r_1 = \frac{\sin 2B}{\sin 2C} \geq 1, r_2 = \frac{\sin 2A}{\sin 2B} \geq 1, r_3 = \frac{\sin 2C}{\sin 2A} \leq 1\right) \tag{3.6}$$

and due to the similarities with the previous cases we dot not continue with the expression of p and q. $\square$

Example 3.5. Recall that the semiperimeter s of Δ is defined through $2s = a + b + c$. The Nagel point (or the center $X(8)$ in the ETC-the Encyclopedia of Triangle Centers; $G = X(2)$, $O = X(3)$ and $H = X(4)$) of Δ ([1]) is given by the Ceva triple:

$$T = \left(r_1 = \frac{s-b}{s-a} \geq 1, r_2 = \frac{s-c}{s-b} \geq 1, r_3 = \frac{s-a}{s-c} \leq 1\right) \tag{3.7}$$

and therefore:

$$p = [\ln(s-b) - \ln(s-a)][\ln(s-c) - \ln(s-b)] + [\ln(s-c) - \ln(s-b)][\ln(s-a) - \ln(s-c)] +$$
$$+ [\ln(s-a) - \ln(s-c)][\ln(s-b) - \ln(s-a)], \tag{3.8}$$

$$q = -[\ln(s-b) - \ln(s-a)][\ln(s-c) - \ln(s-b)][\ln(s-a) - \ln(s-c)]. \tag{3.9}$$

For the particular case of the right-angled triangle of the example 3.2 since $s = 6$ we get an elliptic curve with:

$$\begin{cases} r_1 = 2, \quad r_2 = \frac{3}{2}, \quad r_3 = \frac{1}{3}, \quad p = \ln 2 \ln 3 - (\ln 2)^2 - (\ln 3)^2 \simeq -0.925, \\ q = \ln 2 \ln 3 (\ln 3 - \ln 2) \simeq 0.308, \quad D \simeq -0.604. \end{cases} \tag{3.10}$$

□

Example 3.6. Let $\mathcal{C}(O, R)$ be the circumcircle of Δ and let A' be the point of intersection of the line BC with the tangent line in A to $\mathcal{C}(O, R)$. Similarly, we have the points: B' on the line CA and C' on the line AB. Then these three points are collinear and their defining line is called the *Lemoine line* l_L of Δ. More precisely, l_L is provided by the Menelaus triple $\left(r_1 = \left(\frac{c}{b}\right)^2, r_2 = \left(\frac{a}{c}\right)^2, r_1 = \left(\frac{b}{a}\right)^2\right)$ and hence we have:

$$p(l_L) = 4p(I), \quad q(l_L) = 8q(I). \tag{3.11}$$

The isosceles case $a = b > c$ gives the elliptic curve:

$$\mathcal{C}(l_L) : y^2 = x\left(x - 2\ln\frac{a}{c}\right)\left(x - 2\ln\frac{c}{a}\right). \tag{3.12}$$

□

4. A natural Menelaus-Ceva pair from the 2-dimensional Affine-Euclidean geometry

Suppose now that the given triangle ΔABC is acute and placed in the Affine-Euclidean plane π in which we fix an orthonormal frame $\mathbb{R} = \{O; \bar{i}, \bar{j}\}$. Since the translation is an Euclidean isometry we can suppose that $B = O$ and then the triangle has the vertices $A(u, v)$, $B(0, 0)$, $C(a, 0)$. The equations:

$$u^2 + v^2 = c^2, \quad (u-a)^2 + v^2 = b^2 \tag{4.1}$$

give:

$$u = \frac{a^2 - b^2 + c^2}{2a} (= c\cos B) \in [0, a), \quad v = \frac{\sqrt{[(a+c)^2 - b^2][b^2 - (a-c)^2]}}{2a} = c \sin B > 0. \tag{4.2}$$

Then we have the decompositions:

$$\overrightarrow{BC} = a\bar{i}, \quad \overrightarrow{CA} = (u-a)\bar{i} + v\bar{j}, \quad \overrightarrow{AB} = -u\bar{i} - v\bar{j} \tag{4.3}$$

giving the well-known vectorial relation $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \bar{0}$. Looking at the coefficients of the unit vector $\bar{i}$ we get a logarithmic triple $(k_1, k_2, k_3) = (a > 0, u - a < 0, -u \leq 0)$ and hence we can define:

Definition 4.1. i) The *natural Menelaus line* of Δ is that provided by the Menelaus Theorem applied to the triple $T = (e^u \geq 1, e^{a-u} > 1, e^{-a} < 1)$.
i) The *natural Ceva point* of Δ is that provided by the Ceva Theorem applied to the triple $T = (e^u \geq 1, e^{a-u} > 1, e^{-a} < 1)$.

Example 4.1. Suppose that $B = \frac{\pi}{2}$; then $u = 0$ and $v = c$ which means the Menelaus-Ceva triple $T = T(a) = (1, e^a > 1, e^{-a} < 1)$ where a is one of the legs in the right-angled triangle Δ. Hence:

$$\mathcal{C}(T)(a) : y^2 = x(x^2 - a^2), \quad D(\mathcal{C}(T))(a) = -4a^6 < 0. \tag{4.4}$$

□

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