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DOUBLE-DEPRESSED EUCLIDEAN-DESCARTES POLYNOMIALS AND ASSOCIATED CUBIC CURVES REVISION

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Abstract. The aim of this paper is to continue the study [5] of quartic real polynomials P having the same Euclidean norm as their Descartes resolvent $DR(P)$. The polynomial P is given in its reduced form P_d which means without a cubic term while double-depressed means the lack of the quadratic term. The class of these polynomials is found to consists in two 1-parameter families and the discriminant of cubic curve defined naturally by $DR(P_d)$ is computed. We consider also the double-depressed Descartes resolvent as a self-map of $\mathbb{R}^2$ and its fixed points are determined. Some special angles appear throughout this papers since the coefficients of P_d are trigonometrical functions of the real parameter t. **Keywords**: Quartic real polynomial, Descartes resolvent, Euclidean-Descartes polynomial, Cubic curve, Discriminant.

MSC (2020): 12D05, 51N35, 65H04..

1. Introduction

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The well-known Descartes method associates to a given monic (and reduced) quartic polynomial P_d its Descartes resolvent $DR(P_d)$ as a monic cubic polynomial. In the paper [5] we have considered both polynomials P_d and $DR(P_d)$ as vectors in $\mathbb{R}^3$ and hence we introduce and study the class of *Euclidean-Descartes polynomials* by asking the equality

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$\|P_d\| = \|DR(P_d)\|$ where $\|\cdot\|$ is the usual Euclidean norm of $\mathbb{R}^n$, $n \in \mathbb{N}^*$. The method of Lagrange is the subject of the paper [3].

The defining condition of isometry of the Descartes map $P_d \to DR(P_d)$ was expressed as an equation in terms of the coefficients (p, q, r) of P_d and to solve it we have fixed q in the set $\{0, \pm 1\}$. The present paper continues this line of research with another condition, namely $p = 0$; therefore P_d and $DR(P_d)$ belong now to $\mathbb{R}^2$. Since p appears in the coefficients of x^2 and z^2 of P_d respectively $DR(P_d)$ we call this case as being *double-depressed*. It follows a quadratic equation for the pair (q^2, r) which is naturally interpreted as the equation of an ellipse for which we provide the eccentricity e.

Since an ellipse is usually parametrised with the trigonometrical functions $(\cos t, \sin t)$ we obtain that $q = \cos t$ while r comes in pairs $r_\pm = \pm\frac{\sin 2t}{2\sqrt{15}}$ for the arbitrary real number t. It follows two 1-parametric families of double-depressed quartic polynomials. A second interest in the cubic curves $y^2 = DR(P_d(t))(x)$ for which we compute the discriminant Δ; on this way we continue the studies of [2] and [4]. Hence, we discuss also the vanishing of Δ which gives a special angle $\simeq 80.96°$.

The last section deals with the fixed points of the double-depressed Descartes map $ddDR : (q, r) \in \mathbb{R}^2 \to (-4r, -q^2) \in \mathbb{R}^2$. In addition to the obvious $(0, 0)$ we get the fixed point $(1/4, -1/16)$ implying a second remarkable angle, namely $\arccos\left(\frac{1}{4}\right)$. We point out that the associated cubic curve is not an elliptic one over the field $\mathbb{R}$, see e.q. [6], since its discriminant Δ is strictly positive. Another important conclusion is the non-existence of double-depressed Euclidean-Descartes quartic polynomials with multiple roots.

2. Double depressed Euclidean-Descartes quartic polynomials

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Fix a natural number $n \in \mathbb{N}^*$. The general framework of this paper is provided by the n-dimensional real linear space of monic polynomials of grade n:

$$\mathbb{R}_n^{monic}[x] := \{P(x) = x^n + a_1 x^{n-1} + ... + a_n; a_1, ..., a_n \in \mathbb{R}\}.$$

Since $\mathbb{R}^n$ is an Euclidean space with respect the usual scalar product $\langle \cdot, \cdot \rangle$ the polynomial P has the Euclidean norm:

$$\|P\| := \left(a_1^2 + ... + a_n^2\right)^{\frac{1}{2}}. \tag{2.1}$$

Fix now the quartic polynomial $P \in \mathbb{R}_4^{monic}[x]$:

$$P(x) := y^4 + Ay^3 + By^2 + Cy + D$$

to which we apply the Tschirnhausen transformation $y = x - \frac{A}{4}$ have the *reduced* form (here the subscript d means *depressed*):

$$P_d(x) := x^4 + px^2 + qx + r. \tag{2.2}$$

The Descartes approach for P_d is one of (relative many) methods to deal with P_d; see e.g. [7, p. 174] as well as [8] for an universal method to solve the quartic equation. More precisely, Descartes performs a product decomposition of P_d into two quadratic polynomials:

$$P_d(x) = (x^2 + ax + b)(x^2 + \tilde{a}x + \tilde{b}) \tag{2.3}$$

and therefore:

$$a + \tilde{a} = 0, \quad b + \tilde{b} + a\tilde{a} = p, \quad a\tilde{b} + \tilde{a}b = q, \quad b\tilde{b} = r. \tag{2.4}$$

It follows immediately:

$$4r = 4b\tilde{b} = (b + \tilde{b})^2 - (b - \tilde{b})^2 = (p + a^2)^2 - \left(\frac{q}{a}\right)^2 \tag{2.5}$$

and hence, denoting $a^2 = z$, it follows a cubic equation in u:

$$DR(P_d)(z) := z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0 \tag{2.6}$$

and let us call the polynomial $DR(P_d) \in \mathbb{R}_3^{monic}[z]$ as being the *Descartes resolvent* of P_d. From:

$$DR(P_d)(0) = -q^2 \leq 0 < +\infty = \lim_{z \to +\infty} DR(P_d)(z)$$

the Descartes resolvent have at least one positive root $z_0 = a^2$ which yields the whole data $(a, \tilde{a} = -a, b, \tilde{b})$ through the relations (2.4). In addition to the previous decomposition of P_d the Descartes resolvent is also useful in the following irreducibility characterization: suppose that $P_d \in \mathbb{Q}[x]$. Then P_d is irreducible over $\mathbb{Q}$ if and only if P_d does not have a rational root and $DR(P_d)$ does not have a root that is a square of a rational number.

In a previous study [5] of quartic polynomials we introduce:

Definition 2.1. *i) P (or equivalently P_d) is an Euclidean-Descartes polynomial if it preserves the Euclidean norm with respect to Descartes transformation DR:*

$$\|P_d\| = \|DR(P_d)\|. \tag{2.7}$$

So, the restriction of DR to the class of Euclidean-Descartes polynomials is an Euclidean isometry.
ii) The Descartes-cubic curve associated to (arbitrary) P_d is:

$$DC(P_d) : y^2 = DR(P_d)(x). \tag{2.8}$$

A straightforward computation yields:

Proposition 2.1. *P_d is an Euclidean-Descartes polynomial if and only if:*

$$p^4 + q^4 + 15r^2 - 8p^2r + 3p^2 - q^2 = 0. \tag{2.9}$$

With the 3D-transformation $(X := p^2, Y := q^2, Z = r)$ we have the equation of a two-sheets hyperboloid:

$$H_2(P_d) : X^2 + Y^2 + 15Z^2 - 8XZ + 3X - Y = 0. \tag{2.10}$$

Proof. The hypothesis means that:

$$\|P_d\|^2 = \|DR(P_d)\|^2 \to p^2 + q^2 + r^2 = 4p^2 + (p^2 - 4r)^2 + q^4$$

which is exactly (2.9). □

In [5] we point out an easy case of reduction namely $q \in \{0, \pm 1\}$ and hence P_d which such a coefficient was be called *reduced*. Now, we remark that the coefficient of x^2 in P_d is half the coefficient of z^2 in $DR(P_d)$ and then we introduce a new class:

Definition 2.2. *The depressed polynomial P_d is called double-depressed if $p = 0$.*

Remark 2.1. *i) If we look at the initial quartic polynomial P since $p = B - \frac{3A^2}{8}$ it follows that the double-depressed condition means:*

$$B = \frac{3A^2}{8}. \tag{2.11}$$

Concerning the Descartes coefficients $(a, \tilde{a} = -a, b, \tilde{b})$ the vanishing of p means: $b + \tilde{b} = a^2 = z_0$.

ii) The Descartes-cubic curve is:

$$DC(P_d) : y^2 = x^3 + Px + Q, \quad P = -4r, \quad Q = -q^2 \tag{2.12}$$

and we recall its discriminant: $\Delta := 4P^3 + 27Q^2$.

iii) The intersection of the hyperboloid $H_2(P_d)$ with the plane $X = 0$ is the ellipse:

$$E : Y^2 + 15Z^2 - Y = 0 \tag{2.13}$$

with the eccentricity:

$$e = \sqrt{\frac{14}{15}} \simeq 0.966092. \tag{2.14}$$

□

Since it is easy to gives a parametrization of E it follows a complete characterization of double-depressed quartic polynomials:

Theorem 2.1. *The class of double-depressed quartic polynomials consists in two 1-parametric families with the coefficients:*

$$q(t) = \cos t, \quad r_{\pm}(t) = \pm \frac{\sin 2t}{2\sqrt{15}}, \quad t \in \mathbb{R}. \tag{2.15}$$

The discriminant of the Descartes-cubic curve $DC(P_d)(t)$ is:

$$\begin{cases} DC(P_d)(t) : y^2 = x^3 \mp \left(\frac{2\sin 2t}{\sqrt{15}}\right) x - (\cos t)^2, \\ \Delta_{\pm}(t) = (\cos t)^3 \left[27\cos t \mp \frac{256}{15\sqrt{15}} (\sin t)^3\right]. \end{cases} \tag{2.16}$$

Proof. The equation (2.9) with $p = 0$ means:

$$15r^2 = q^2(1 - q^2) \tag{2.17}$$

forcing q to be a cosine of a real parameter t. Hence, the claimed expressions (2.15) follow immediately. Also, the value of the discriminant being:

$$\Delta = 27q^4 - 16^2 r^3 \tag{2.18}$$

the formulae (2.16) is directly. □ □

We are interested now in the vanishing of $\Delta_{\pm}$:
i) $\Delta_{-} = 0$ implies $\cos t \in \{0, \simeq 0.1572\}$; this means $t \in \{90^{\circ}, \simeq 80.96^{\circ}\}$
ii) $\Delta_{+} = 0$ implies $\cos t \in \{0, \simeq -0.1572\}$.

3. The fixed points of the double-depressed Descartes map

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A second way to find double-depressed Euclidean-Descartes polynomials is to find the fixed points of the *dd-Descartes map*:

$$ddDR : (q, r) \in \mathbb{R}^2 \to (-4r, -q^2) \in \mathbb{R}^2 \tag{3.1}$$

or, in a matrix form:

$$ddDR : \begin{pmatrix} q \\ r \end{pmatrix} \in \mathbb{R}^2 \to \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q^2 \\ r^2 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} \in \mathbb{R}^2. \tag{3.2}$$

By denoting Γ_1 and Γ_2 the 2×2 matrices above it results that $\Gamma_2 = 4(\Gamma_1)^T$. The Descartes map is a bijective map on its image $\mathbb{R}^2_{-} := \{(A, B) \in \mathbb{R}^2; B \leq 0\}$ with:

$$(ddDR)^{-1}(A, B) = \left(\pm\sqrt{-B}, -\frac{A}{4}\right) \tag{3.3}$$

and the iteration of the map $ddDR$ produces the interesting quadratic map:

$$(q, r) \in \mathbb{R}^2 \to (4q^2, -16r^2) \in \mathbb{R}^2. \tag{3.4}$$

A direct computation yields:

Proposition 3.1. *The dd-Descartes map* (3.1) *has only two fixed points:* $(0, 0)$ *which corresponds via* (2.15) *to* $t_1 = \frac{\pi}{2}$ *and* $\left(\frac{1}{4}, -\frac{1}{16}\right)$ *which corresponds to* $r_{-}(t_2 = \arccos(0.25) \simeq 75.52^{\circ})$.

It results:
a) $DC(P_d)(t_1) : y^2 = x^3$, which is the *semicubical parabola*, $\Delta(t_1) = 0$ for both signs $\pm$,
b) $DR(P_d(t_2))(z) = z^3 + \frac{z}{4} - \frac{1}{16}, \|P_d(t_2)\| = \frac{\sqrt{17}}{16}, DC(P_d)(t_2) : y^2 = x^3 + \frac{x}{4} - \frac{1}{16}, \Delta_{-}(t_2) = \frac{43}{256} > 0$. $DR(P_d(t_2))$ has an unique real root $z_0 \simeq 0.2119$. We recall that a real polynomial P with all real roots is called *hyperbolic*, see e.g. [1]; so, $DR(P_d(t_2))$ is not a hyperbolic polynomial.

The fixed point $\left(\frac{1}{4}, -\frac{1}{16}\right)$ corresponds to the point $(Y, Z) = \left(\frac{1}{16}, -\frac{1}{16}\right)$ which is the intersection of the ellipse E with the second bisectrix $B_2 : y = -x$.

It is worth to point out $P_d(t_1)(x) = x^4$ have multiple roots. We finish this section with the problem of existence of double-depressed Euclidean-Descartes polynomials with multiple roots. In [5] we have obtained:

Proposition 3.2. *Excepting $P_d(x) = x^4$ there are only two Euclidean-Descartes polynomials with multiple roots:*

$$\begin{cases} P_d^{\pm}(x) = x^4 \pm 4\sqrt{3}x^2 + 12 = (x^2 \pm 2\sqrt{3})^2, \\ DR(P_d^{\pm})(z) = z^2(z \pm 8\sqrt{3}), \quad \|P_d^{\pm}\| = 8\sqrt{3}. \end{cases} \tag{3.4}$$

The value $p = \pm 4\sqrt{3} \neq 0$ means that there are no double-depressed Euclidean-Descartes polynomials with multiple roots.

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